Comm. Korean Math. Soc. 7(1992), No. 1, pp. 15–24

ON THE RELATIVE HOMOTOPY JIANG SUBGROUPS

SEOUNG HO LEE

D.H.Gottlieb ([2],[4]) defined and studied the evaluation subgroups $G_n(X)$ of the homotopy groups $\pi_n(X)$ of a topological space X. Recently, $G_n(X)$ was generalized by Woo and Kim [9] as following ; Let (X, A, x_0) be a triplet. Consider the class of continuous maps $F : A \times S^n \to X$ such that $F(a, s_0) = f(a)$, then the map $h : (S^n, s_0) \to (X, x_0)$ defined by $h(s) = F(x_0, s)$ represent an element $[h] \in \pi_n(X, x_0)$ for a given $f : (A, x_0) \to (X, x_0)$. The set of all elements $[h] \in \pi_n(X, x_0)$ obtained in the above manner from some F was denoted by $G_n^f(X, A, x_0)$.

The exact sequence of homotopy groups combined with relative homotopy groups for a pair (X, A) plays an important role in algebraic topology. Woo and Lee [11] defined the relative evaluation subgroup $G_n^{Rel}(X, A, x_0)$ of $\pi_n(X, A, x_0)$ as following; an element $\alpha \in \pi_n(X, A, x_0)$ is in $G_n^{Rel}(X, A, x_0)$ if there exists a map $H : (X \times I^n, A \times \partial I^n, x_0 \times J^{n-1}) \to (X, A, x_0)$ such that $[H(x_0, \cdot)] = \alpha$ and $H(x, u) = x$ for every $u \in J^{n-1}$. They also showed that the following sequence

$$\cdots \longrightarrow G_n(A) \longrightarrow G_n^i(X, A) \longrightarrow G_n^{Rel}(X, A) \longrightarrow G_{n-1}(A) \longrightarrow \cdots$$

is exact if the inclusion $i : A \to X$ has a left homotopy inverse.

Jiang subgroups of the fundamental group play an important role in the fixed point theory. In this paper, we introduce the relative homotopy Jiang subgroup $G_n^{Rel}(f, x_0)$, show the exactness of the sequence of those groups for some topological pair (X, A) and investigate the effect of changing of base points on $G_n^{Rel}(f, x_0)$ .We also show that $\bar{f}_*\pi_1(A, x_0)$ also operates trivially on the G-sequence of f for (X, A) and study the relationship between $G_n(\tilde{X}, \tilde{A})$ and $G_n(X, A)$, where $(\tilde{X}, \tilde{A})$ is a pair covering space of (X, A). Throughout this paper, $f : (X, A) \to (X, A)$ is a self-map of a pair of CW-complexes with A path connected and we will follow the notations and terminology of [5].

Received June 25, 1991.

Consider the class of continuous functions

$$F : X \times S^n \longrightarrow X$$

such that $F(x, s_0) = x$, where $x \in X$ and s_0 is a base point of S^n. Then the map $f : S^n \to X$ defined by $f(s) = F(x_0, s)$, where x_0 is a base point of X, represents an element $\alpha = [f] \in \pi_n(X, x_0)$.

DEFINITION. The set of all elements $\alpha \in \pi_n(X, x_0)$ obtained in the above manner from some F will be denoted by $G_n(X, x_0)$. Thus for every $\alpha \in G_n(X, x_0)$, there is at least one map $F : X \times S^n \to X$ which satisfies the above conditions such that $[f] = \alpha$. We say that F is an affiliated map to α [4].

Let (X, A) be a topological pair and $f : (X, A) \to (X, A)$ be a self-map. We shall write $\bar{f} : A \to A$ for the restriction of f to A and $f_A : A \to X$ for the composition of $\bar{f}$ and the inclusion $i : A \to X$. The initial $(n-1)$-face of I^n defined by $t_n = 0$ will be identified with I^{n-1}. The union of all remaining $(n-1)$-faces of I^n is denoted by J^{n-1}. Then we have $\partial I^n = I^{n-1} \cup J^{n-1}$.

