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# SOME CONCEPTS OF MULTIVARIATE NEGATIVE DEPENDENCE

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## 1. Introduction

Alam and Wallenius(1976) have derived stochastically increasing properties, concepts of positive likelihood ratio dependence, and $m^*$-positive dependence, Block and Ting(1981) have showed that the Alam and Wallenius concept of $m^*$-positive dependence is actually the same as the concept of totally positive of order $2(TP_2)$ in pairs, a property satisfied by a wide variety of multivariate distributions. Furthermore, they also have showed that several other concepts are equivalent to concepts of $TP_2$ in pairs and have obtained a general result of ALLP(1978) on the positive dependence, a result which required a much more involved proof originally.

In this paper, we extend positive dependence and stochastically increasing properties in Alam and Wallenius(1976) as well as several types of positive notions in Block and Ting(1981) to the negative case. In Section 2, stochastically decreasing properties and negative likelihood ratio dependence notions corresponding to the stochastically increasing properties and positive likelihood ratio dependence notions are presented. In Section 3, we show that several other concepts are equivalent to the concept of reverse rule of order $2(RR_2)$ in pairs, that is, we prove following: $m^*$-negative dependence $\Leftrightarrow$ reverse rule by deletion (RRD) $\Leftrightarrow$ $RR_2$ in pairs $\Leftrightarrow$ multivariate reverse rule of order $2(MRR_2)$ we obtain a generalization of the bivariate implication $RR_2 \Rightarrow RCSD$.

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## 2. Stochastically decreasing and negatively likelihood ratio dependent

Let $\underline{X} = (X_1, \cdots, X_n)$ be an $n$-variate random vector. A random variable $Y$ is stochastically increasing in $\underline{X}$ if for every $y$, $P\{Y > y|\underline{X} = \underline{x}\}$ is increasing in each component of $\underline{x}$. Following Barlow and Proschan(1975) we denote this property by $Y \uparrow$ st $\underline{X}$. It is well known (Alam, Wallenius, 1976) that $Y \uparrow$ st $\underline{X}$ if and only if $E[\psi(Y)|\underline{X} = \underline{x}]$ is increasing in each component of $\underline{x}$ for every increasing integrable function $\psi$. Motivated by this positive notion we introduce the following negative dependence.

DEFINITION 2.1(EBRAHIMI AND GHOSH, 1981). A random variable $Y$ is 'stochastically decreasing' in random variables $X_1, \cdots, X_n$ if for every real $y$

$$P(Y > y|X_1 = x_1, \cdots, X_n = x_n) \tag{2.1}$$

is decreasing in $x_1, \cdots, x_n$. We write $Y \downarrow$ st in $x_1, \cdots x_n$.

The following theorem provides a characterization of stochastically decreasing.

THEOREM 2.2. *Let $\underline{X} = (X_1, \cdots, X_n)$ be an $n$-variate random vector. Then $Y \downarrow$ st $\underline{X} \Leftrightarrow P(Y > y|\underline{X} = \underline{x}) \downarrow$ in $\underline{x}$ for all real $y$ $\Leftrightarrow E(\psi(Y)|\underline{X} = \underline{x}) \downarrow$ in $\underline{x}$ for all increasing integrable function $\psi$.*

*Proof.* $Y \downarrow st\underline{X} \Leftrightarrow P(Y > y|\underline{X} = \underline{x}) \downarrow$ in $\underline{x}$ for all real $y$ by definition $\Leftrightarrow P(Y \geq y|\underline{X} = \underline{x}) \downarrow$ in $\underline{x}$ for all real $y$.

Now assume $E(\psi(Y)|\underline{X} = \underline{x}) \downarrow$ in $\underline{x}$ for all real valued increasing integrable function $\psi$. Putting $\psi(Y) = I_{[Y \geq y]}$ it follows that $P(Y \geq y|\underline{X} = \underline{x}) \downarrow$ in $\underline{x}$ where $I_{[Y \geq y]}$ is an indicator function. Conversely, suppose $E[I_{[Y \geq y]}|\underline{X} = \underline{x}] \downarrow$ in $\underline{x}$. Then $E[\psi(Y)|\underline{X} = \underline{x}] \downarrow$ in $\underline{x}$ for every nonnegative simple function $\psi$. Using the monotone convergence theorem, the same is true for every nonnegative increasing function $\psi$. Subsequently, the result holds for every increasing function $\psi$.

DEFINITION 2.3(EBRAHIMI AND GHOSH, 1981). The random variables $X_1, \cdots, X_n$ are said to be stochastically decreasing in sequence if for $i = 2, 3, \cdots, n$ and all real number $x_i$,

$$P(X_i > x_i|X_1 = x_1, \cdots, X_{i-1} = x_{i-1}) \text{ is decreasing in } x_1, \cdots, x_{i-1}.$$

From Theorem 2.2 and Definition 2.3 we immediately obtain following corollary.

