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A NOTE ON THE TOPOLOGICAL INDICES

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Let MSO_{4n} be the $4n$-dimensional Thom's bordism group. Then $MSO_* \otimes Q$ (Q : rationals) is generated by $p^2(C)$, $p^4(C), \cdots, p^{2n}(C)$ over Q ([2]), where $p^k(C)$ is the k-dimensional complex projective space.

In [4] the authors intimated that there exists an elliptic operator D_0 such that

$$i_t(D_0) = (2k+1)L[p^{2k}(C)] = (2k+1)I(p^{2k}(C))$$

where $i_t(D_0)$ is the topological index of D_0, $L[p^{2k}(C)]$ the L-genus of $p^{2k}(C)$ and $I(p^{2k}(C))$ the index of $p^{2k}(C)$. We also proved that $L[p^{2k}(C)] = I(p^{2k}(C)) = 1$ in [4].

In this paper we shall show that for the k-dimensional complex projective space $p^k(C)$ there exists an elliptic operator

$$D : C^\infty(\sum_{i \equiv 0 \bmod 2} \Lambda^i(T^*(X) \otimes C)) \to C^\infty(\sum_{i \equiv 0 \bmod 2} \Lambda^i(T^*(X) \otimes C))$$

such that $i_t(D) = k+1$, where $T^*(X)$ is the cotangent bundle over $X = X^{2k} = p^k(C)_{\mathbf{R}}$ and $p^k(C)_{\mathbf{R}}$ is the underlying real oriented C^∞ manifold of $p^k(C)$ (Theorem 7). Hence for $X^{4n} = p^{2k}(C)_{\mathbf{R}}$ it is true that

$$i_t(D) = (2k+1)L[X^{4k}].$$

Eventually, we shall verify that the topological indices and the cobordism theory are deeply related with [4] and this paper.

By the definition of the Chern classes and the Euler class with respect to a complex vector bundle $\omega = (E(\omega), \pi, B(\omega))$ we have

$$C_n(\omega) = e(\omega_{\mathbf{R}})$$

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where $\dim_C \omega = n$ (C : complexes), $\omega_{\mathbf{R}}$, $\omega_{\mathbf{R}}$ the underlying real vector bundle of ω, $e(\omega_{\mathbf{R}})$ the Euler class of $\omega_{\mathbf{R}}$ and $C_i(\omega)$ the ith Chern class of ([2]).

For $p^k(C)$, let $(\tau(p^k(C)),\ \pi,\ p^k(C))$ be the tangent bundle over $p^k(C)$ and let us put

$$C_i(p^k(C) = \text{the } i\text{th Chern class of } (\tau(p^k(C)),\ \pi,\ p^k(C)),$$

and

$$C(p^k(C) = 1 + C_1(p^k(C)) + \cdots + C_k(p^k(C)),$$

which is called the <u>total Chern class</u>.

Since $H^2(p^k(C); Q) \cong Q$ we can take the generator $\alpha \in H^2(p^k(C), Q)$ such that α^k is the fundamental cohomology class of $p^k(C)$. Moreover we have the following properties

(i) $C(p^k(C)) = (1+\alpha)^{k+1} (\alpha^{k+1} = 0)$

(ii) $C_i(p^k(C)) = (k+1)^{C_i} \alpha^i$.

PROPOSITION 1. *For the Euler class $e(p^k(C)_{\mathbf{R}})$ of the underlying real vector bundle of $(\tau(p^k(C)),\ \pi,\ p^k(C))$ we have $e(p^k(C)_{\mathbf{R}}) = (k+1)\alpha^k$.*

Proof. Since

$$e(p^k(C)_{\mathbf{R}}) = C_k(p^k(C))$$

([2]) by (ii) above it is clear that

$$e(p^k(C)_{\mathbf{R}}) = (k+1)\alpha^k.$$

Since the tangent vector bundle $(\tau(p^k(C)),\ \pi,\ p^k(C))$ is a complex bundle

$$\tau(X) = (\tau(X),\ \pi,\ X)$$

is a $2k$-dimensional oriented real vector bundle ([5]) over X, where $X = X^{2k} = p^k(C)_{\mathbf{R}}$. We put as follows.