DEFINITION 1. An element $\alpha \in \pi_n(X, A, f(x_0))$ is said to be in the relative homotopy Jiang subgroup $G_n^{Rel}(f, x_0)$ of f if there is a map $H : (X \times I^n, A \times \partial I^n) \to (X, A)$ such that $[H(x_0, \cdot)] = \alpha$ and $H(x, u) = f(x)$ for every $u \in J^{n-1}$. We say that H is an affiliated map to α with respect to f and $H(x_0, \cdot)$ is the trace of H. We will abbreviate an affiliated map to α w.r.t. f to an affiliated map to α if no confusion arise.

Let X^X be the mapping space from X to itself with compact open topology, $A(X^X)$ be the subspace of X^X which consists of all maps $g \in X^X$ such that $g(A) \subset A$ and $[A^A]$ be the subspace of X^A which consists of all maps $g \in X^A$ such that $g(A) \subset A$. Consider the evaluation map $p : X^X \to X$ such that $p(g) = g(x_0)$, then p induces a homomorphism

$$p_* : \pi_n(X^X, A(X^X), f) \to \pi_n(X, A, f(x_0)).$$

It is easy to see that $p_*(\pi_n(X^X, A(X^X), f)) = G_n^{Rel}(f, x_0)$.

THEOREM 1. *$G_n^{Rel}(f,x_0)$ is a subgroup of $\pi_n(X,A,f(x_0))$ for $n>1$.*

Proof. Let $\alpha,\beta \in G_n^{Rel}(f,x_0)$. Then there exist affiliated maps H,G to α,β respectively. Define

$$F:(X\times I^n, A\times \partial I^n)\to (X,A)$$

by

$$F(x,t_1,\cdots,t_n)=\begin{cases} H(x,2t_1,\cdots,t_n), & 0\le t_1\le 1/2\\ G(x,2t_1-1,\cdots,t_n), & 1/2\le t_1\le 1.\end{cases}$$

Then F is an affiliated map to $\alpha+\beta$ w.r.t. f. Define

$$K:(X\times I^n, A\times \partial I^n)\to (X,A)$$

by $K(x,t_1,\cdots,t_n)=H(x,(1-t_1),\cdots,t_n)$. Then K is an affiliated map to α^{-1} w.r.t. f.

An element $\alpha\in\pi_n(X,f(x_0))$ is said to be in the homotopy Jiang subgroup $G_n^{f_A}(X,A,x_0)$ of f w.r.t. A if there is a map $F:A\times S^n\to X$ such that $h(s)=F(x_0,s)$ is an element of α and $F(a,s_0)=f_A(a)$, where s_0 is a base point of S^n. An equivalent definition of $G_n^{f_A}(X,A,x_0)$ is the following : Define $p:X^A\to X$ by $p(g)=g(x_0)$; then p induces a homomophism $p_*:\pi_n(X^A,f_A)\to\pi_n(X,f(x_0))$. The $G_n^{f_A}(X,A,x_0)$ is the image of $\pi_n(X^A,f_A)$ under the homomorphism p_*. We will write $G_n^f(X,X,x_0)$ by $G_n(f,x_0)$ and also call the homotopy Jiang subgroup of f. If $n=1$, then $G_1(f,x_0)$ is the Jiang subgroup $J(f)$ of f [1]. In the usual sense , $G_n^{Rel}(f,x_0)$, viewed as a subgroup of $\pi_n(X,A,f(x_0))$, is independent of the base point in A. Let $\sigma:I\to A$ be a path such that $\sigma(0)=x_0$ and $\sigma(1)=x_1$. Then $f\sigma$ induces an isomomorphism $(f\sigma)_*:\pi_n(X,A,f(x_1))\cong\pi_n(X,A,f(x_0))$ given by $(f\sigma)[g]=[g']$, where g' is homotopic to g by a homotopy $F:I^n\times I\to X$ that sends $\partial I^n\times I$ to A and satisfies $F(u,t)=f\sigma(1-t)$ for all $u\in J^{n-1}$ [7]. It is easy to show that $(f\sigma)_*:G_n^{Rel}(f,x_1)\cong G_n^{Rel}(f,x_0)$.

