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THE CANONICAL DECOMPOSITION OF SIEGEL MODULAR FORMS II*

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0. Introduction

Let n, k, q be positive integers. Let Γ be a congruene subgroup of the symplectic group $\Gamma^n = Sp_n(\mathbf{Z})$ of level q and let χ be a congruence character on Γ. We introduced [1] the canonical decomposition of the space $\mathcal{M}_k^n(\Gamma, \chi)$ of Siegel modular forms of degree n, weight k, and character χ relative to Γ into $n+1$ subspaces $\mathcal{M}_k^{n,r}(\Gamma, \chi)$, $r = 0, 1, \ldots, n$.

In this article, we apply the canonical decomposition to the space $\mathcal{M}_k^n = \mathcal{M}_k^n(\Gamma^n, 1)$ to recapture the existence of an eigen-basis for the space [2, 3]. More precisely, we decompose $\mathcal{M}_k^n$ into $n+1$ canonical subspaces $\mathcal{M}_k^{n,r}$, $r = 0, 1, \ldots, n$, and show that each $\mathcal{M}_k^{n,r}$ is invariant under the Hecke operators from a certain Hecke ring and has a simultaneous eigen-basis with respect to those Hecke operators.

Let $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ be the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over A, a commutative ring with 1, and let $M_n(A) = M_{n,n}(A)$. Let $GL_n(A)$ and $SL_n(A)$ be the group of invertible matrices in $M_n(A)$ and its subgroup consisting of matrices of determinant 1, respectively.

For $g \in M_m(A)$, $h \in M_{m,n}(A)$, let $g[h] = {}^t hgh$, where ${}^t h$ is the transpose of h. Let I_n and 0_n be the identity and the zero matrices, respectively. Let $\det g$ be the determinant of g. For $g \in M_{2n}(A)$, we let A_g, B_g, C_g, and D_g be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of g, respectively, and we write $g = (A_g, B_g; C_g, D_g)$. Let $\mathrm{diag}(N_1, N_2, \ldots, N_r)$ be the matrix with block matrieces $N_1, N_2, \ldots N_r$ on its main diagonal and

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zeroes outside. Let $\mathcal{N}_m$ be the set of all semi–positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues > 0) matrices.

Let $\mathcal{G}_n = GSp_n^+(\mathbf{R}) = \{g \in M_{2n}(\mathbf{R}); J_n[g] = rJ_n,\ r > 0\}$ where $J_n = (0_n, I_n; -I_n, 0_n)$ and r is a real number determined by g and we denote it by $r(g)$. Let $\Gamma^n = Sp_n(\mathbf{Z}) = \{M \in M_{2n}(\mathbf{Z}); J_n[M] = J_n\}$ and $S^n = S_p^n = \{g \in M_{2n}(\mathbf{Z}[\frac{1}{p}]); J_n[g] = p^\delta J_n,\ \delta \in \mathbf{Z}\}$ where p is a prime number and δ is an integer determined by g, which we denote by $\delta(g)$. Let $\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbf{C}); {}^tZ = Z,\ Y > 0\}$ where $Y > 0$ means Y is positive definite.

For an arbitrary complex-valued function F on $\mathcal{H}_n$ and $g \in \mathcal{G}_n$, we set

$$(0.1) \qquad (F|_k g)(Z) = (\det g)^{k-(n+1)/2}(\det(C_g Z + D_g))^{-k} F(g\langle Z\rangle)$$

where $Z \in \mathcal{H}_n$ and $g\langle Z\rangle = (A_g Z + B_g)(C_g Z + D_g)^{-1} \in \mathcal{H}_n$.

Finally, for $Z \in M_n(\mathbf{C})$, we set $e(Z) = e^{2\pi i\sigma(Z)}$ where $\sigma(Z)$ is the trace of Z. For other standard notations and basic facts, we refer the readers [4, 5, 6].

1. Hecke Rings

In this section, we introduce some Hecke rings and the Ψ-operator.

Let G be a multiplicative group and Γ be its subgroup. Let $\tilde{\Gamma}$ be the commensurator subgroup of Γ in G, i.e., $\tilde{\Gamma} = \{g \in G; g^{-1}\Gamma g \cap \Gamma$ is of finite index in both $g^{-1}\Gamma g$ and $\Gamma\}$. Let S be a semi-group containing Γ and contained in $\tilde{\Gamma}$. Let (Γ, S) be a Hecke pair, i.e., $\Gamma S = S\Gamma = S$. Let $L(\Gamma, S)$ be the vector space over $\mathbf{C}$ spanned by the formal left cosets (Γg), $g \in S$. For $X = \sum_{i=1}^{\mu} a_i(\Gamma g_i) \in L(\Gamma, S)$, $a_i \in \mathbf{C}$, and $g \in S$, we set $X \cdot g = \sum_{i=1}^{\mu} a_i(\Gamma g_i g) \in L(\Gamma, S)$. We define $D(\Gamma, S) = \{X \in L(\Gamma, S); X \cdot M = X$ for any $M \in \Gamma\}$. $D(\Gamma, S)$ is in fact a ring under the multiplication defined by $X \cdot Y = \sum_{i,j} a_i b_j(\Gamma g_i h_j) \in D(\Gamma, S)$ for any $X = \sum a_i(\Gamma g_i)$, $Y = \sum b_j(\Gamma h_j) \in D(\Gamma, S)$, $a_i, b_j \in \mathbf{C}$. $D(\Gamma, S)$ is called the Hecke ring of the Hecke pair (Γ, S).

