FIXED POINT THEORY OF MULTIFUNCTIONS
IN TOPOLOGICAL VECTOR SPACES

SEHIE PARK

1. Introduction

In [47], S.Simons gave an existence theorem for certain families of quasiconcave functions on a compact convex set and its applications to locally convex spaces. On the other hand, in [17], Ky Fan obtained a similar (but not comparable) result for a paracompact convex set and its applications. A common generalization of those results was due to J.C.Bellenger [4]. Recently, the author and J.S.Bae [39] removed the paracompactness assumption in the existence results of Fan and Bellenger.

In the present paper, we apply our existence theorem to obtain new coincidence, fixed point, and surjectivity theorems, and existence theorems on critical points for a larger class of multifunctions than upper hemicontinuous ones defined on convex sets. Moreover, by using J.Jiang's recent method in [25], we also deal with a larger class of multifunctions having more general boundary conditions than weakly inward (outward) ones.

In Section 3, Theorem 1 is our version of Bellenger's theorem on the existence of maximizable quasiconcave real functions on convex spaces, and Theorems 2 and 3 are direct consequences of Theorem 1, and which seem to be most convenient for the applications in the rest of the present paper.

In Section 4 we show that Theorem 3 can be used to obtain generalizations of various results on coincidence, fixed point, surjectivity, and existence of critical points due to Fan, Simons, Jiang, Shih and Tan,

Cellina, Cornet, and many others. Note that these recent results generalize well-known classical theorems due to Brouwer, Schauder, Tychonoff, Rothe, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Browder, Halpern, Halpern and Bergman, Reich, Kaczynski, Granas and Liu, Park, and many others. For details, see [32], [38]. Our new results generalize such Brouwer or Kakutani type fixed point theorems to a class of multifunctions more general than weakly inward (outward) upper hemicontinuous ones defined on a convex subset of a topological vector space having sufficiently many linear functionals.

2. Preliminaries

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$ [30]. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$.

Let E be a real Hausdorff topological vector space (t.v.s.) and E^* its topological dual. A multifunction $F : X \to 2^E \backslash \{\emptyset\}$ is said to be *upper hemicontinuous* (u.h.c.) or a *Cornet-Lasry-Robert* (CLR) function if for each $f \in E^*$ and for any real α, the set $\{x \in X : \sup f(Fx) < \alpha\}$ is open in X. The concept goes back to Cornet [11], [12] and Lasry and Robert [29]. Note that an upper semicontinuous (u.s.c.) function is upper demicontinuous (u.d.c.), and that an u.d.c. function is u.h.c. [46].

Let $cc(E)$ denote the set of nonempty closed convex subsets of E and $kc(E)$ the set of nonempty compact convex subsets of E. Bd, Int, and — will denote the boundary, interior, and closure, resp., with respect to E.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

A function $F : X \to 2^E$ is said to be *weakly inward* [*outward*, resp.] if $Fx \cap \overline{I}_X(x) \neq \emptyset$ [$Fx \cap \overline{O}_X(x) \neq \emptyset$, resp.] for each $x \in \mathrm{Bd}\, X \backslash Fx$.

For $f \in E^*$ and $U, V \subset E$, let

$$d_f(U, V) := \inf\{|f(u - v)| : u \in U, \ v \in V\}.$$

Recall that a real-valued function $f : X \to \mathbf{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is open for each $r \in \mathbf{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is convex for each $r \in \mathbf{R}$.

In this paper all topological spaces are assumed to be Hausdorff.

3. The existence theorem and its direct consequences

We begin with the following form of Bellenger's existence theorem [4, Theorem 1] of maximizable quasiconcave functions on convex spaces, due to the author and Bae [39].

THEOREM 1. *Let X be a convex space. Suppose that*

(1.1) *for each $x \in X$, Sx is a convex set of u.s.c. quasiconcave real functions on X (that is, every convex combination of any two functions in Sx is in Sx);*

(1.2) *for each $x \in X$, Tx is a nonempty subset of Sx;*

(1.3) *for each u.s.c. quasiconcave real function g on X, $T^{-1}g$ is compactly open in X; and*

(1.4) *there exist a c-compact set $L \subset X$ and a nonempty compact set $K \subset X$ such that for every $x \in X \backslash K$ and $g \in Sx$, $gx < \max g[x, L]$.*

Then there exist an $\overline{x} \in K$ and $g \in \mathrm{co}\,T\overline{x} \subset S\overline{x}$ such that $g\overline{x} = \max g(X)$.