Recall that the boundry operator $\partial:\pi_n(X,A,f(x_0))\to\pi_{n-1}(A,f(x_0))$ is a map given by $\partial[g]=[g\mid_{I^{n-1}}]\in\pi_{n-1}(A,f(x_0))$. Let $(X,A,f(x_0))$ be any triplet. The inclusion maps $i:(A,f(x_0))\to(X,f(x_0))$, j :

$(X, f(x_0), f(x_0)) \to (X, A, f(x_0))$ induce homomorphisms i_* and j_* for each $n > 0$. Together with the boundary operator ∂, they form a long exact sequence ;

$$\xrightarrow{j_*} \pi_{n+1}(X, A, f(x_0)) \xrightarrow{\partial} \pi_n(A, f(x_0)) \xrightarrow{i_*} \pi_n(X, f(x_0)) \to$$
$$\cdots \xrightarrow{j_*} \pi_2(X, A, f(x_0)) \xrightarrow{\partial} \pi_1(A, f(x_0)) \xrightarrow{i_*} \pi_1(X, f(x_0))$$

which is called the homotopy exact sequence of a triplet $(X, A, f(x_0))$.

We can easily prove the following Lemmas by some modifications of Lemma 9,10,11 of [11].

LEMMA 1. *∂ carries $G_n^{Rel}(f, x_0)$ into $G_{n-1}(\bar{f}, x_0)$ for $n > 2$.*

LEMMA 2. *j_* carries $G_n^{f_A}(X, A, x_0)$ into $G_n^{Rel}(f, x_0)$.*

LEMMA 3. *If the inclusion $i : A \to X$ has a left homotopy inverse, then $i_*(G_n(\bar{f}, x_0)) = i_*(\pi_n(A, f(x_0))) \cap G_n^{f_A}(X, A, x_0)$.*

Consequently, we obtain a sequence for a selp-map $f : (X, A) \to (X, A)$

$$\longrightarrow G_{n+1}^{Rel}(f, x_0) \xrightarrow{\partial} G_n(\bar{f}, x_0) \xrightarrow{i_*} G_n^{f_A}(X, A, x_0) \xrightarrow{j_*} G_n^{Rel}(f, x_0) \longrightarrow$$

$$\cdots \longrightarrow G_2^{Rel}(f, x_0) \xrightarrow{\partial} G_1(\bar{f}, x_0) \xrightarrow{i_*} G_1^{f_A}(X, A, x_0)$$

This sequence will be called the G-sequence of f for the pair (X, A). Is the G-sequence of f exact? It may not be true in general. But the G-sequence of f is exact for some pairs. The natural map $p : X^X \to X^A$ given by $p(g) = g_A$ is a fiber map [9]. Since $A(X^X) = p^{-1}([A^A])$,

$$p : \pi_n(X^X, A(X^X), f) \to \pi_n(X^A, [A^A], f_A)$$

is one to one fashion for $n > 0$. We have the following Theorem by modifications of Theorem 12 of [11].

THEOREM 2. *If the inclusion $i : A \to X$ has a left homotopy inverse, then the G-sequence of f for (X, A) is exact.*

Two maps $a_0, a_1 : (X, A) \to (Y, B)$ are given. If w is a path in B from $a_0(x_0)$ to $a_1(x_0)$, an w-homotopy from a_0 to a_1 is a homotopy $H : (X, A) \times I \to (Y, B)$ such that $H(x, 0) = a_0(x)$, $H(x, 1) = a_1(x)$, and $H(x_0, t) = w(t)$. If (X, A) has a nondegenerate base point, there is a map

$$h_{[w]} : [X, A, x_0; Y, B, w(1)] \to [X, A, x_0; Y, B, w(0)]$$

characterized by the property $h_{[w]}[a_1] = [a_0]$ if and only if a_0 is w-homotopic to a_1. If (X, A) is a suspension, the map is a homomorphism.