We define a formal double coset $(\Gamma g \Gamma)$, $g \in S$, by $(\Gamma g\Gamma) = \sum_{i=1}^{\mu}(\Gamma g_i)$ when $\Gamma g \Gamma$ is a disjoint union of $\Gamma g_1, \ldots, \Gamma g_\mu$, $g_i \in S$. Since S is contained in $\tilde{\Gamma}$, $g^{-1}\Gamma g \cap \Gamma$ is of finite index in Γ and the index is exactly μ. If $M_1, \dots, M_\mu$ are the left coset representatives of $g^{-1}\Gamma g \cap \Gamma$ in Γ, then $\Gamma g \Gamma$ is a disjoint union of $\Gamma g M_1, \dots, \Gamma g M_\mu$, $gM_i \in S$, so that $(\Gamma g \Gamma) = \sum_{i=1}^{\mu} (\Gamma g M_i)$.

It is known [7] that the formal left cosets $(\Gamma g \Gamma)$, $g \in S$, form a basis for $D(\Gamma, S)$.

Let (Γ, S), (Γ', S') be Hecke pairs satisfying the conditions

$$(1.1) \qquad \Gamma' \subset \Gamma,\ \Gamma S' = S, \text{ and } \Gamma \cap S' S'^{-1} \subset \Gamma'.$$

Then for $X = \sum a_i (\Gamma g_i) \in D(\Gamma, S)$, g_i can be replaced by $g_i' \in S'$ because of the second condition so that X can be written in the form $X = \sum a_i (\Gamma g_i')$. We define a map

$$(1.2) \qquad \varepsilon : D(\Gamma, S) \to D(\Gamma', S') \text{ by } \varepsilon(X) = \sum a_i (\Gamma' g_i') \in D(\Gamma', S').$$

Then ε is an injective ring homomorphism. Moreover, ε is an isomorphism if $[\Gamma : {g'}^{-1} \Gamma g' \cap \Gamma] = [\Gamma' : {g'}^{-1} \Gamma' g' \cap \Gamma']$ for every $g' \in S'$.

Let $\Gamma_0^n = \{M \in \Gamma^n; C_M = 0_n\}$ and $S_0^n = \{g \in S^n : C_g = 0_n\}$. It is well known [7] that (Γ^n, S^n) and (Γ_0^n, S_0^n) are Hecke pairs. Note that S^n, S_0^n are groups. We denote the Hecke rings $D(\Gamma^n, S^n)$ and $D(\Gamma_0^n, S_0^n)$ by $\mathcal{L}^n = \mathcal{L}_p^n$ and $\mathcal{L}_0^n = \mathcal{L}_{0,p}^n$, respectively.

Since (Γ^n, S^n), (Γ_0^n, S_0^n) satisfy the conditions (1.1), we have an injective ring homomorphism

$$(1.3) \qquad \varepsilon_0^n : \mathcal{L}^n \to \mathcal{L}_0^n$$

defined as in (1.2). We set

$$(1.4) \qquad \mathbf{L}_0^n = \mathbf{L}_{0,p}^n = \varepsilon_0^n(\mathcal{L}^n),$$

which is a subring of $\mathcal{L}_0^n$.

We now introduce the Ψ-operator. Let $X = \sum a_i (\Gamma_0^n g_i) \in \mathcal{L}_0^n$. $g_i \in S_0^n$ can be written in the form $g_i = (p^{\delta_i} D_i^*, B_i; 0_n, D_i)$, where $D_i = \begin{pmatrix} D_i' & * \\ 0 & p^{d_i} \end{pmatrix}$ with $D_i' \in M_{n-1}(\mathbf{Z}[\frac{1}{p}])$, $d_i \in \mathbf{Z}$, and $D_i^* = {}^t D_i^{-1}$. We set for $n \geq 1$

$$(1.5) \qquad \Psi_0(X, u) = \sum a_i p^{-n d_i} (\Gamma_0^{n-1} g_i') u^{d_i - \delta_i} \in \mathcal{L}_0^{n-1}[u^{\pm 1}]$$

where $g_i' = (p^{\delta_i}(D_i')^*, B_i'; 0_{n-1}, D_i')$ with B_i' the $(n-1)\times(n-1)$ block in the upper left corner of B_i, and u is an independent variable. We make a convention that $\mathcal{L}_0^0 = \mathbf{C}$ so that $\Psi_0(X,u) = \sum a_i p^{-d_i} u^{d_i-\delta_i} \in \mathbf{C}[u^{\pm 1}]$ when $n=1$. The map $\Psi_0(-,u) : \mathcal{L}_0^n \to \mathcal{L}_0^{n-1}[u^{\pm 1}]$ is a well-defined ring homomorphism.

It is known [8] that $\Psi_0(-,p^{n-k}) : \mathbf{L}_0^n \to \mathbf{L}_0^{n-1}$ is a surjective ring homomorphism. We define

$$\Psi : \mathcal{L}^n \to \mathcal{L}^{n-1} \tag{1.6}$$

to make the following diagram commutes:

$$\begin{array}{ccccccc} X & \in & \mathcal{L}^n & \xrightarrow{\varepsilon_0^n} & \mathbf{L}_0^n \subset \mathcal{L}_0^n & \ni & \varepsilon_0^n(X) \\ & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi_0(-,p^{n-k})} & & \\ \Psi X & \in & \mathcal{L}^{n-1} & \xrightarrow{\varepsilon_0^{n-1}} & \mathbf{L}_0^{n-1} \subset \mathcal{L}_0^{n-1} & \ni & \Psi_0(\varepsilon_0^n(X), p^{n-k}) \end{array} \tag{1.7}$$

Obviously, Ψ is also a well-defined surjective ring homomorphism. We set Ψ^0 =the identity operator and $\Psi^r = \Psi \circ \Psi^{r-1}$ for $1 \le r \le n$.