COROLLARY 2.4. *Let* $(X_1, \cdots, X_n)$ *be stochastically decreasing in sequence. Then* $E(\psi(X_i)|X_1 = x_1, \cdots, X_{i-1} = x_{i-1})$ *is decreasing in* $x_1, \cdots, x_{i-1}$ *for every increasing integrable function* $\psi$.

The family of conditional distributions of $Y$ given $\underline{X} = \underline{x}$ is said to be positive likelihood ratio dependent(PLRD) on $\underline{X}$ if the conditional density(conditional probability mass function) $f(y|\underline{X} = \underline{x})$ exists such that, whenever $y' \geq y$ and $\underline{x}' \geq \underline{x}$

$$(2.2) \qquad f(y'|\underline{X} = \underline{x}')f(y|\underline{X} = \underline{x}) - f(y'|\underline{X} = \underline{x})f(y|\underline{X} = \underline{x}') \geq 0.$$

Following Dyskstra, Hewett and Thompson(1973), we denote the property by $Y$ plrd $\underline{X}$. Motivated by (2.2) we present another negative dependence.

DEFINITION 2.5. The family of conditional distributions of $Y$ given $\underline{X} = \underline{x}$ is said to be negatively likelihood ratio dependent(NLRD) on $\underline{X}$ if the conditional density (conditional probability mass function) $f(y|\underline{X} = \underline{x})$ exists such that, whenever $y' \geq y$ and $\underline{x}' \geq \underline{x}$,

$$(2.3) \qquad f(y'|\underline{X} = \underline{x}')f(y|\underline{X} = \underline{x}) - f(y'|\underline{X} = \underline{x})f(y|\underline{X} = \underline{x}') \leq 0.$$

We denote the property by $Y$ nlrd $X$.

Let $\underline{X}^* = (X_{i_1}, X_{i_2}, \cdots, X_{i_k})$ be $k$-dimensional subvector of $\underline{X} = (X_i, \cdots, X_n)$, $k < n$. Then $Y \downarrow$ st $\underline{X}(Y$ nlrd $\underline{X})$ does not necessarily imply $Y \downarrow$ st $\underline{X}^*(Y$ nlrd $\underline{X}^*)$. The following examples illustrate these.

Let $\underline{X} = (X_1, X_2)$ be given by the following joint probability mass function (Table I).

| | | $X_2$ = 0 | | $X_2$ = 1 | |
|---|---|---|---|---|---|
| | | $X_1$ | | $X_1$ | |
| | | 0 | 1 | 0 | 1 |
| Y | 0 | 0 | 0.01 | 0.01 | 0.2 |
| | 1 | 0.01 | 0.56 | 0.01 | 0.2 |

Table I

Hence $P[Y = 1|X_1 = 0,\ X_2 = 0] = 1$, $P[Y = 1|X_1 = 1,\ X_2 = 0] = 0.56/0.57 = 0.98$, and $P[Y = 1|X_1 = 0,\ X_2 = 1] = 0.5$, $P[Y = 1|X_1 = 1,\ X_2 = 1] = 0.5$ so that $Y \downarrow$ st $(X_1, X_2)$. However, $P[Y = 1|X_1 = 0] = 0.02/0.03 = 0.67 < 0.78 = 0.76/0.97 = P[Y = 1|X_1 = 1]$ and therefore it is not the case $Y \downarrow$ st $X_1$. Similarly, $Y$ nlrd $\underline{X}$ does not necessarily imply $Y$ nlrd $\underline{X}^*$. The following example illustrates this: Let $Y, X_1, X_2, (Y \leq X_1 \leq X_2)$ be random variables with joint pdf

$$f(y, x_1, x_2) = \begin{cases} 48yx_1x_2, & 0 \leq y \leq x_1 \leq x_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

It is easy to check from (2.4) that $Y$ nlrd $(X_1, X_2)$. However, the joint pdf $h$ of $X_1$ and $X_2$ is $h(x_1, x_2) = 24x_1^3x_2$, $0 \leq x_1 \leq x_2 \leq 1, h(x_1, x_2) = 0$, otherwise, pdf $h'$ of $Y$ and $X_2$ is $h'(y, x_2) = 24yx_2(x_2^2 - y^2)$, $0 \leq y \leq x_2 \leq 1$, $h'(y, x_2) = 0$, otherwise, pdf $g$ of $X_2$ is $g(x_2) = 6x_2^5$, $0 \leq x_2 \leq 1$, and the conditional pdf of $Y$ given $X_2$ is

$$f(y|x_2) = \begin{cases} 4y(x_2^2 - y^2)/x_2^4, & 0 \leq y \leq x_2 \leq 1. \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Hence for $y < y'$, $x_2 < x_2'$ it follows from (2.5) that