THEOREM[8]. *For any pair (X, A) and any $n \geq 1$, there is a covariant functor from the fundamental groupoid of A to the category of pointed sets if $n = 1$ and the category of groups if $n > 1$ which assigns $\pi_n(X, A, x_0)$ to $x_0 \in A$ and to a path class $[w]$ in A the map*

$$h_{[w]} : \pi_n(X, A, w(1)) \to \pi_n(X, A, w(0)).$$

In the way, $\pi_1(A, x_0)$ acts as a group of operators on the left on $\pi_n(X, A, x_0)$, and if A is path connected and $x_0, x_1 \in A$, then $\pi_n(X, A, x_0)$ and $\pi_n(X, A, x_1)$ are isomorphic by an isomorphism determined up to the action of $\pi_1(A, x_0)$.

Between $[(I^n, \partial I^n, J^{n-1}), (X, A, f(x_0))]$ and $[(B^n, S^{n-1}, s_0), (X, A, f(x_0))]$ there is a one-to-one correspondence. We will identify the corresponding elements .

LEMMA 4. *There is an equivalent definition of definition 1 : $\alpha \in G_n^{Rel}(f, x_0)$ iff there exists a map*

$$F : (X \times B^n, A \times S^{n-1}) \to (X, A)$$

such that $[F(x_0, \cdot)] = \alpha$ and $F(x, s_0) = f(x)$ for some $\alpha \in \pi_n(X, A, f(x_0))$.

Proof. Let $\alpha \in G_n^{Rel}(f, x_0)$. Then there exists an affiliated map $G : (X \times I^n, A \times \partial I^n) \to (X, A)$ such that $[G(x_0, \cdot)] = \alpha$ and $G(x, u) = f(x)$ for $u \in J^{n-1}$. Consider a homeomorphism $h : (B^n, S^{n-1}, s_0) \equiv$

$(I^n, \partial I^n, z_0)$ and a homotopy equivalence $i : (I^n, \partial I^n, z_0) \to (I^n, \partial I^n, J^{n-1})$ where $z_0 = (0, 0, \cdots, 0)$. Define $F : X \times B^n \to X$ by $F(x, u) = G(1 \times (ih))(x, u)$. Then $F(a, s) = G(a, ih(s)) \in A$ for $(a, s) \in A \times S^{n-1}$, $F(x, s_0) = G(x, ih(s_0)) = f(x)$. Thus $F : (X \times B^n, A \times S^{n-1}, x_0 \times s_0) \to (X, A, f(x_0))$ and $[F(x_0, \cdot)] = \alpha \circ i \circ h \equiv \alpha$ and $F(x, s_0) = G(x, ih(s_0)) = f(x)$.

Conversely, assume there exists a map $F : (X \times B^n, A \times S^{n-1}, x_0 \times s_0) \to (X, A, f(x_0))$ such that $[F(x_0, \cdot)] = \alpha$ and $F(x, s_0) = f(x)$. Define $H : X \times I^n \to X$ by $H(x, u) = F(1 \times (h^{-1}j))(x, u)$, where j is a homotopy inverse of i. Then $H(a, u) = F(a, h^{-1}j(u)) \in A$ for $(a, u) \in A \times \partial I^n$, $H(x, u) = F(x, h^{-1}j(u)) = f(x)$ for $u \in J^{n-1}$. Then we have $H : (X \times I^n, A \times \partial I^n) \to (X, A)$ and $[H(x_0, \cdot)] = [F(x_0, h^{-1}j)] = \alpha$ and $H(x, u) = F(x, h^{-1}j(u)) = f(x)$ for $u \in J^{n-1}$. Therefore H is an affiliated map to α w.r.t. f.