2. The Canonical Decomposition

In this section, we briefly discuss on the canonical decomposition of Siegel modular forms (see [1]).

Let n, k, q be positive integers. Let $\Gamma^n(q) = \{M \in \Gamma^n; M \equiv I_{2n}(mod q)\}$ and call it the principal congruence subgroup of Γ^n of level q. Let Γ be a congruence subgroup of Γ^n of level q, i.e., $\Gamma^n(q) \subset \Gamma \subset \Gamma^n$. Let χ be a congruence character on Γ, i.e., $\chi : \Gamma \to \mathbf{C}^\times$ is a character satisfying $\chi(\Gamma^n(q)) = 1$. We define $\mathcal{M}_k^n(\Gamma,\chi)$ to be the set of all $F : \mathcal{H}_n \to \mathbf{C}$ satisfying the conditions:

$$F \text{ is holomorphic on } \mathcal{H}_n, \tag{2.1}$$

$$F|_k M = \chi(M)F \text{ for any } M \in \Gamma, \text{ and} \tag{2.2}$$

$$\begin{gathered} \text{when } n=1, (cz+d)^{-k}F(M\langle z\rangle) \text{ is bounded as } \mathrm{Im}(z) \to \infty \\ \text{for any} M = (a,b;c,d) \in \Gamma^1 = SL_2(\mathbf{Z}),\ z \in \mathcal{H}_1. \end{gathered} \tag{2.3}$$

$F \in \mathcal{M}_k^n(\Gamma, \chi)$ is called the Siegel modular form of degree n, weight k, level q, and character χ relative to Γ. $\mathcal{M}_k^n(\Gamma, \chi)$ is a finite dimensional vector space over $\mathbf{C}$ [9]. The boundedness condition (2.3) for $n > 1$ follows from (2.1) and (2.2), which is known as Köcher's effect [10]. For the simplicity, we write $\mathcal{M}_k^n(\Gamma) = \mathcal{M}_k^n(\Gamma, 1)$, where 1 is the trivial character on Γ. In particular, we write $\mathcal{M}_k^n = \mathcal{M}_k^n(\Gamma^n)$ and $\mathcal{M}_k^n(q) = \mathcal{M}_k^n(\Gamma^n(q))$.

Note that if Γ' is a congruence subgroup of Γ^n of level q contained in Γ and if $\chi' : \Gamma' \to \mathbf{C}^\times$ is the restriction of χ to Γ', then $\mathcal{M}_k^n(\Gamma, \chi) \subset \mathcal{M}_k^n(\Gamma', \chi')$. In particular, $\mathcal{M}_k^n(\Gamma, \chi) \subset \mathcal{M}_k^n(q)$.

It is known [4] that every $F \in \mathcal{M}_k^n(q)$, hence every $F \in \mathcal{M}_k^n$ has a Fourier series expansion of the form

$$F(Z) = \sum_{N \in \mathcal{N}_n} f(N) e(NZ/q), \quad Z \in \mathcal{H}_n. \tag{2.4}$$

We now define the Siegel operator Φ. Let $F : \mathcal{H}_n \to \mathbf{C}$ be an arbitrary function with a Fourier series expansion (2.4). We define

$$(\Phi F)(Z') = \lim_{\lambda \to +\infty} F\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}, \quad Z' \in \mathcal{H}_{n-1}. \tag{2.5}$$

The limit exists and

$$(\Phi F)(Z') = \sum_{N' \in \mathcal{N}_{n-1}} f\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e(N'Z'/q), \; Z' \in \mathcal{H}_{n-1}. \tag{2.6}$$

We set $\Phi^0 =$ the identity operator and $\Phi^r = \Phi \circ \Phi^{r-1}$ for $1 \le r \le n$.

We now consider the image of $\mathcal{M}_k^n(\Gamma, \chi)$ under Φ^r, $r = 0, 1, \ldots, n$ (see [11]).

Let G_r^n be the r-th Satake group [12] for $0 \le r \le n$, i.e.,

(2.7)

$$G_r^n = \Big\{ M \in \Gamma^n ; A_M = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}, B_M = \begin{pmatrix} B_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix}, \\ C_M = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, D_M = \begin{pmatrix} D_1 & D_{12} \\ 0 & D_2 \end{pmatrix} \\ \text{where } A_1, B_1, C_1, D_1 \in M_r(\mathbf{Z}) \Big\}.$$

Then the map

(2.8) $w_r^n : G_r^n \to \Gamma^r$ defined by $w_r^n(M) = M_1 = (A_1, B_1; C_1, D_1) \in \Gamma^r$

is a well-defined surjective group homomorphism. We set $\Gamma_r = w_r^n(\Gamma \cap G_r^n)$. Note that $(\Gamma^n)_r = \Gamma^r$ and $(\Gamma^n(q))_r = \Gamma^r(q)$. Therefore, Γ_r is again a congruence subgroup of Γ^r of level q.