Let $\hat{X}$ denote the set of all u.s.c. quasiconcave real functions on X and $M : \hat{X} \to 2^X$ a multifunction defined by $Mg = \{x \in X : gx = \max g(X)\}$ for $g \in \hat{X}$. Then Theorem 1 can be restated as follows:

THEOREM 1'. *If X is a convex space and if $S = T : X \to 2^{\hat{X}}$ satisfies (1.1)–(1.4) of Theorem 1, then the composite multifunction $MS : X \to 2^X$ has a fixed point $\overline{x} \in K$.*

REMARKS. 1. Here, Mg may be empty. From now on, we denote

$$M_g := Mg \cap K = \{x \in K : gx = \max g(X)\}.$$

2. Note that Theorem 1 includes Simons [**47**, Theorem 0.1], [**48**, Theorem 1], Fan [**16**, Theorem 8], and Bellenger [**4**, Theorem 1].

The following is a direct consequence of Theorem 1:

THEOREM 2. *Let X, $\hat{X}$, L, and K be as in Theorem 1. Let Y be a nonempty convex subset of $\hat{X}$, and $\alpha, \beta : X \times Y \to \overline{\mathbf{R}} = [-\infty, +\infty]$ functions such that $\{g \in Y : \alpha(x,g) > \beta(x,g)\}$ is convex for each $x \in X$. Suppose that, for each $g \in Y$,*

(2.1) $X_g = \{x \in X : \alpha(x,g) \le \beta(x,g)\}$ *is compactly closed;*
(2.2) $X_g \supset M_g$; *and*
(2.3) *for each $x \in X \backslash K$, $gx = \max g[x, L]$ implies $x \in X_g$.*

Then there exists an $x \in X_g$ for all $g \in Y$.

Proof. Define $T : X \to 2^{\hat{X}}$ by

$$Tx := \{g \in Y : \alpha(x,g) > \beta(x,g)\}$$

for $x \in X$. Then Tx is convex. Suppose that $Tx \ne \emptyset$ for each $x \in X$. Then

$$T^{-1}g = \begin{cases} \emptyset & \text{if } g \in \hat{X} \backslash Y \\ \{x \in X : \alpha(x,g) > \beta(x,g)\} & \text{if } g \in Y \end{cases}$$

is open in X. Therefore, (1.1) and (1.2) are satisfied. Further, (2.3) implies (1.3). Hence, by Theorem 1, there exist $x \in M_g$ and $g \in Tx$. This contradicts (2.2). This completes our proof.

REMARKS. 1. If α is concave and β is convex in their second variables, then $\{g \in Y : \alpha(x,g) > \beta(x,g)\}$ is convex for each $x \in X$. If α is l.s.c. and β is u.s.c. in their first variables, then (2.1) holds automatically. Therefore, for $X = K$, Theorem 2 reduces to Simons [**47**, Theorem 2.1].

2. Because of (2.3), the condition (2.2) is actually equivalent to $X_g \supset Mg$.

Motivated by a recent result of Bellenger and Simons [**5**, Corollary 3], we obtain the following:

THEOREM 3. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. with topological dual E^*, $B : E^* \to 2^{\hat{X}} \setminus \{\emptyset\}$ a multifunction with convex graph, and $P, Q : X \to 2^E \setminus \{\emptyset\}$. Suppose that for each $f \in E^*$,*

(3.1) *$X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ is compactly closed;*

(3.2) *$MBf \cap K \subset X_f$; and*

(3.3) *for each $x \in X \setminus K$ and $g \in Bf$, $gx = \max g[x, L]$ implies $x \in X_f$.*

Then there exists an $x \in \bigcap \{X_f : f \in E^*\}$.

Proof. We use Theorem 1'. For each $x \in X$, define

$$Sx = B(\{f \in E^* : \sup f(Px) < \inf f(Qx)\}).$$

Since $\{f \in E^* : \sup f(Px) < \inf f(Qx)\}$ is convex and B has convex graph, $S : X \to 2^{\hat{X}}$ is a well-defined convex-valued multifunction. For each $g \in \hat{X}$, by (3.1),

$$S^{-1}g = \bigcup \{X \setminus X_f : f \in B^{-1}g\}$$

is compactly open. Moreover, by (3.3), for each $x \in X \setminus K$ and $g \in Sx$, we have $gx < \max g[x, L]$. Suppose that, for each $x \in X$, $\{f \in E^* : \sup f(Px) < \inf f(Qx)\} \neq \emptyset$. Then $Sx \neq \emptyset$, and hence, S satisfies all of the requirements of Theorem 1'. Therefore,

$$\exists \overline{x} \in K \quad \text{such that} \quad \overline{x} \in MS\overline{x},$$

that is,

$$\exists f \in E^* \quad \text{such that} \quad \overline{x} \in MBf \cap K \quad \text{and} \quad \overline{x} \in X \setminus X_f.$$

This contradicts (3.2). Therefore, there exists an $x \in X$ such that $\{f \in E^* : \sup f(Px) < \inf f(Qx)\} = \emptyset$, that is, $x \in X_f$ for all $f \in E^*$. This completes our proof.