THEOREM 3. *If $f, g : (X, A) \to (X, A)$ are homotopic maps, then $G_n^{Rel}(f, x_0)$ and $G_n^{Rel}(g, x_0)$ are isomorphic .*

Proof. Let H be a homotopy from f to g. Then $\sigma(t) = H(x_0, t)$ is a path from $f(x_0)$ to $g(x_0)$. It is sufficient to show that

$$h_{[\sigma^{-1}]}(G_n^{Rel}(g, x_0)) \subseteq G_n^{Rel}(f, x_0).$$

Let $\alpha \in G_n^{Rel}(g, x_0)$. Then there exists an affiliated map $G : (X \times B^n, A \times S^{n-1}) \to (X, A)$ such that $[G(x_0, \cdot)] = \alpha$ and $G(\cdot, s_0) = g$ by the above Lemma . Define

$$\begin{aligned} \phi :& A \times S^{n-1} \times 0 \cup A \times s_0 \times I \to A \quad \text{by} \\ & \phi(a, s, 0) = G(a, s) \qquad \text{if } (a, s, 0) \in A \times S^{n-1} \times 0, \\ & \phi(a, s_0, t) = H(a, 1 - t) \qquad \text{if } (a, s_0, t) \in A \times s_0 \times I. \end{aligned}$$

Then ϕ is well-defined and continuous. By the absolute homotopy extension property, there is an extension $\bar{\phi} : A \times S^{n-1} \times I \to A$. Define a map

$$\begin{aligned} \psi :& X \times B^n \times 0 \cup X \times s_0 \times I \cup A \times S^{n-1} \times I \to X \quad \text{by} \\ & \psi(a, s, t) = \bar{\phi}(a, s, t) \qquad \text{if } (a, s, t) \in A \times S^{n-1} \times I, \\ & \psi(x, u, 0) = G(x, u) \qquad \text{if } (x, u, 0) \in X \times B^n \times 0, \\ & \psi(x, s_0, t) = H(x, 1 - t) \qquad \text{if } (x, s_0, t) \in X \times s_0 \times I. \end{aligned}$$

Then we have an extension $\bar{\psi} : X \times B^n \times I \to X$. Let $F(x,u) = \bar{\psi}(x,u,1)$. Then $F(a,s) = \bar{\psi}(a,s,1) = \bar{\phi}(a,s,1) \in A$ for $(a,s) \in A \times S^{n-1}$. Thus we have $F : (X \times B^n, A \times S^{n-1}) \to (X,A)$. Since $F(x,s_0) = \bar{\psi}(x,s_0,1) = H(x,0) = f(x)$, $[F(x_0,\cdot)] \in G_n^{Rel}(f,x_0)$ by Lemma 4.

Moreover we can take $\bar{\psi}(x_0,\cdot,\cdot) : (B^n, S^{n-1}) \times I \to (X,A)$ such that $\bar{\psi}(x_0,u,0) = \psi(x_0,u,0) = G(x_0,u)$, $\bar{\psi}(x_0,u,1) = F(x_0,u)$. Therefore $h_{[\sigma^{-1}]}([G(x_0,\cdot)]) = [F(x_0,\cdot)]$.

COROLLARY 1. *If $f,g : X \to X$ are homotopic , then $G_n(f,x_0)$ and $G_n(g,x_0)$ are isomorphic.*

LEMMA 5. *Given a map $f : (X,A) \to (X,A)$ and any two points $x_0, x_1 \in A$, there exists a map $g : (X,A) \to (X,A)$ which is homotopic to f and $\bar{f}^{-1}(x_0)$ and x_1 are in $g^{-1}(x_0)$.*

Proof. For $\bar{f} : A \to A$ the restriction of f to A, there exists a map $\bar{g} : A \to A$ homotopic to $\bar{f}$ such that $\bar{f}^{-1}(x_0)$ and x_1 are in $\bar{g}^{-1}(x_0)$, where the homotopy $\bar{G}$ from $\bar{f}$ to $\bar{g}$ is an extension of $G : (A \times 0) \cup (T \times I) \to A$ defined by

$$G(x,t) = \begin{cases} \bar{f}(x) & \text{if } t = 0, \\ x_0 & \text{if } x \in \bar{f}^{-1}(x_0), \\ C(t) & \text{if } x = x_1, \end{cases}$$

where C is a path from $\bar{f}(x_1)$ to x_0 and $T = \bar{f}^{-1}(x_0) \cup x_1$.[1]