Let $F : \mathcal{H}_n \to \mathbf{C}$ and $M \in \Gamma \cap G_r^n$ is of the form (2.7). It follows immediately from (0.1), (2.7), and (2.8) that

$$\Phi^{n-r}(F|_k M) = (\det D_2)^{-k}(\Phi^{n-r}F)|_k M_1. \tag{2.9}$$

So for $F \in \mathcal{M}_k^n(\Gamma, \chi)$ and $M_1 \in \Gamma^r$, we have

$$(\Phi^{n-r}F)|_k M_1 = (\det D_2)^k \chi(M) \Phi^{n-r} F \tag{2.10}$$

where $M \in \Gamma \cap G_r^n$ is of the form (2.7) such that $w_r^n(M) = M_1$. If we set $\chi_r(M_1) = (\det D_2)^k \chi(M)$, then since χ_r is independent of the choice of $M \in \Gamma \cap G_r^n$ with $w_r^n(M) = M_1$, $\chi_r : \Gamma_r \to \mathbf{C}^\times$ is a well-defined character such that $\chi_r(\Gamma^r(q)) = 1$. Therefore, from (2.10) we have

$$\Phi^{n-r}F \in \mathcal{M}_k^r(\Gamma_r, \chi_r) \text{ if } F \in \mathcal{M}_k^n(\Gamma, \chi). \tag{2.11}$$

For the case $r = 0$, we make the convention that $\mathcal{M}_k^0(\Gamma, \chi) = \mathbf{C}$, $\Gamma^0 = \Gamma^0(q) = \{1\}$, and $\mathcal{H}_0$ is a single point set.

For $M \in \Gamma^n$, let $\Gamma^M = M^{-1}\Gamma M$, which is again a congruence subgroup of Γ^n of level q, and let χ^M be the character : $\Gamma^M \to \mathbf{C}^\times$ defined by $\chi(\hat{M}) = \chi(M\hat{M}M^{-1})$ for $\hat{M} \in \Gamma^M$. Obviously $\chi^M(\Gamma^n(q)) = 1$ and

$$F|_k M \in \mathcal{M}_k^n(\Gamma^M, \chi^M) \text{ if } F \in \mathcal{M}_k^n(\Gamma, \chi). \tag{2.12}$$

Combining (2.11) and (2.12), we have for $M \in \Gamma^n$

$$\Phi^{n-r}(F|_k M) \in \mathcal{M}_k^r(\Gamma_r^M, \chi_r^M) \text{ if } F \in \mathcal{M}_k^n(\Gamma, \chi) \tag{2.13}$$

where $\Gamma_r^M = (\Gamma^M)_r$ and $\chi_r^M = (\chi^M)_r$. Note that if $\Gamma = \Gamma^n(q)$, then

$$\mathcal{M}_k^r(\Gamma_r^M, \chi_r^M) = \mathcal{M}_k^r(\Gamma_r, \chi_r) = \mathcal{M}_k^r(q), \tag{2.14}$$

and similary if $\Gamma = \Gamma^n$, then

$$\mathcal{M}_k^r(\Gamma_r^M, \chi_r^M) = \mathcal{M}_k^r(\Gamma_r, \chi_r) = \mathcal{M}_k^r \tag{2.15}$$

while χ is trivial in both (2.14) and (2.15).

Let $F \in \mathcal{M}_k^n(\Gamma, \chi)$. F is called a cusp form if $\Phi(F|_k M) = 0$ for all $M \in \Gamma^n$. For $F, G \in \mathcal{M}_k^n(\Gamma, \chi)$, at least one of which is a cusp form, we set

$$(F, G)_0 = \int_{D(\Gamma)} F(Z)\overline{G(Z)}(\det Y)^k \widetilde{dZ} \tag{2.16}$$

where $D(\Gamma)$ is a fundamental domain of Γ in $\mathcal{H}_n$, $Z \in X + iY \in \mathcal{H}_n$, and $\widetilde{dZ} = (\det Y)^{-n-1} dX dY$ is the $\mathcal{G}_n$-invariant volume element on $\mathcal{H}_n$. (2.16) is a well-defined positive definite Hermitian inner product [12] and is called the Maass-Petersson inner product on $\mathcal{M}_k^n(\Gamma, \chi)$. Unfortunately, however, (2.16) is meaningless if neither of F, G is a cusp form.

We now introduce a positive definite Hermitian inner product which is meaningful on the whole space $\mathcal{M}_k^n(\Gamma, \chi)$. Let $\mathcal{S}_k^n(\Gamma, \chi)$ be the subspace of $\mathcal{M}_k^n(\Gamma, \chi)$ consisting of all the cusp forms. Every $F \in \mathcal{M}_k^n(\Gamma, \chi)$ can be written in the form $F = F^* + F_n$ where $F_n \in \mathcal{S}_k^n(\Gamma, \chi)$ and F^* is contained in the orthogonal complement of $\mathcal{S}_k^n(\Gamma, \chi)$ in $\mathcal{M}_k^n(\Gamma, \chi)$ with respect to the Maass-Petersson inner product, which we denote by $\mathcal{E}_k^n(\Gamma, \chi)$. We call F_n the cusp part of F. We set

$$(F, G) = \sum_{r=0}^{n} \sum_{M \in \Gamma \backslash \Gamma^n} [\Gamma^r : \Gamma_r^M]^{-1} \Big((\Phi_M^{n-r} F)_r, (\Phi_M^{n-r} G)_r \Big)_0 \tag{2.17}$$

where $\Phi_M^{n-r} F = \Phi^{n-r}(F|_k M)$, $(\Phi_M^{n-r} F)_r$ is the cusp part of $\Phi_M^{n-r} F$, and $(-,-)_0$ is the Maass-Petersson inner product on $\mathcal{M}_k^r(\Gamma_r^M, \chi_r^M)$.