$$p_i(X) = p_i(\tau(X)) = \text{the } i^{th}\text{Pontrijagin class of } \tau(X)$$

and

$$p(X) = p(\tau(X)) = \text{the total Pontrijagin class of } \tau(X).$$

For the generator α of $H^2(X : Q) \cong Q$ we have the following ([1], [2]) :
(iii) $P(X) = (1 + \alpha^2)^{k+1}$
(iv) $P_i(X) = {}^{(k+1)}C_i \alpha^{2i}$.
Note that $\dim_R(X) = 2k$ and for $m > [\frac{k}{2}] p_m(X) \in H^{4m}(X^{2k} : Q) = 0$. Thus for $m = [\frac{k}{2}]$

$$\begin{aligned} P(X) =& 1 + P_1(X) + \cdots + p_m(X) \\ &= 1 + (k+1)\alpha^2 + \cdots + {}_{(k+1)}C_m \alpha^{2m}. \end{aligned}$$

PROPOSITION 2. $e(\tau(X) \otimes C) = 0$.

Proof. By (iv) above and $e((p^k(C) \otimes C)_{\mathbf{R}}) = \{e(p^k(C))_{\mathbf{R}}\}^2$ ([5]) we have the following :

$$\begin{aligned} e((\tau(X) \otimes C)_{\mathbf{R}}) &= \{e(p^k)(C)_{\mathbf{R}})\}^2 \\ &= ((k+1)\alpha^k)^2 (\text{by Proposition 1}) \\ &= (k+1)^2 \alpha^{2k} (\in H^{4k}(X^{2k} : Q)) \\ &= 0. \end{aligned}$$

As in [4] for the L-genus $L[p^{2k}(C)]$ of $p^{2k}(C)$ we have the following :

$$(*) \qquad L[p^{2k}(C)] = 1$$

DEFINITION 3. A SO($2k$)-structure on $X^{2k} = p^k(C)_{\mathbf{R}}$ is defined by an isomorphism

$$\varphi : P \times_{\mathrm{SO}(2k)} \mathbf{R}^{2k} \cong T(X)$$

which is preserving orientation, where P is a principal SO($2k$)-bundle ever $X = X^{2k}$, $\mathbf{R}^{2k}$ has the usual Riemannian metric and $T(X)$ the tangent bundle of X which has the usual orientation ([5]).

Let $B_{\mathrm{SO}(2k)}$ be the classifying space of SO($2k$) ([1]) and let $E_{\mathrm{SO}(2k)}$ be the universal principal SO($2k$)-bundle ([3]). Then there exists a continuous map

$$f : X \longrightarrow B_{\mathrm{SO}(2k)}$$

such that

$$f^*(E_{\mathrm{SO}(2k)}) = P,$$

where f is called a classifying map for P. For

$$\tilde{\mathbf{R}}^{2k} = E_{\mathrm{SO}(2k)\times \mathrm{SO}(2k)} R^{2k}$$

we also have

$$f^*(\tilde{\mathbf{R}}^{2k}) \cong T(X).$$

Let $T^*(X)$ be the dual bundle of $T(X)$ (i.e., the cotangent bundle of X), and let us put for $V = \tilde{\mathbf{R}}^{2k}$ such that

$$B(\tilde{V}^*) = \{e \in T^*(X) | \ \|e\| \leq 1\} \ (V^* = (\mathbf{R}^{2k})^*)$$

and

$$S(\tilde{V}^*) = \{e \in T^*(X) | \ \|e\| = 1\}.$$

Let M and N be complex $\mathrm{SO}(2k)$-modules which are isomorphic to each other ($\dim_{\mathbf{R}} M_{\mathbf{R}} = 2k = \dim_R N_{\mathbf{R}}$). We assume that there is a $\mathrm{SO}(2k)$-equivalent map

$$\sigma : S(\tilde{V}^*) \longrightarrow \mathrm{Iso}(M, N)$$

where $\mathrm{Iso}(M, N)$ is the set of all $\mathrm{SO}(2k)$-isomorphisms between M and N. For

$$E = P \times_{\mathrm{SO}(2k)} M \text{ and } F = P \times_{\mathrm{SO}(2k)} N$$