Define $H : X \times 0 \cup A \times I \to X$ by

$$H(x,0) = f(x) \quad \text{for } (x,0) \in X \times 0,$$
$$H(a,t) = \bar{G}(a,t) \quad \text{for } (a,t) \in A \times I.$$

Then H is well-defined and continuous. By the homotopy extension property, there is an extension $\bar{H} : X \times I \to X$ of H. Let $g(x) = \bar{H}(x,1)$. Then g is an extension of $\bar{g}$, $g(x) = x_0$ for $x \in \bar{f}^{-1}(x_0)$ and $g(x_1) = x_0$.

Lemma 5 implies that, given $f : (X,A,x_0) \to (X,A,f(x_0))$, there is a map $g : (X,A,x_0) \to (X,A,x_0)$ homotopic to f. Theorem 3 and Lemma 5 permit us to assume that the base point $x_0 \in X$, that we choose, is a fixed point of the map with which we are working.

THEOREM 4. *For any map $f : (X, A) \to (X, A)$,*

$$G_n^{Rel}(X, A, x_0) \subseteq G_n^{Rel}(f, x_0).$$

Proof. Let $\alpha \in G_n^{Rel}(X, A, x_0)$. Then there exists an affiliated map $H : (X \times I^n, A \times \partial I^n, x_0 \times J^{n-1}) \to (X, A, x_0)$ such that $[H(x_0, \cdot)] = \alpha$ and $H(x, J^{n-1}) = x$. Define $H' : (X \times I^n, A \times \partial I^n) \to (X, A)$ by $H'(x,t) = H(f(x), t)$. Then, it follows that $[H'(x_0, \cdot)] = [H(f(x_0), \cdot)] = \alpha$ and $H'(x, u) = H(f(x), u) = f(x)$ for any $u \in J^{n-1}$.Thus $\alpha \in G_n^{Rel}(f, x_0)$.

COROLLARY 2. *For any map $f : X \to X$, $G_n(X) \subseteq G_n(f)$.*

In particular , if $n = 1$,then we have

COROLLARY 3. *For any map $f : X \to X$,*

$$J(X) \subseteq J(f)$$

where $J(X)$ is the Jiang subgroup of the identity map.

THEOREM 5. *$(\bar{f})_* \pi_1(A, x_0)$ operates trivially on the G-sequence of f for (X, A).*

Proof. In diagrammatic terms, theorem means that, for each $\eta = i_*(\xi)$ where $\xi = (\bar{f})_*([\sigma])$ and $[\sigma] \in \pi_1(A, x_0)$, all vertical maps in the following diagram are the identity maps :

$$\begin{array}{ccccccccc}
\longrightarrow & G_{n+1}^{Rel}(f, x_0) & \xrightarrow{\partial} & G_n(\bar{f}, x_0) & \xrightarrow{i_*} & G_n^{f_A}(X, A, x_0) & \xrightarrow{j_*} & G_n^{Rel}(f, x_0) & \longrightarrow \\
 & \downarrow{\scriptstyle \tau'_\xi} & & \downarrow{\scriptstyle \tau_\xi} & & \downarrow{\scriptstyle \tau_\eta} & & \downarrow{\scriptstyle \tau'_\xi} & \\
\longrightarrow & G_{n+1}^{Rel}(f, x_0) & \xrightarrow{\partial} & G_n(\bar{f}, x_0) & \xrightarrow{i_*} & G_n^{f_A}(X, A, x_0) & \xrightarrow{j_*} & G_n^{Rel}(f, x_0) & \longrightarrow
\end{array}$$

this theorem can be proved by modifications of Theorem 2 in [10].