We have the following theorem.

THEOREM 2.1. *The pairing (2.17) is a well-defined positive definite Hermitian inner product on the whole space $\mathcal{M}_k^n(\Gamma, \chi)$, which is called the canonical inner product on the space.*

Proof. See [1].

We set

$$\mathcal{M}_k^{n,n}(\Gamma,\chi) = \mathcal{S}_k^n(\Gamma,\chi) \tag{2.18}$$

and for $0 \le r < n$

$$\begin{aligned}\mathcal{M}_k^{n,r}(\Gamma,\chi) = \Big\{F \in \mathcal{M}_k^n(\Gamma,\chi);\ &\Phi_M^{n-r}F \text{ is a cusp form in} \\ &\mathcal{M}_k^r(\Gamma_r^M,\chi_r^M) \text{ for any } M \in \Gamma^n \text{ such that} \\ &\Big(F, \perp_{s=r+1}^n \mathcal{M}_k^{n,s}(\Gamma,\chi)\Big) = 0\Big\}.\end{aligned} \tag{2.19}$$

When $n = 1$, $\mathcal{M}_k^{1,0}(\Gamma,\chi)$ and $\mathcal{M}_k^{1,1}(\Gamma,\chi)$ coincide with the subspaces of classical Eisenstein series [13, 14] and cusp forms [15], respectively, which are orthogonal to each other with respect to the canonical inner product as well as the Maass-Petersson inner product. This can be extended to aribitrary n as the following:

THEOREM 2.2. *The space $\mathcal{M}_k^n(\Gamma,\chi)$ is decomposed into $n+1$ subspaces $\mathcal{M}_k^{n,r}(\Gamma,\chi)$, $0 \le r \le n$, which are pairwise orthogonal with respect to the canonical inner product.*

Proof. See [1].

We write

$$\mathcal{M}_k^n(\Gamma,\chi) = \perp_{r=0}^n \mathcal{M}_k^{n,r}(\Gamma,\chi) \tag{2.20}$$

and call it the canonical decomposition of $\mathcal{M}_k^n(\Gamma,\chi)$. The subspace $\mathcal{M}_k^{n,r}(\Gamma,\chi)$ is called the r-th canonical subspace of $\mathcal{M}_k^n(\Gamma,\chi)$ for each $r = 0, 1, \ldots, n$. Note that every $F \in \mathcal{M}_k^n(\Gamma,\chi)$ can be written in the form

$$F = F_0 + F_1 + \cdots + F_n \tag{2.21}$$

where $F_r \in \mathcal{M}_k^{n,r}(\Gamma,\chi)$. We call F_r the r-th canonical part of F. In particular, F_0 is called the Eisenstein series part and F_n the cusp part of F. Note that $F^* = F_0 + F_1 + \cdots + F_{n-1}$.

We close this section with the following theorem.

THEOREM 2.3. *For* $F, G \in \mathcal{M}_k^n(\Gamma, \chi)$, *at least one of which is a cusp form, the canonical inner product coincide with the Maass-Petersson inner product.*

Proof. See [1].

3. Main Theorems

In this section, we prove our main results:

(1) For each $0 \leq r \leq n$, $\mathcal{M}_k^{n,r}$ is invariant under the Hecke operators in $\mathcal{L}^n$ (Theorem 3.5.),
(2) Hecke operators in $\mathcal{L}^n$ are Hermitian with respect to the canonical inner product on $\mathcal{M}_k^n$ (Theorem 3.6), and as a consquence
(3) $\mathcal{M}_k^n$ has a simultaneous eigen-basis with respect to all Hecke operators in $\mathcal{L}^n$ (Theorem 3.7.).

Let $F \in \mathcal{M}_k^n$ and $X = \sum a_i(\Gamma^n g_i) \in \mathcal{L}^n$, $g_i \in S^n$. We set

$$F|_k X = \sum a_i F|_k g_i. \tag{3.1}$$

The action (3.1) is independent of the choice of the left coset representatives g_i, and hence is well-defined. We call $X \in \mathcal{L}^n$ acting on $\mathcal{M}_k^n$ as in (3.1) a Hecke operator.

Note that $F|_k X|_k Y = F|_k X \cdot Y$ for any $X, Y \in \mathcal{L}^n$. It follows easily from the definition of $\mathcal{M}_k^n$ that $F|_k X \in \mathcal{M}_k^n$.

Let $t = 0$ or 1. We define $\mathcal{M}^n(t)$ to be the set of all $F : \mathcal{H}_n \to \mathbf{C}$ satisfying the conditions:

$$F \text{ is holomorphic on } \mathcal{H}_n, \tag{3.2}$$

$$F(M\langle Z\rangle) = (\det D_M)^t F(Z), \ Z \in \mathcal{H}_n, \text{ for any } M \in \Gamma_0^n, \text{ and} \tag{3.3}$$

$$\text{when } n = 1,\ F(z) \text{ is bounded as } \operatorname{Im} z \to \infty,\ z \in \mathcal{H}_1. \tag{3.4}$$

$F \in \mathcal{M}^n(t)$ is called an even or odd Siegel modular form when $t = 0$ or 1, respectively.

