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EIGENVALUES AND DIOPHANTINE APPROXIMATION

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Put $||x|| = \text{dist}(x, \mathbf{Z})$ for real x, wherer $\mathbf{Z}$ is the set of integers. Let a be an irrational number and b any real number. H. Kesten [3] proved that the sum

$$\sum_{n=1}^{\infty} \frac{||nb||^2}{n^2||na||^2}$$

is finite if and only if $b \equiv k\,a \pmod 1$ for some integer k. This is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} \frac{|1 - e^{2\pi inb}|^2}{n^2|1 - e^{2\pi ina}|^2}$$

since there exist two positive constants A, B satisfying

$$A||x|| \leq |1 - e^{2\pi ix}| \leq B||x||.$$

Later K. Peterson [4] found a simple proof using Hilbert space methods. And H. Helson [1],[2] obtained the same result while studying the properties of cocycles defined on the unit circle. In this article, by further exploiting the idea in K. Peterson's paper [5], it is shown that in two dimensional case there is a similar Diophantine approximations of irrational numbers.

We identify the 2-torus $\mathbf{T}^2$ which is a compact abelian group with $[0,1) \times [0,1)$ where $[0,1)$ is the half-open unit interval in the real line. Choose two irrational numbers α, $\beta \in [0,1)$ such that $1, \alpha, \beta$ are linearly independent over the field of rational numbers. For real number x, let $\{x\}$ denote the fractional part of x. Let Γ be the subgroup of $\mathbf{T}^2$ generated by $\gamma_0 = (\alpha, \beta) \in \mathbf{T}^2$.

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A real valued function v on $\Gamma \times \mathbf{T}^2$ is called an *additive coboundary* if

$$v(\gamma, g) = w(g+\gamma) - w(g)$$

for some real valued measurable function w where $\gamma \in \Gamma$, $g \in \mathbf{T}^2$. And if w is square-integrable, then v is called an L^2 – *coboundary*. Note that the relation

$$v(\gamma_0, g) = w(g+\gamma_0) - w(g)$$

defines a coboundary v uniquely, and v satisfies $v(n\gamma_0, g) = w(g+n\gamma_0) - w(g)$.

Choose δ, $\epsilon \in (0,1)$ and put $E = \{(x,y) : 0 \le x < \delta, 0 \le y < \epsilon\} \subset \mathbf{T}^2$. Consider the function $v = \chi_E - \delta \cdot \epsilon$ where χ_E is the characteristic function of E. We will see that v is not an L^2-coboundary for some δ, ϵ. We expand $v(x,y)$ into its Fourier series:

$$v(x,y) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} a_{mn}\, e^{2\pi i(mx+ny)}$$

where

$$a_{mn} = -\frac{1}{(2\pi)^2} \cdot \frac{1}{mn}(e^{-2\pi i m\delta} - 1)(e^{-2\pi i n\epsilon} - 1), \quad mn \ne 0,$$

$$a_{m0} = -\frac{i\epsilon}{2\pi} \cdot \frac{1}{m}(e^{-2\pi i m\delta} - 1), \quad m \ne 0,$$

$$a_{0n} = -\frac{i\delta}{2\pi} \cdot \frac{1}{n}(e^{-2\pi i n\epsilon} - 1), \quad n \ne 0, \quad \text{and}$$

$$a_{00} = 0.$$

Suppose that there exists a square-integrable functionw on $\mathbf{T}^2$ such that $v(x,y) = w(x+\alpha, y+\beta) - w(x,y)$, that is, v is an L^2-coboundary. Now let

$$w(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} b_{mn} e^{2\pi i(mx+ny)}$$

be the formal Fourier expansion of w. Then $a_{mn} = (e^{2\pi i(m\alpha+n\beta)}-1)b_{mn}$. So the necessary and sufficient condition for the existence of such a square-integrable function w is the convergence of the following three sums:

$$\sum_{m\neq 0}\sum_{n\neq 0}\frac{|e^{-2\pi im\delta}-1|^2|e^{-2\pi in\epsilon}-1|^2}{m^2n^2|e^{-2\pi i(m\alpha+n\beta)}-1|^2},$$