We will investigate the relationship between $G_n(X, A)$ and $G_n(\tilde{X}, \tilde{A})$ for the covering spaces $\tilde{X}$ and $\tilde{A}$.

DEFINITION 2. A map $p : (\tilde{X}, \tilde{A}) \to (X, A)$ is a pair covering of (X, A) if $p : \tilde{X} \to X$ and $p\,|_{\tilde{A}} : \tilde{A} \to A$ are covering maps .

THEOREM 6. *Let $p : (\tilde{X}, \tilde{A}) \to (X, A)$ be a pair covering of (X, A). If $p_*(\alpha) \in G_1(X, A)$, then $\alpha \in G_1(\tilde{X}, \tilde{A})$.*

Proof. Since $p_*(\alpha) \in G_1(X,A)$, there exists an affiliated map $\phi : A \times S^1 \to X$ to $p_*(\alpha)$ such that the closed path $\sigma : t \longmapsto H(x_0, t)$ represents $p_*(\alpha)$, where H is the composition of $(1 \times q)$ and ϕ , $q : I \to S^1$ given by $q(t) = e^{2\pi i t}$. By the homotopy extension property, there exists a homotopy extension $\tilde{H} : \tilde{A} \times I \to \tilde{X}$ of $H(p|_{\tilde{A}} \times 1)$ such that $p\tilde{H} = H(p|_{\tilde{A}} \times 1)$ and $\tilde{H}(\tilde{x}, 0) = i(\tilde{x})$, where $i : \tilde{A} \times 0 \to \tilde{X}$ is the inclusion map. Let $\tilde{\sigma}(t) = \tilde{H}(\tilde{x_0}, t)$, then $\tilde{\sigma}$ is a lifting of σ, where $\tilde{x_0}$ is a base point of $\tilde{X}$ such that $p(\tilde{x_0}) = x_0$. By the Monodromy Theorem, $[\tilde{\sigma}] = \alpha \in \pi_1(\tilde{X}, \tilde{x}_0)$. Therefore we see that $\tilde{\phi} = \tilde{H}(1 \times q)^{-1}$ is an affiliated map to α w.r.t.A because q is an identification.

THEOREM 7. *Let $p : (\tilde{X}.\tilde{A}) \to (X, A)$ be a pair covering of (X, A). If $n > 1$, then $p_*^{-1}(G_n(X, A)) \subseteq G_n(\tilde{X}, \tilde{A})$. In other words, if we identify $\pi_n(X)$ with $\pi_n(\tilde{X})$ under the isomorphism, then $G_n(\tilde{X}, \tilde{A}) \supseteq G_n(X, A)$.*

Proof. Let $\alpha \in G_n(X, A)$. Then there exists an affiliated map $\phi : A \times S^n \to X$ to α such that $\phi(\cdot, s_0) = i : A \to X$. Since $\phi_*(p|_{\tilde{A}} \times 1)_*[\pi_1(\tilde{A} \times S^n)] \subseteq p_*\pi_1(\tilde{X})$ where $\tilde{x_0}$ is a base point of $\tilde{X}$ such that $p(\tilde{x_0}) = x_0$, there exists a lifting $\tilde{\phi} : \tilde{A} \times S^n \to \tilde{X}$ of ϕ such that $p\tilde{\phi} = \phi(p|_{\tilde{A}}) \times 1)$.

Now $\tilde{\phi}\ |_{\tilde{A}} = i_{\tilde{A}}$ (as $\tilde{\phi}(\tilde{x}_0, s_0) = \tilde{x}_0$) and $\tilde{\phi}\ |_{S^n}$ represents $p_*^{-1}(\alpha)$ (as $p\tilde{\phi}\ |_{S^n} = \phi(p \times 1)\ |_{S^n} = \phi\ |_{S^n} = \alpha$). Thus $p_*^{-1}(\alpha) \in G_n(\tilde{X}, \tilde{A})$.

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Department of Mathematics Education
Korea University
Seoul 136–701, Korea