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GELFAND–KIRILLOV DIMENSION OF AN ALGEBRA WITH EXTENSION OF THE BASE FIELD

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1. Introduction

Let A be an algebra over a field k. Let B be the k-algebra $A \otimes_k k(x)$ where $k(x)$ is the rational function field. I have been interested in Krull dimension of B (in the sence of Gabriel and Rentschler). The Gelfand-Kirillov dimension and Krull dimension in the sence of Gabriel and Rentschler share many formal properties. This raise the question about the Gelfand-Kirillov dimension of the k-algebra B.

The Krull dimension of B is between the Krull dimension of A and the Krull dimension of A plus one. We obtained the exact formula for the Krull dimension of B when A is right Noetherian and has finite right Krull dimension ([2]). Furthermore, some sufficient conditions for $\mathrm{Kdim}(B)$ to be $\mathrm{Kdim}(A)$ and to be $\mathrm{Kdim}(A) + 1$ were obtained ([3]).

However, the problem of Gelfand-Kirillov dimension has a very simple solution : $\mathrm{GKdim}(B)$ is equal to $\mathrm{GKdim}(A)+1$ (from now on $\mathrm{GKdim}(R)$ denotes the Gelfand-Kirillov dimension of a k-algebra R). It is easily shown that $\mathrm{GKdim}(AS^{-1}) = \mathrm{GKdim}(A)$ where S is a multiplicatively closed subset of regular elements in the center of a k-algebra A. $A \otimes_k k(x_1, \cdots, x_n)$ is isomorphic to $(A[x_1, \cdots, x_n])S^{-1}$ as a k-algebra where S is the set of nonzero polynomials in $k[x_1, \cdots, x_n]$. So $\mathrm{GKdim}(A \otimes_k k(x_1, \cdots, x_n)) = \mathrm{GKdim}(A[x_1, \cdots, x_n]) = \mathrm{GKdim}(A)+n$. But the Krull dimension of $A \otimes_k k(x_1, \cdots, x_n)$ is still unknown when $n > 1$.

When $n = 1$, it can be shown that the $\mathrm{GKdim}(A \otimes_k k(x))$ is equal to $\mathrm{GKdim}(A) + 1$ interpreting $A \otimes_k k(x)$ as a tensor product.

Unless explicity stated to the contrary, A is an algebra over a field k. For the definition and basic properties of Gelfand-Kirillov dimension, the readers are referred to [1].

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2. The Gelfand-Kirillov dimension of $A \otimes_k k(x_1, \cdots, x_n)$ as a localization

Let S be the set of nonzero polynomials in $k[x_1, \cdots, x_n]$.

PROPOSITION 2.1. *(4.2 Proposition, [1]) If Ω is a multiplicatively closed subset of regular elements in the center of A, then* GKdim $(A\Omega^{-1})$ *is equal to* GKdim(A).

Proof. Obviously, Ω is an automatically an Ore set. Since A is a subalgebra of $A\Omega^{-1}$, GKdim$(A) \leq$ GKdim$(A\Omega^{-1})$. So, all we need to show is that GKdim$(A\Omega^{-1}) \leq$ GKdim(A).

Let W be a finite dimensional subspace of $A\Omega^{-1}$ and let $s \in \Omega$ be a common denominator for the basis elements of W (such an element s exists !). Then $Ws \subseteq A$. So, $V = Ws + ks + k1$ is a finite dimensional subspace of A and satisfies $W^n \subseteq V^n s^{-n}$ for each natural number n. Thus $\dim(W^n) \leq \dim(V^n)$, and hence $\dim_k(k + W + W^2 + \cdots + W^n) \leq \dim_k(k + V + V^2 + \cdots + V^n)$ for all n. This shows that GKdim$(A\Omega^{-1}) \leq$ GKdim(A).

PROPOSITION 2.2. *$A \otimes_k k(x_1, \cdots, x_n)$ is isomorphic to $(A[x_1, \cdots, x_n])S^{-1}$.*

Proof. Consider the map

$$\theta : A[x_1, \cdots, x_n] \longrightarrow A \otimes_k k(x_1, \cdots, x_n)$$

given by

$$\theta(\sum_i a_i x_1^{k_{i1}} x_2^{k_{i2}} \cdots x_n^{k_{in}}) = \sum_i a_i \otimes x_1^{k_{i1}} x_2^{k_{i2}} \cdots x_n^{k_{in}}).$$

Then θ is an embedding. Moreover $\theta(s)$ is invertable in $A \otimes_k k(x_1, \cdots, x_n)$ for each s in S. Hence there is a homomorphism

$$\theta : (A[x_1, \cdots, x_n])S^{-1} \longrightarrow A \otimes_k k(x_1, \cdots, x_n)$$

such that $\theta\iota = \theta$, where

$$\iota : A[x_1, \cdots, x_n] \longrightarrow (A[x_1, \cdots, x_n])S^{-1}$$

is the inclusion of $A[x_1, \cdots, x_n]$ into $(A[x_1, \cdots, x_n])S^{-1}$, by the universal mapping property. It is easy to show that θ is one-to-one and onto. Thus θ is a $k-$algebra isomorphism.

COROLLARY 2.3. $\mathrm{GKdim}((A[x_1, \cdots, x_n])S^{-1}) = \mathrm{GKdim}(A[x_1, \cdots, x_n])$.

Proof. Since S is a multiplicatively closed subset of regular elements and every element in S is in the center of $A[x_1, \cdots, x_n]$, it follows from Proposition 2.1.

COROLLARY 2.4. $\mathrm{GKdim}(A \otimes_k k(x_1, \cdots, x_n)) = \mathrm{GKdim}(A) + n$.

Proof. By Proposition 2.2,

$$\mathrm{GKdim}(A \otimes_k k(x_1, \cdots, x_n)) = \mathrm{GKdim}((A[x_1, \cdots, x_n])S^{-1}).$$

Hence

$$\mathrm{GKdim}(A \otimes_k k(x_1, \cdots, x_n)) = \mathrm{GKdim}(A[x_1, \cdots, x_n]) = \mathrm{GKdim}(A) + n.$$

It is a very desirable result that $\mathrm{GKdim}(A \otimes_k k(x_1, \cdots, x_n))$ depends on the $\mathrm{GKdim}(A)$ and the number of parameters.

3. The Gelfand-Kirillov dimension of $A \otimes_k k(x)$ as a tensor product

Note that

$$\mathrm{GKdim}\{k(x_1, \cdots, x_n)\} = \mathrm{GKdim}(k) + n$$

because $k(x_1, \cdots, x_n)$ is a localization of $k[x_1, \cdots, x_n]$. Moreover, we have the following proposition ;

PROPOSITION 3.1. *(Proposition 3.12 [1]) If* $\mathrm{GKdim}(A) \leq 2$, *then* $\mathrm{GKdim}(A \otimes_k B) = \mathrm{GKdim}(A) + \mathrm{GKdim}(B)$.

Proof. (See the proof in [1].)

We can apply this proposition to $A \otimes_k k(x)$ since $\mathrm{GKdim}(k(x)) = 1 < 2$, and we obtain the following ;

COROLLARY 3.2. $\mathrm{GKdim}(A \otimes_k k(x_1, \cdots, x_n)) = \mathrm{GKdim}(A) + n$ *for* $n = 1, 2$.

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