$$\sum_{m\neq 0}\frac{|e^{-2\pi im\delta}-1|^2}{m^2|e^{-2\pi im\alpha}-1|^2},$$

$$\sum_{n\neq 0}\frac{|e^{-2\pi in\epsilon}-1|^2}{n^2|e^{-2\pi in\beta}-1|^2}.$$

By Kesten's theorem, the second and the third sums are finite if and only if $\delta \equiv k\alpha$, $\epsilon \equiv \ell\beta \pmod 1$ for some integers k, ℓ. Now suppose that the first sum is also finite. That is,

$$S_{\alpha,\beta}(k,\ell) \equiv \sum_{m\neq 0}\sum_{n\neq 0}\frac{|e^{-2\pi imk\alpha}-1|^2|e^{-2\pi in\ell\beta}-1|^2}{m^2n^2|e^{-2\pi i(m\alpha+n\beta)}-1|^2} < \infty.$$

Put $q = e^{2\pi iw}$. Then we get $q(x+\alpha, y+\beta) = e^{2\pi iv(x,y)}q(x,y) = e^{2\pi i(-\delta\epsilon)}q(x,y) = e^{2\pi i(-\{k\alpha\}\{\ell\beta\})}q(x,y)$. Now we define a unitary operator U on $L^2(\mathbf{T})$ by $(Uf)(x,y) = f(x+\alpha, y+\beta)$. Then the eigenvalues of U are of the form $e^{2\pi i(m\alpha+n\beta)}$ where m, n are integers. Since $Uq = e^{2\pi i\{k\alpha\}\{\ell\beta\}}q$, we have $\{k\alpha\}\{\ell\beta\} \equiv m_1\alpha + n_1\beta \pmod 1$ for some m_1, n_1. Hence $k\alpha\cdot\ell\beta \equiv m\alpha + n\beta \pmod 1$ for some m, n.

By summarizing the previous results, we can answer the question raised by K. Peterson [5].

PROPOSITION 1. *Let* $E = \{(x,y) : 0 \le x < \gamma,\quad 0 \le y < \delta\} \subset \mathbf{T}^2$. *Then we have the following:*

(i) The function $\chi_E - \gamma\cdot\delta$ *is not an* L^2*-coboundary with possible exceptions when* γ, δ *are integral multiples of* α, β *(mod 1), respectively.*

(ii) Suppose $\gamma \equiv k\alpha$, $\delta \equiv \ell\beta$ *and* $k\ell\alpha\beta \not\equiv m\alpha + n\beta$ *(mod 1) for some integers* k, ℓ, $k\ell \neq 0$. *Then the functions* $\chi_E - \gamma\cdot\delta$ *is not an* L^2*-coboundary.*

(iii) If $S_{\alpha,\beta}(k,\ell) < \infty$, then $k\ell\alpha\beta \in \mathbf{Z}\cdot\alpha + \mathbf{Z}\cdot\beta + \mathbf{Z}$.

COROLLARY. *If $\alpha\beta$ is a rational number, then $S_{\alpha,\beta}(1,1) = \infty$.*

Proof. Let $r = \alpha\beta$. Then $0 < r < 1$ and $r = m\alpha + n\beta + \ell$ for some integers m, n, ℓ. Hence the linear independence of the three numbers 1, α, β implies that $r = \ell$, which is a contradiction.

PROPOSITION 2. *Suppose $S_{\alpha,\beta}(k,\ell)$ is finite for some $k \neq 0$, $\ell \neq 0$. If one of α, β is an algebraic number, then the other is also algebraic and $\deg(\alpha) = \deg(\beta) > 2$.*

Proof. Let $\mathbf{Q}(a)$ denote the smallest extension field of $\mathbf{Q}$ containing an irrational number a. Since $k\ell\alpha\beta \equiv m\alpha + n\beta \pmod 1$, $\alpha(k\ell\beta - m) = n\beta + p$, $\alpha = (n\beta + p)/(k\ell\beta - m)$ for some integers m, n and p. Hence α belongs to $\mathbf{Q}(\beta)$. Similarly, β belongs to $\mathbf{Q}(\alpha)$, too. Hence $\mathbf{Q}(\alpha) = \mathbf{Q}(\beta)$, and α and β have the same algebraic degree. Now suppose that $\deg(\alpha) = \deg(\beta) = 2$. Since $\mathbf{Q}(\alpha)$ has dimension 2 over the field $\mathbf{Q}$ and since 1, α, β are all in $\mathbf{Q}(\alpha)$, they are linearly dependent over $\mathbf{Q}$, which is a contradiction.

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