σ induces an isomorphism

$$\sigma_p : \pi^* E \cong \pi^* F$$

where $\pi : B(\tilde{V}^*) \to X$ is the projection and we put $\pi | S(\tilde{V}^*) = \pi$. If we put

$$C^\infty(E) = \{ \ g : X \to E | g \text{ is a } C^\infty \text{ cross section of } E\}$$

and take a differential operator

$$D : C^\infty(E) \longrightarrow C^\infty(F)$$

such that the symbol $\sigma(D)$ of D is just σ_p i.e.,

$$(**) \qquad \sigma(D) = \sigma_p.$$

Hence

$$d(\pi^*E, \pi^*F, \sigma_p) \in K(B(\tilde{V}^*),\ S(\tilde{V}^*))$$

where $\pi : B(\tilde{V}^*) \to X$ is the projection and $K(B(\tilde{V}^*),\ S(\tilde{V}^*))$ is the relative K-group on $(B(\tilde{V}^*),\ S(\tilde{V}^*))$. We define such that

$$(***) \qquad \mathrm{Ch}(D) = f^{**}\{(-1)^k \frac{\mathrm{Ch}(E) - \mathrm{Ch}(F)}{e(\tilde{V}^*)}\}$$

([3], [6]) where $\mathrm{Ch}(E)$ is the Chern character of E, $f : X \to B)_{\mathrm{SO}(2k)}$ is the classifying map of P ;

$$H^{**}(X : Q) = \prod_{j=1}^{\infty} H^j(X : Q)$$

and

$$f^{**} : H^{**}(B_{\mathrm{SO}(2k)} : Q) \longrightarrow H^{**}(X : Q)$$

is induced from the classifying map f.

DEFINITION 4. For $X = p^k(C)$ there exist $x_1, \cdots, x_k$ in $H^{**}(X : Q)$ such that

$$1 + C_1(T(X)) + \cdots + C_k(T(X)) = \prod_{i=1}^{k}(1 + x_i)$$

where $C_i(T(X)) = C_i(p^k(C))$ ([3]). In this case the Todd class $\mathcal{T}(T(X))$ of $T(X)$ is defined as

$$\mathcal{T}(T(X)) = \prod_{i=1}^{k}(1 - e^{-x_i}).$$

In particular, if we use the same notations for $T^*(X)$ ($\cong T(X)$) as above we have

$$\mathcal{T}(X) = \mathcal{T}(T^*(X)) = \prod_{i=1}^{k}(1 - e^{-x_i})$$

DEFINITION 5. With the above notations for an elliptic operator

$$D : C^{\infty}(E) \longrightarrow C^{\infty}(F)$$

with the symbol $\sigma(D) = \sigma_p$ as in $(**)$ the topological index $i_t(D)$ of D is defined as

$$i_t(D) = (\mathrm{Ch}(D)\mathcal{T}(X))[X]$$

where $[X]$ is the fundamental homology class of $X^{2k}(=X) = p^k(C)_{\mathbf{R}}$.