PROPOSITON 3.1. *Let* Γ *be a congruence subgroup of* Γ^n *of level* q *containing* Γ_0^n *and let* $\chi : \Gamma \to \mathbf{C}^\times$ *be a character such that* $\chi(\Gamma^n(q)) =$

1 and $\chi(M) = (\det D_M)^{t-k}$ for any $M \in \Gamma_0^n$. Then $\mathcal{M}_k^n(\Gamma,\chi) \subset \mathcal{M}^n(t)$. In particular, $\mathcal{M}_k^n \subset \mathcal{M}^n(t)$ if k and t are of the same parity.

Proof. Let $F \in \mathcal{M}_k^n(\Gamma,\chi)$, $M = (A,B;0_n,D) \in \Gamma_0^n$. $(F|_kM)(Z) = (\det D)^{-k}F(M\langle Z\rangle) = \chi(M)F(Z)$. So $F(M\langle Z\rangle) = (\det D)^k\chi(M)F(Z) = (\det D)^tF(Z)$. Therefore, $F \in \mathcal{M}^n(t)$. The second assertion follows immediately because $\det D_M = \pm 1$ for any $M \in \Gamma_0^n$.

Let k and t have the same parity where $t = 0$ or 1. For $F \in \mathcal{M}^n(t)$ and $X = \sum a_i(\Gamma_0^n g_i) \in \mathcal{L}_0^n$, $g_i \in S_0^n$, we set

$$F|_kX = \sum a_iF|_kg_i. \tag{3.5}$$

The action (3.5) is well-defined and $F|_kX|_kY = F|_kX \cdot Y$ for any $X, Y \in \mathcal{L}_0^n$. Since $\mathcal{M}_k^n \subset \mathcal{M}^n(t)$, it follows easily from (3.1) and (3.5) that

$$F|_kX = F|_k\varepsilon_0^n(X) \tag{3.6}$$

where ε_0^n is the injective ring homomorphism (1.3).

Proposition 3.2. *Let $F \in \mathcal{M}_k^n$ and $X \in \mathcal{L}^n$. Then $\Phi(F|_kX) = \Phi F|_k\Psi X$. More generally, $\Phi^r(F|_kX) = \Phi^rF|_k\Psi^rX$ for $0 \le r \le n$, where Φ and Ψ are the operators in (2.5) and (1.6), respectively.*

Proof. Let $Y = \varepsilon_0^n(X) \in \mathbf{L}_0^n$. Regarding $F \in \mathcal{M}^n(t)$ where k and t and of the same parity with $t = 0$ or 1, we have [7] $\Phi(F|_kY) = \Phi F|_k\Psi_0(Y,p^{n-k})$ (see (1.5) and (1.7)). Since $\varepsilon_0^{n-1}(\Psi X) = \Psi_0(Y,p^{n-k})$ by (1.7) and $\Phi F \in \mathcal{M}_k^{n-1}$ by (2.11) and (2.15), it follows from (3.6) that $\Phi(F|_kX) = \Phi F|_k\Psi X$. The second assertion follows immediately.

We now give the following new discription of $\mathcal{M}_k^{n,r}(\Gamma,\chi)$.

Proposition 3.3. *For $0 \le r < n$,*

$$\mathcal{M}_k^{n,r}(\Gamma,\chi) = \Big\{F \in \mathcal{M}_k^n(\Gamma,\chi); \Phi_MF \in \perp_{s=r}^{n-1}\mathcal{M}_k^{n-1,s}(\Gamma_{n-1}^M,\chi_{n-1}^M)$$
$$\text{such that } \Big(F, \perp_{s=r+1}^{n}\mathcal{M}_k^{n,s}(\Gamma,\chi)\Big) = 0\Big\}.$$

Proof. Let the sets on the left and right be A and B, respectively (see 2.19). It is a well known result for $n = 1$ [13, 14, 15]. Assume $n > 1$.

Let $F \in B$. Since $\Phi_M F \in \perp_{s=r}^{n-1} \mathcal{M}_k^{n-1,s}(\Gamma_{n-1}^M, \chi_{n-1}^M)$, $\Phi_{L'}^{(n-1)-r}(\Phi_M F)$ is a cusp form for any $L' \in \Gamma^{n-1}$ by (2.18) and (2.19). But $\Phi_M F|_k L' = \Phi(F|_k M)|_k L' = d^k \Phi(F|_k ML)$ for some $L \in G_{n-1}^n$ such that $w_{n-1}^n(L) = L'$, where $D_L = \begin{pmatrix} D_{L'} & * \\ 0 & d \end{pmatrix}$ (see (2.7), (2.8), and (2.9)). So $\Phi_{ML}^{n-r} F = d^{-k}\Phi_{L'}^{(n-1)-r}(\Phi_M F)$ turns out to be a cusp form, which implies $F \in A$ by the definition (2.19).