$$\begin{aligned} & f(y'|x_2')f(y|x_2) - f(y|x_2')f(y'|x_2) \\ =& (4y'(x_2'^2 - y'^2)/x_2'^4)(4y(x_2^2 - y^2)/x_2^4) \\ & - (4y'(x_2^2 - y'^2)/x_2^4)(4y(x_2'^2 - y^2)/x_2'^4) \\ =& (16yy'/(x_2^4x_2'^4))\{(x_2'^2 - y'^2)(x_2^2 - y^2) - (x_2^2 - y'^2)(x_2'^2 - y^2)\} \geq 0 \end{aligned}$$

since $(x_2'^2 - y'^2)(x_2^2 - y^2) - (x_2^2 - y'^2)(x_2'^2 - y^2) = (x_2'^2 - x_2^2)(y'^2 - y^2) \geq 0$, so that not $Y$ nlrd $X_2$ but $Y$ plrd $X_2$.

Table II is a discrete distribution exhibiting intransitivity of the $\downarrow$ st and the nlrd relations.

| | Z | 0 | | 1 | |
|---|---|---|---|---|---|
| | Y | 0 | 1 | 0 | 1 |
| X | 0 | 0.1 | 0.0 | 0.1 | 0.1 |
| X | 1 | 0.1 | 0.1 | 0.5 | 0.0 |

Table II

It is easy to show $X \downarrow$ st $Y$ and $Y \downarrow$ st $Z$ but not $X \downarrow$ st $Z$:

$$P[X = 1|Y = 0] = 0.6/0.8 = 0.75 > P[X = 1|Y = 1] = 0.1/0.2 = 0.5$$
$$P[Y = 1|Z = 0] = 0.1/0.3 = 0.33 > P[Y = 1|Z = 1] = 0.1/0.7 = 0.14,$$

so that $X \downarrow$ st $Y$ and $Y \downarrow$ st $Z$. However, $P[X = 1|Z = 0] = 0.2/0.3 = 0.67 < P[X = 1|Z = 1] = 0.5/0.7 = 0.7$ and therefore it is not the case that $X \downarrow$ st $Z$. Similarly $X$ nlrd $Y$ and $Y$ nlrd $Z$ but not $X$ nlrd $Z$: $P[X = 1|Y = 1] = 0.1/0.2$ $P[X = 1|Y = 0] = 0.6/0.8, P[X = 0|Y = 1] = 0.1/0.2$, and $P[X = 0|Y = 0] = 0.3/0.8$ imply $(0.1/0.2)(0.3/0.8) < (0.6/0.8)(0.1/0.2)$, so that $X$ nlrd $Y$ and $P[Y = 1|Z = 1] = 0.1/0.7$, $P[Y = 1|Z = 0] = 0.1/0.3$, $P[Y = 0|Z = 1] = 0.6/0.7$ and $P[Y = 0|Z = 0] = 0.2/0.3$ imply $(0.1/0.7)(0.1/0.3) < (0.6/0.7)(0.2/0.3)$, so that $Y$ nlrd $Z$.

However $P[X = 1|Z = 1] = 0.5/0.7, P[X = 1|Z = 0] = 0.2/0.3, P[X = 0|Z = 1] = 0.2/0.7$, and $P[X = 0|Z = 0] = 0.1/0.3$ imply $(0.5/0.7)(0.2/0.3) < (0.2/0.7)(0.1/0.3)$, so that it is not $X$ nlrd $Z$.

Alam and Wallenius(1976) have introduced another notion of positive dependence generalizing the idea of positive regression dependence of Lehmann(1966). For any random vector $\underline{X} = (X_1, \cdots, X_n)$ let $\underline{X}^{(i)}$ denote the random vector $(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ obtained by deleting the ith component of $\underline{X}$.

DEFINITION 2.6. The components of $\underline{X}$ are $s^*$-positively dependent if $X_i \uparrow$ st $\underline{X}^{(i)}$ for $i = 1, 2, \cdots, n$.

Motivated by Definition 2.6 we introduce the following notion of negative dependence.

DEFINITION 2.7. The components of $X$ are $s^*$-negatively dependent if $X_i \downarrow$ st $\underline{X}^{(i)}$ for $i = 1, \cdots, n$.

It is obvious that if $\underline{X} = (X_1, \cdots, X_n)$ satisfies $s^*$-negative dependence then $X_n$ is stochastically decreasing in $X_1, \cdots, X_{n-1}$ and we obtain immediately the following theorem.

THEOREM 2.8. *Let the components of $\underline{X}$ be $s^*$-negatively dependent and let the components of every subvector of $\underline{X}$ be $s^*$-negatively dependent. Then the random vector $\underline{X}$ is stochastically decreasing in sequence.*

## 3. Concepts of m*-negative dependence

We start with the definitions of total positivity of order $2(TP_2)$ and reverse rule of order $2(RR_2)$, which may be found in Karlin(1968).