LEMMA 6. *For the oriented Riemannian manifold*

$$X = X^{2k} = p^k(C)_R$$

let

$$D : C^{\infty}(\sum_{l\equiv 0 \mathrm{mod}\ 2} \Lambda^l(T^*(X)\otimes C) \to C^{\infty}(\sum_{l\equiv 1 \bmod 2} \Lambda^l(T^*(X)\otimes C))$$

be an elliptic operator with its symbol $\sigma(D) = \sigma_p$ as in $(**)$. Then for the Euler class $e(X)$ of $T(X)$ and the fundamental homology class $[X]$ of $X = X^{2k}$

$$i_t(D) = e(X)[X].$$

Proof. We put such that

$$V = \mathbf{R}^{2k},\ \tilde{V}^* = p \times_{SO(2k)} V^*$$

and

$$\tilde{M} = \sum_{l\equiv 0 \bmod 2} \Lambda^l(\tilde{V}^*\otimes C),\ \tilde{N} = \sum_{l\equiv 0 \bmod 2} \Lambda^l(\tilde{V}^*\otimes C)$$

where,

$$P \times_{SO(2k)} V \cong T(X)$$

is a SO$(2k)$-structure on X.

Since $V \cong V^*$ as SO$(2k)$-modules, if V has the weights $y_1, \cdots, y_k$ then $V\otimes C \cong V^*\otimes C$ has the weights $\pm y_1, \cdots, \pm y_k$ ([3], [6]). Moreover, from

$$\sum_{i=1}^{k}(-1)^i \mathrm{Ch}(\Lambda^i(\tilde{V}^*\otimes C)) = \prod_{i=1}^{k}(1-e^{y_i})(1-e^{-y_i})$$

([3], [6]), we have

$$\mathrm{Ch}(\tilde{M}) - \mathrm{Ch}(\tilde{N}) = \prod_{i=1}^{k}(1 - e^{y_i})(1 - e^{-y_i}).$$

Thus by $(* * *)$ above we have the following ;

$$\begin{aligned}\mathrm{Ch}(D) &= f^{**}((-1)^k \frac{\mathrm{Ch}(\tilde{M}) - \mathrm{Ch}(\tilde{N})}{e(\tilde{V}^*)}) \\ &= f^{**}(\prod_{i=1}^{k} \frac{(1 - e^{y_i})(1 - e^{-y_i})}{-y_i})\end{aligned}$$

because that $e(\tilde{V}^*) = y_1 \cdots y_k$ ([3], [6]).

On the other hand, by Definition 4

$$\begin{aligned}\mathcal{T}(X) &= \mathcal{T}(T^*(X) \otimes C) = \mathcal{T}(\tilde{V}^* \otimes C) \\ &= f^{**}(\prod_{i=1}^{k} \frac{y_i(-y_i)}{(1 - e^{y_i})(1 - e^{-y_i})}).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Ch}(D)\ (X) &= f^{**}(\prod_{i=1}^{k} \frac{(1 - e^{y_i})(1 - e^{-y_i})}{-y_i} \cdot \frac{y_i(-y_i)}{(1 - e^{y_i})(1 - e^{-y_i})}) \\ &= f^{**}(\prod_{i=1}^{k} y_i) = f^{**}(e(\tilde{V}^*)) \\ &= e(X)\end{aligned}$$

and thus

$$i_t(D) = (\mathrm{Ch}(D)\mathcal{T}(X))[X] = e(X)[X].$$

THEOREM 7. *For the elliptic operator D as in Lemma 6 $i_t(D) = (k+1)L(p^k(C)) = k+1$.*

Proof. For $X^{2k} = p^k(C)_{\mathbf{R}}$ we have

$$e(X) = (k+1)\alpha^k$$

by Proposition 1, where α is the generator of $H^2(X^{2k} : Q) \cong Q$. The fundamental homology class $[X]$ is the generator of $H_{2k}(X^{2k} : Q) \cong Q$ such that

$$\alpha^k([X]) = 1.$$

Hence $e(X)[X] = (2k+1)\alpha^{2k}([X]) = 2k+1$.

Since

$$L[p^{2k}(C)] = 1$$

as in [4] we have

$$i_t(D) = (2k+1)L[p^{2k}(C)] = 2k+1$$

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