Let $F \in A$ and $\Phi_M F = (\Phi_M F)_* + (\Phi_M F)_{**}$, where $(\Phi_M F)_* \in \perp_{s=0}^{r-1} \mathcal{M}_k^{n-1,s}(\Gamma_{n-1}^M, \chi_{n-1}^M)$ and $(\Phi_M F)_{**} \in \perp_{s=r}^{n-1} \mathcal{M}_k^{n-1,s}(\Gamma_{n-1}^M, \chi_{n-1}^M)$. Then $\Phi_{L'}^{(n-1)-r}(\Phi_M F)_* = \Phi_{L'}^{(n-1)-r}(\Phi_M F - (\Phi_M F)_{**}) = d^k \Phi_{ML}^{n-r} F - \Phi_{L'}^{(n-1)-r}(\Phi_M F)_{**}$ is a cusp form for any $L' \in \Gamma^{n-1}$ because $F \in A$. From the obvious fact that $(\Phi_M F)_*$ is orthogonal to $\perp_{s=r}^{n-1} \mathcal{M}_k^{n-1,s}(\Gamma_{n-1}^M, \chi_{n-1}^M)$, we have $(\Phi_M F)_* = 0$. Therefore $F \in B$ and the proposition follows.

Let $F, G \in \mathcal{M}_k^n(\Gamma, \chi)$, at least one of which is a cusp form. Then from the definition (2.16) of the Maass-Petersson inner product follows that

$$(F, G)_0 = (F|_k g,\ G|_k g)_0 \tag{3.7}$$

for any $g \in Sp_n(\mathbf{R})$ where the Maass-Petersson inner product on the right is on $\mathcal{M}_k^n(\Gamma^g, \chi^g)$, which can be defined similary to (2.12).

PROPOSITION 3.4. *Let $F, G \in \mathcal{M}_k^n$, at least one of which is a cusp form, and let $X = (\Gamma^n g \Gamma^n) \in \mathcal{L}^n$, $g \in S^n$. Then $(F|_k X, G)_0 = (F, G|_k X)_0$.*

Proof. Recall that X can be written in the form $X = \sum_{i=1}^{\mu}(\Gamma^n g M_i)$, where $M_1, \ldots, M_\mu \in \Gamma^n$ are the left coset representatives of $g^{-1}\Gamma^n g \cap \Gamma^n$ in Γ^n. From (2.16) follows

$$\begin{aligned}(F|_k X, G)_0 &= \sum_{i=1}^{\mu}(F|_k g M_i, G)_0 \\ &= \sum_{i=1}^{\mu}(F|_k g, G|_k M_i^{-1})_0 = \sum_{i=1}^{\mu}(F|_k g, G)_0\end{aligned}$$

because $G \in \mathcal{M}_k^n$ and $M_i^{-1} \in \Gamma^n$, where $(-,-)_0$ are the Maass-Petersson inner products on proper spaces. Let $\delta = \delta(g)$, i.e., $J_n[g] = p^\delta J_n$. Since $\det g = p^{\delta n}$, we have $p^{\delta/2} g^{-1} \in Sp_n(\mathbf{R})$. Therefore, $(F|_k g, G)_0 = (F|_k g \cdot p^{\delta/2} g^{-1}, G|_k p^{\delta/2} g^{-1})_0 = (F, G|_k p^\delta g^{-1})_0$ where the last equality follows from (0.1). So $(F|_k X, G)_0 = \sum_{i=1}^{\mu} (F, G|_k p^\delta g^{-1})_0$. But g and $p^\delta g^{-1}$ have the same diagonalization under the left and right multiplication of Γ^n so that $\Gamma^n g \Gamma^n = \Gamma^n p^\delta g^{-1} \Gamma^n$. Therefore, $X = (\Gamma^n p^\delta g^{-1} \Gamma^n) = \sum_{i=1}^{\mu} (\Gamma^n p^\delta g^{-1} L_i)$ for some $L_i \in \Gamma^n$ so that $(F|_k X, G)_0 = \sum_{i=1}^{\mu} (F|_k L_i, G|_k p^\delta g^{-1} L_i)_0 = (F, G|_k X)_0$.

We now prove our main theorems.

THEOREM 3.5. *Let $F \in \mathcal{M}_k^{n,r}$, $0 \le r \le n$ and $X \in \mathcal{L}^n$. Then $F|_k X \in \mathcal{M}_k^{n,r}$.*

Proof. We may assume that $X = (\Gamma^n g \Gamma^n)$, $g \in S^n$, because such double cosets form a basis for $\mathcal{L}^n$. We use induction on n and $u = n - r$. When $n = 1$, it is a well known result [13, 14, 15]. Assume $n > 1$. Let $u = 0$. Since F is a cusp form in this case, we have $\Phi F = 0$ and hence $\Phi(F|_k X) = \Phi F|_k \Psi X = 0$, which means $F|_k X$ is also a cusp form. This proves the theorem for $r = n$. We now assume $u > 0$, i.e., $0 \le r < n$. From Proposition 3.2 follows that $\Phi^{n-r}(F|_k X) = \Phi^{n-r} F|_k \Psi^{n-r} X$ is a cusp form since $\Phi^{n-r} F$ is. Let $G \in \perp_{s=r+1}^{n} \mathcal{M}_k^{n,s}$. From the definition (2.17) of the canonical inner product, we have $(F|_k X, G) = \sum_{s=0}^{n} \Big((\Phi^{n-s}(F|_k X))_s, (\Phi^{n-s} G)_s \Big)_0$. For each s,