DEFINITION 3.1. A nonnegative real function $f : R^2 \to R$ is $TP_2(RR_2)$ if

$$(3.1) \qquad f(x_1, x_2) f(x_1', x_2') \geq (\leq) f(x_1, x_2') f(x_1', x_2)$$

whenever $x_1' \geq x_1$, and $x_2' \geq x_2$.

The definitions below are appeared in Karlin and Rinott(1980 a,b). For every $\underline{x}, \underline{y} \in R^k$, denote : $\underline{x} \geq \underline{y}$ if $x_i \geq y_i$ for all $i = 1, \cdots, k$, $\underline{x} \vee \underline{y} = (\max(x_1, y_1), \cdots, \max(y_k, y_k))$, and $\underline{x} \wedge \underline{y} = (\min(x_1, y_1), \cdots, \min(x_k, y_k))$. The following definition is the natural generalization of Definition 3.1.

DEFINITION 3.2 (KARLIN AND RINOTT, 1980). Consider a nonnegative real function $f : R^k \to R$. We say that $f(\underline{x})$ is multivariate totally positive of order 2 or $MTP_2$ (multivariate reverse rule of order 2 or $MRR_2$) if

$$(3.2) \qquad f(\underline{x} \vee \underline{y}) f(\underline{x} \wedge \underline{y}) \geq (\leq) f(\underline{x})(\underline{y}) \text{ for every } \underline{x}, \underline{y} \in R^k.$$

DEFINITION 3.3 (KARLIN, 1968). Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector with density $f(\underline{x}) = f(x_1, \cdots, x_n)$. We say $\underline{X}$ or $f$ is $TP_2$ in pairs ($RR_2$ in pairs) for any $i \neq j$ and for $x_i < x'_i, x_j < x'_j$

(3.3)

$$f(x_i, x_j, \underline{x}^{(i,j)}) f(x'_i, x'_j, \underline{x}^{(i,j)}) \geq (\leq) f(x'_i, x_j, \underline{x}^{(i,j)}) f(x_i, x'_j, \underline{x}^{(i,j)})$$

for all $\underline{x}^{(i,j)} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ where $(x_i, x_j, \underline{x}^{(i,j)})$ is notation for the $n$-tuple with $x_i$ in the $i$-th position, $x_j$ in the $j$-th position and $\underline{x}^{(i,j)}$ in the remaining positions.

Multivariate distributions which satisfy (3.3) can be found in Ebrahimi and Ghosh(1981) and also in Karlin and Rinott (1980 a.b). Alam and Wallenius(1976) defined the concept of $m^*$-positive dependence as follows.

DEFINITION 3.4. The components of the random vector $\underline{X} = (X_1, X_2, \cdots, X_n)$ is said to be $m^*$-positively, if for $i = 1, \cdots, n$, the conditional distribution of $X_i$ given $\underline{X}^{(i)}$ is positively likelihood ratio dependent on $X^{(i)}$, i,e., if the conditional density $g(x_1|\underline{x}^{(i)})$ exists such that, whenever $x'_i \geq x_i$ and $\underline{x}'^{(i)} \geq \underline{x}^{(i)}$

(3.4)

$$g(x'_i|\underline{X}^{(i)} = \underline{x}'^{(i)}) g(x_i|\ aa^{(i)} = \underline{x}^{(i)}) \geq g(x'_i|\underline{X}^{(i)} = \underline{x}^{(i)}) g(x_i|\underline{X}^{(i)} = \underline{x}'^{(i)}),$$

for $i = 1, 2, \cdots, n$.

Motivated by (3.4), the components of $\underline{X}$ are $m^*$-negative dependence is defined as follows.

DEFINITION 3.5. The components of the random vector $\underline{X} = (X_1, \cdots, X_n)$ are said to be $m^*$-negatively dependent($m^*$-ND) if for all $i = 1, \cdots, n$ the conditional distribution of $X_i$ given $\underline{X}^{(i)}$ is negatively likelihood ratio dependent(NLRD) on $\underline{X}^{(i)}$, *i.e.*, if the conditional density $g(x_i|\underline{x}^{(i)})$ exists such that whenever $x'_i \geq x_i$ and $\underline{x}'^{(i)} \geq \underline{x}^{(i)}$

$$g(x'_i|\underline{X}^{(i)} = \underline{x}'^{(i)}) g(x_i|\underline{X}^{(i)} = \underline{x}^{(i)}) \leq g(x'_i|\underline{X}^{(i)} = \underline{x}^{(i)}) g(X_i|\underline{X}^{(i)} = \underline{x}'^{(i)}),$$

for $i = 1, 2, \cdots, n$.