$$\begin{aligned}
\Big((\Phi^{n-s}(F|_k X))_s, (\Phi^{n-s} G)_s \Big)_0 &= \Big(\Phi^{n-s}(F|_k X), (\Phi^{n-s} G)_s \Big)_0 \\
&= \Big(\Phi^{n-s} F|_k \Psi^{n-s} X, (\Phi^{n-s} G)_s \Big)_0 \\
&= \Big(\Phi^{n-s} F, (\Phi^{n-s} G)_s|_k \Psi^{n-s} X \Big)_0
\end{aligned}$$

where the last two equalities follow from Propositions 3.2 and 3.4, respectively. From Propositions 3.2, 3.3, and the induction hypothesis, we have $(\Phi^{n-s} G)_s|_k \Psi^{n-s} X = (\Phi^{n-s} G|_k \Psi^{n-s} X)_s = (\Phi^{n-s}(G|_k X))_s$. Thus $(F|_k X, G) = \sum_{s=0}^{n} \Big(\Phi^{n-s} F, (\Phi^{n-s}(G|_k X))_s \Big)_0 = \sum_{s=0}^{n} \Big((\Phi^{n-s} F)_s, (\Phi^{n-s}(G|_k X))_s \Big)_0 = (F, G|_k X)$ by (2.17). Again by the induction hypothesis, we have $(F, G|_k X) = 0$. From (2.19) follows the theorem.

THEOREM 3.6. *Let* $F, G \in \mathcal{M}_k^n$ *and* $X \in \mathcal{L}^n$. *Then* $(F|_k X, G) = (F, G|_k X)$. *In other words, Hecke operators* $X \in \mathcal{L}^n$ *are Hermitian with respect to the canonical inner product on* $\mathcal{M}_k^n$.

Proof. Again we may assume that $X = (\Gamma^n g \Gamma^n)$, $g \in S^n$. From (2.17), we have $(F|_k X, G) = \sum_{s=0}^n \left((\Phi^{n-s}(F|_k X)\right)_s, (\Phi^{n-s}G)_s)_0$. We have $\left((\Phi^{n-s}(F|_k X))_s, (\Phi^{n-s}G)_s\right)_0 = \left(\Phi^{n-s}F, (\Phi^{n-s}G)_s|_k \Psi^{n-s}X\right)_0$ for each s by the same reasoning as in Theorem 3.5. But here, the next step $(\Phi^{n-s}G)_s|_k \Psi^{n-s}X = (\Phi^{n-s}G|_k \Psi^{n-s}X)_s = (\Phi^{n-s}(G|_k X))_s$ follows from Theorem 3.5 and Proposition 3.2. Therefore, we have $(F|_k X, G) = (F, G|_k X)$ as asserted.

We now obtain the following theorem as an easy corollary of Theorems 3.5 and 3.6.

THEOREM 3.7. *For each* r, $0 \leq r \leq n$, $\mathcal{M}_k^{n,r}$ *has a simultaneous eigen-basis with respect to all the Hecke operators* $X \in \mathcal{L}^n$, *and hence so does* $\mathcal{M}_k^n$.

Proof. See [5, Theorem 4.2.20].

Above results on $\mathcal{M}_k^n$ are proved by Maass [2] and Andrianov [3] in a different manner: They decomposed $\mathcal{M}_k^n$ into $n+1$ subspaces $M_k^{n,r}$, $0 \leq r \leq n$, which were defined inductively by using Φ-operator, so that $\mathcal{M}_k^n = \oplus_{r=0}^n M_k^{n,r}$. More precisely, $M_k^{n,n} = \{F \in \mathcal{M}_k^n; \Phi F = 0\}$ and $M_k^{n,r} = \{F \in \mathcal{M}_k^n; \Phi F \in M_k^{n-1,r}$ such that $(F, M_k^{n,n})_0 = 0\}$ for $0 \leq r < n$. Maass proved that Hecke operators $X \in \mathcal{L}^n$ are Hermitian and Andrianov proved that for any $0 \leq r \leq n$, $M_k^{n,r}$ is invariant under the Hecke operators and that $M_k^{n,r}$, hence $\mathcal{M}_k^n$ has a simultaneous eigen-basis with respect to them. The Maass-Petersson inner product, however, is meaningless on $\oplus_{r=0}^{n-1} M_k^{n,r}$ so that the orthogonality between subspaces $M_k^{n,r}$ cannot be determined except that $M_k^{n,r} \perp_0 M_k^{n,n}$ for $0 \leq r < n$, where $\perp_0$ is the orthogonality with respect to the Maass-Petersson inner product.

The following quesiton arises naturally: $M_k^{n,r} = \mathcal{M}_k^{n,r}$? At the present time, we know:

(1) $M_k^{n,n} = \mathcal{M}_k^{n,n}$ and $\oplus_{r=0}^{n-1} M_k^{n,r} = \mathcal{E}_k^n = \perp_{r=0}^{n-1} \mathcal{M}_k^{n,r}$ (see Theorem 2.3),

(2) $M_k^{n,r} = \mathcal{M}_k^{n,r}$ for any $0 \leq r \leq n$ if Φ is surjective. (Φ is surjective, for example, when k is even $> 2n$ [4].)

Evdokimov [11] erroneously claimed the proof of the above theorems for a general space $\mathcal{M}_k^n(\Gamma, \chi)$. His mistake was on the definition of the canonical inner product (2.17), where he used $(\Phi^{n-r}F_r,\ \Phi^{n-r}G_r)_0$, as he noticed later. We adopt, however, many of his ideas in this article.

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