A necessary and sufficient condition for $\underline{X} = (X_1, \cdots, X_n)$ to have $m^*$-negative dependence in the multivariate normal case is that the regression coefficients of $\underline{X}_i$ on $X^{(i)}$ be nonpositive for $i = 1, \cdots, n$.

DEFONITION 3.6. (AHMED, ETAL., 1978). A random vector $\underline{X} = (X_1, \cdots, X_2)$ is said to be totally positive by deletion (TPD) if $\underline{X}$ has a density $f$ such that the determinant

(3.5)

$$\begin{vmatrix} f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) & f(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n) \\ f(x'_1, \cdots, x'_{i-1}, x_i, x'_{i+1}, \cdots, x'_n) & f(x'_1, \cdots, x'_{i-1}, x'_i, x'_{i+1}, \cdots, x'_n) \end{vmatrix} \geq 0$$

for all $x_i \leq x'_i$, $i = 1, 2, \cdots, n$.

Motivated by (3.5) we introduce the following notion of negative dependence.

DEFINITION 3.7. $\underline{X}$ is said to be reverse rule by deletion (RRD) if $\underline{X}$ has a density $f$ such that for all $x_i \leq x'_i$, $i = 1, \cdots, n$,

(3.6)

$$\begin{vmatrix} f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) & f(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n) \\ f(x'_1, \cdots, x'_{i-1}, x_i, x'_{i+1}, \cdots, x'_n) & f(x'_1, \cdots, x'_{i-1}, x'_i, x'_{i+1}, \cdots, x'_n) \end{vmatrix} \leq 0$$

THEOREM 3.8. *Let $X_1, \cdots, X_n$ have joint density $f(\underline{x})$, where $f(\underline{x}) > 0$ for all $\underline{x} \in \Omega = \prod_{i=1}^n \Omega_i, \Omega_i \in R^1$, then the following four statements are equivalent:*

*(1) $X_1, X_2, \cdots, X_n$ are $m^*$-negatively dependent,*

*(2) $X_1, X_2, \cdots, X_n$ are RRD,*

*(3) $f(x_1, x_2, \cdots, x_n)$ is $RR_2$ in pairs,*

*(4) $f(\underline{x})$ is $MRR_2$, that is, $f(\underline{x} \vee \underline{y}) f(\underline{x} \wedge \underline{y}) \leq f(\underline{x}) f(\underline{y}) \underline{x}, \underline{y} \in R^n$.*

*Proof.* (1) $\Rightarrow$ (2). Assume that $X_1, \cdots, X_n$ are $m^*$-negatively dependent. Then for all $i = 1, \cdots, n$ and $x'_i \geq x_i$ and $\underline{x}'^{(i)} \geq \underline{x}^{(i)}$

$$g(x'_i | \underline{X}^{(i)} = \underline{x}'^{(i)}) g(x_i | \underline{X}^{(i)} = \underline{x}^{(i)} \leq g(x'_i | \underline{X}^{(i)} = \underline{x}^{(i)}) g(x_i | \underline{X}^{(i)} = \underline{x}'^{(i)})$$

$$\Rightarrow \frac{g(x'_1, \cdots, x'_{i-1}, x'_i, x'_{i+1}, \cdots, x'_n)}{g(x'_1, \cdots, x'_{i-1}, x'_{i+1}, \cdots, x'_n)} \frac{g(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)}{g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)}$$

$$\leq \frac{g(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n)}{g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)} \frac{g(x'_1, \cdots, x'_{i-1}, x_i, x'_{i+1}, \cdots, x'_n)}{g(x'_1, \cdots, x'_{i-1}, x'_{i+1}, \cdots, x'_n)}$$

$$\Rightarrow g(x'_1, \cdots, x'_{i-1}, x'_i, x'_{i+1}, \cdots, x'_n) g(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \leq$$

$$g(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n) g(x'_1, \cdots, x'_{i-1}, x_i, x'_{i+1}, \cdots, x'_n)$$

so that $X_1, X_2, \cdots, X_n$ are RRD.
(2) $\Rightarrow$ (3). In the definition of RRD take any $x_i \leq x_i'$, $x_j \leq x_j'$ for $i, j (j \neq i)$ and take $x_k = x_k'$ for $k = 1, 2, \cdots, n$ with $k \neq j$, $k \neq i$, then $X_1, X_2, \cdots, X_n$ are $RR_2$ in pairs.
(3) $\Rightarrow$ (4). By permuting the indices if necessary, we may consider $\underline{x}, \underline{y} \in R^n$ such that $x_k \geq y_k$ for $k = 1, \cdots, r$ and $x_k \leq y_k$ for $k = r+1, \cdots, n$. Let $s = n - r$. Now for $0 \leq i \leq r$, $0 \leq j \leq s$ define:

$$\underline{z}^{i,j} = (x_1 \vee y_1, \cdots, x_i \vee y_i, x_{i+1} \wedge y_{i+1}, \cdots, x_r \wedge y_r, x_{r+1} \vee y_{r+1}, \cdots, \\ x_{r+j} \vee \cdots, x_{r+j} \vee y_{r+j}, x_{r+j+1} \wedge y_{r+j+1}, \cdots, x_n \wedge y_n)$$

where $\underline{x}_k \vee \underline{y}_k = \max\{\underline{x}_k, \underline{y}_k\}, \underline{x}_k \wedge \underline{y}_k = \min\{\underline{x}_k, \underline{y}_k\}$ then $\underline{z}^{r,0} = \underline{x}, \underline{z}^{0,s} = \underline{y}, \underline{z}^{0,0} = \underline{x} \wedge \underline{y}, \underline{z}^{r,s} = \underline{x} \vee \underline{y}$. Since $\underline{z}^{i+1,j}$ and $\underline{z}^{i,j+1}$ differ only at the $(i+1)$th and the $(r+j+1)$th coordinates, $\underline{z}^{i+1,j} \wedge \underline{z}^{i,j+1} = \underline{z}^{i,j}, \underline{z}^{i,j+1} \vee \underline{z}^{i,j+1} = \underline{z}^{i+1,j+1}$, and since the density is $RR_2$ in pairs

$$1 \leq \prod_{i=0}^{r-1} \prod_{j=0}^{s-1} \frac{f(\underline{z}^{i+1,j}) f(\underline{z}^{i,j+1})}{f(\underline{z}^{i,j}) f(\underline{z}^{i+1,j+1})} = \frac{f(\underline{z}^{0,s} f(\underline{z}^{r,0})}{f(\underline{z}^{0,0}) f(\underline{z}^{r,s})} = \frac{f(\underline{x}) f(\underline{y})}{f(\underline{x} \wedge \underline{y}) f(\underline{x} \vee \underline{y})}.$$

(4) $\Rightarrow$ (1). Take $\underline{x} = (x_i, \underline{x}'^{(i)})$, $\underline{y} = (x_i', \underline{x}^{(i)})$ where $x_k \leq x_k'$, then $\underline{x} \wedge \underline{y} = (x_i, \underline{x}^{(i)}$, $\underline{x} \vee \underline{y} = (x_i', \underline{x}'^{(i)})$ and $f(\underline{x}) f(\underline{y}) \geq f(\underline{x} \wedge \underline{y}) f(\underline{x} \vee \underline{y})$. (1) follows just by dividing through on both sides by appropriate marginal densities which must be positiveon the appropriate projections of $\Omega$.

REMARK. It should be noticed that (3) does not imply (2) without the aassumption that $f(\underline{x}) > 0$ on $\Omega$ of the form in the theorem. The simple counter example given is

$$f(x_1, x_2, x_3) = \begin{cases} 1/2 & \text{if } 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1,\ 0 \leq x_3 \leq 1, \\ 1/2 & \text{if } 1 < x_1 \leq 2,\ 1 < x_2 \leq 2,\ 1 < x_3 \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 1. Let $(X_1, \cdots, X_n)$ have a multinormal distribution with joint pdf

$$f(x_1, \cdots, x_n) = (2\pi)^{-(n/2)} |\underset{\sim}{\sum}|^{-(1/2)} \exp[-(1/2) \sum_{1 \le i,j \le n} r_{ij}(x_i - \mu_i)(x_j - \mu_j)]$$

where $(\mu_1, \cdots, \mu_n)$ is a mean vector, $\underset{\sim}{\sum}$ is a variance - covariance matrix which is positive definite, and $\underset{\sim}{\sum}^{-1} = ((r_{ij}))$. Suppose $r_{ij} > 0$ for all $1 \le i, \ j \le n$. Then $(X_1, \cdots, X_n)$ satisfies Theorem 3.8.

EXAMPLE 2. Let $(X_1, \cdots, X_n)$ have a multinomial distribution witrh joint pdf

$$f(x, \cdots, x_k) = \frac{N!}{\prod_1^k x_i!(N - \sum_1^k x_i)!} \prod_1^k p_i^{xi} (1 - \sum_1^k p_i)^{N - \sum_1^k x_i}$$

$$x_i \ge 0, \ \sum_1^k xi \le N, \ \sum_1^k p_i \le 1.$$

Then $(X_1, \cdots, X_n)$ satisfies Theorem 3.8.

THEOREM 3.9. *If components of* $\underline{X} = (X_1, \cdots, X_n)$ *are* $m^*$*-negatively dependent then the components of* $\underline{X}$ *are* $s^*$*-negatively dependent.*

*Proof.* By definition of $m^*$-negative dependence there exists the conditional density function $g(x_i | \underline{x}^{(i)})$ such that for $x_i' \ge x_i$ and $x'^{(i)} \ge x^{(i)}$

$$g(x_i' | \underline{X}^{(i)} = x'^{(i)}) g(x_i | \underline{x}^{(i)}) \le g(x_i' | \underline{X}^{(i)} = \underline{x}^{(i)}) g(x_i | \underline{X}^{(i)} = \underline{x}'^{(i)})$$

which implies

$$
\begin{aligned}
&\frac{g(x'_1,\cdots,x'_n)}{g_1(x'_1,\cdots,x'_{i-1},x'_{i+1},\cdots,x'_n)}\cdot\frac{g(x_1,\cdots,x_n)}{g_1(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}\\
&\le\frac{g_1(x_1,\cdots,x_{i-1},x'_i,x_{i+1},\cdots,x_n)}{g_1(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}\cdot\frac{g_1(x'_1,\cdots,x'_{i-1},x_i,x'_{i+1},\cdots,x'_n)}{g_1(x'_1,\cdots,x'_{i-1},x'_{i+1},\cdots,x_n)}\\
\Rightarrow&\begin{vmatrix} g(x'_1,\cdots,x'_n) & g(x_1,\cdots,x_{i-1},x'_i,x_{i+1},\cdots,x_n)\\ g(x'_1,\cdots,x'_{i-1},x_i,x'_{i+1},\cdots,x'_n) & g(x_1,\cdots,x_n)\end{vmatrix}\le 0\\
\Rightarrow&\begin{vmatrix} \int_t^\infty g(x'_1,\cdots,x'_{i-1},u_i,x'_{i+1},\cdots,x'_n)du_i & \int_t^\infty g(x_1,\cdots,x_{i-1},u_i,x_{i+1},\cdots,x_n)du_i\\ \int_{-\infty}^t g(x'_1,\cdots,x'_{i-1},u_i,x'_{i+1},\cdots x'_n)du_i & \int_{-\infty}^t g(x_1,\cdots,x_{i-1},u_i,x_{i+1},\cdots,x_n)du_i\end{vmatrix}\le 0\\
\Rightarrow&\begin{vmatrix} \int_t^\infty g(x'_1,\cdots,u_i,\cdots,x'_n)dx'_i & \int_t^\infty g(x_1,\cdots,x_{i-1},u_i,x_{i+1},\cdots,x_n)dx'_i\\ g_1(x'_1,\cdots,x'_{i-1},x'_{i+1},\cdots,x'_n) & g_1(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)\end{vmatrix}\le 0\\
\Rightarrow&P(X_i>t|X_1=x'_1,\cdots,X_{i-1}=x'_{i-1},X_{i+1}=x'_{i+1},\cdots,X_n=x'_n)\\
&\le P(X_i>t|X_1=x_1,\cdots,X_{i-1}=x_{i-1},X_{i+1}=x_{i+1},\cdots,X_n=x_n),
\end{aligned}
$$

so that $X_i\downarrow$ st $\underline{X}^{(i)}$.

REMARK. If $\underline{X}=(X_1,\cdots,X_n)$ satisfies $m^*$-negative dependence and every subvector of $\underline{X}$ satisfies $m^*$-negative dependence then $(X_1,\cdots,X_n)$ is stochastically decreasing in sequence.

Motivated by Corollary 1 in Block and Ting(1981) we obtain the following theorem.

THEOREM 3.10. *Let $(X_1,\cdots,X_n)$ have a density $f(x_1,\cdots,x_n)$ which is $RR_2$ in pairs, on the set $\Omega$ in Theorem 3.8 and let every subector $(X_{i_1},\cdots,X_{i_k})$ have a density which is $RR_2$ in pairs. Then for every integrable increasing function $\Phi$, $E(\Phi(X_1,\cdots,X_n)|X_1>a_1,\cdots,X_n>a_n)$ is decreasing in $a_1,\cdots,a_n$.*

*Proof.* As in the proof of Corollary 1 of [4], we prove as follows: Let $\underline{X}=(X_1,\cdots,X_n)$. Since $X_i$ is stochastically decreasing in sequence by assumption(see Theorem 3.8 and Remark) it is easy to show the stronger condition that for $i=2,\cdots,n$

$$E(\Phi(\underline{X})|X_1=x_1,\cdots,X_{i-1}=x_{i-1},X_i>a_i,\cdots,X_n>a_n)\tag{3.7}$$

is decreasing in $x_1,\cdots,x_{i-1}$ for all increasing $\Phi$, for all $a_i,\cdots,a_n$. Now we can write

$$E(\Phi(\underline{X})|X_i > a_i, i = 1, \cdots, n) = E(\Phi_1(\underline{X}_1)|X_i > a_i, i = 1, \cdots, n)$$

where

$$\Phi_1(X_1) = E(\Phi_2(X_1, X_2)|X_1, X_i > a_i, i = 2, \cdots, n)$$

and

$$\Phi_i(X_1, \cdots, X_i) = \begin{cases} E(\Phi_{i+1}(X_1, \cdots, X_{i+1}|X_1, \cdots, X_i, X_{i+1} > a_{i+1}, \\ \qquad \cdots, X_n > a_n), \ i = 2, \cdots, n-1, \\ \Phi(\underline{X}), \ i = n. \end{cases}$$

Thus, from (3.7)

$$\Phi_{n-1}(x_1, \cdots, x_{n-1}) = E(\Phi(\underline{X})|X_1 = x_1, \cdots, X_{n-1} = x_{n-1}, X_n > a_n)$$

decreases in $x_1, \cdots, x_{n-1}$ and in general

$$\Phi_{i-1}(x_1, \cdots, x_{i-1}) = E(\Phi_i(\underline{X})|X_1 = x_1, \cdots, X_{i-1} = x_{i-1}, \\ X_i > a_i, \cdots, X_n > a_n)$$

decreases in $x_1, \cdots, x_{i-1}$ for $i = 2, \cdots, n$. Since $\Phi_1(x_1)$ decreases in $x_1$, it is not hard to show that

$$E(\Phi(\underline{X})|X_1 > a_1, \cdots, X_n > a_n) = E(\Phi_1(\underline{X}_1)|X_1 > a_1, \cdots, X_n > a_n)$$

decreases in $a_1$ (i.e., for increasing $\Phi$ and any r.v. $Y$, $E(\Phi(Y)|Y > a)$ decreases in $a$). Similarly the previous quantity decreases in $a_2, \cdots, a_n$.

DEFINITION 3.11 (BRINDLEY AND THOMPSON, 1972). Random variables $X_1, \cdots, X_n$ are right corner set decreasing (written RCSD $(X_1, \cdots, X_n)$) if $P[X_1 > x'_1, \cdots, X_n > x'_n|X_1 > x_1, \cdots, X_n > x_n]$ is decreasing in $\{x_i : x_i > x'_i\}$ for every choice of $x'_1, \cdots, x'_n$.

From Thorem 3.10 we obtain the following theorem which is generalization of the bivariate implication $RR_2(X, Y) \Rightarrow RCSD(X, Y)$.

THEOREM 3.12. *If the components of $\underline{X}$ are $m^*$-negatively dependent and every subvector of $\underline{X}$ are $m^*$-negatively dependent, then $P\{X_1 > x_1, \cdots, X_n > x_n | X_1 > x'_1, \cdots, X_n > x'_n\}$ is decreasing in $\{x'_i : x'_i \geq x_i\}$ for every choice of $x_1, \cdots, x_n$*

*Proof.* By Theorems 3.8 and 3.10 for every increasing function $\Phi(\underline{X})$

$$E[\Phi(\underline{X})|X_1 > x'_1, \cdots, X_n > x'_n] \text{ is decreasing in } x'_1, \cdots, x'_n. \tag{3.8}$$

Taking $\Phi(\underline{X}) = 1_{[\underline{X} > \underline{x}]}$ we obtain

$$\begin{aligned} &P[X_1 > x_1, \cdots, X_n > x_n | x_1 > x'_1, \cdots, X_n > x_n] \\ &\qquad = E(\Phi(X)|X_1 > x'_1, \cdots, X_n > x'_n]. \end{aligned} \tag{3.9}$$

From (3.8) the right hand side of (3.9) is decreasing in $\{x'_i : x'_i \geq x_i\}$ for every choice of $x_1, \cdots, x_n$, so that $RCSD(X_1, \cdots, X_n)$.

THEOREM 3.13. *$Y \uparrow st\ \underline{X}$ and the components of $\underline{X}$ are $m^*$-negatively dependent and the components of every subvector of $\underline{X}$ are $m^*$-negatively dependent, then $P\{Y > y | X_1 > a_1, \cdots, X_n > a_n\}$ is decreasing in $a_1, \cdots, a_n$.*

*Proof.* Taking $\Phi(\underline{X}) = P[Y > y | X_1 = x_1, \cdots, X_n = x_n]$ we have

(3.10)

$$P[Y > y | x_1 > a_1, \cdots, X_n > a_n] = E(\Phi(\underline{X})|X_1 > a_1, \cdots, X_n > a_n)$$

Now $\Phi(\underline{X})$ is increasing in $x_1, \cdots, x_n$ since $Y \uparrow$ st $\underline{X}$ so that, by Theorem 3.10 the right hand side of (3.10) is decreasing in $a_1, \cdots, a_n$.

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