Theorem 3 does not require any explicit connection between X and E. However, an important special case is the following:

COROLLARY 3.1. *Let X, L, and K be as in Theorem 3, E a t.v.s. containing X such that the topology of X is finer than its relative topology w.r.t. E, and $P, Q : X \to 2^E \backslash \{\emptyset\}$. Suppose that, for each $f \in E^*$,*

    (1) *$X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ is compactly closed in X;*

    (2) *$X_f \supset M_f$; and*

    (3) *for each $x \in X \backslash K$, $fx = \max f[x, L]$ implies $x \in X_f$.*

Then there exists an $x \in \bigcap \{X_f : f \in E^*\}$.

Proof using Theorem 2. Note that, for each $f \in E^*$, $f|X$ belongs to $\hat{X}$. Now, our conclusion is immediate from Theorem 2 with $Y := \{f|X : f \in E^*\}$, $\alpha(x, f) := \inf f(Qx)$, and $\beta(x, f) := \sup f(Px)$.

Proof using Theorem 3. Note that, for each $f \in E^*$, $f|X$ belongs to $\hat{X}$. Therefore, we may choose $B : E^* \to \hat{X}$ to be the single-valued function $f \mapsto f|X$. Then (2) implies (3.2) in Theorem 3. Since (3) implies (3.3) in Theorem 3, the conclusion follows from Theorem 3.

REMARKS. 1. In fact, we can choose any topology on X such that X is a convex space and $f|X$ is u.s.c. for each $f \in E^*$. For example, X can be a topological subspace of E; or X can have the relative original topology of E and E can have the $w(E, E^*)$-topology.

2. If P and Q are u.h.c., then (1) holds automatically. In this case, for $X = K$, Corollary 3.1 reduces to Simons [47, Theorem 2.2].

3. An example of functions P and Q which satisfy (1) but not u.h.c. can be given as follows: Let $P, Q : [0, \infty) \to 2^{\mathbf{R}} \backslash \{\emptyset\}$ be defined by

$$Px = \begin{cases} [1 + x, 2 + x] & \text{if } x \neq 1 \\ 1/2 & \text{if } x = 1, \end{cases}$$

$$Qx = \begin{cases} [-x - 2, -x - 1] & \text{if } x \neq 1 \\ -1/2 & \text{if } x = 1. \end{cases}$$

4. In [2], for $f \in E^*$, Mf is denoted by $\partial(X; -f)$ and is called a "supporting set."

4. Coincidence, fixed point, and surjectivity theorems

In this section, we apply Theorem 3 to obtain various coincidence, fixed point, and surjectivity theorems and existence theorems on critical points (or zeros) of multifunctions.

The following coincidence theorem is basic in this section.

THEOREM 4. *Suppose that, in addition to the hypothesis of Theorem 3, for each $x \in X$, Px and Qx are convex, and that either*

(A) *E^* separates points of E and, for each $x \in X$, $\overline{Px}$ and $\overline{Qx}$ are compact; or*

(B) *E is locally convex and, for each $x \in X$, one of $\overline{Px}$ and $\overline{Qx}$ is compact.*

Then there exists an $x \in X$ such that $\overline{Px} \cap \overline{Qx} \neq \emptyset$.

Proof. Suppose that $\overline{Px} \cap \overline{Qx} = \emptyset$ for all $x \in X$. Then, under conditions (A) or (B), the separation theorems in a t.v.s. [44, p.70 or p.58, resp.] imply that for each $x \in X$ there is an $f \in E^*$ with $\inf f(\overline{Qx}) > \sup f(\overline{Px})$. This contradicts Theorem 3.

By putting $Bf = f \,|\, X$ for each $f \in E^*$, Theorem 4 reduces to the following:

COROLLARY 4.1. *Let X, L, K, and E be as in Corollary 3.1. Let $P, Q : X \to 2^E$ be functions such that either*

(A) *E^* separates points of E and $P, Q : X \to kc(E)$, or*

(B) *E is locally convex, $P, Q : X \to cc(E)$, and Px or Qx is compact in E for each $x \in X$.*

Suppose that (1), (2), and (3) in Corollary 3.1 hold. Then there exists an $x \in X$ such that $Px \cap Qx \neq \emptyset$.

REMARK. For $X = K$, Corollary 4.1 is due to Park [38, Theorem 15].

Putting $-f$ instead of arbitrary $f \in E^*$ in Corollary 4.1, we have the following:

COROLLARY 4.2. *Let X, L, K, E, and P, Q be as in Corollary 4.1. Suppose that*

(1) *P and Q are u.h.c.;*

(2) *for each $x \in K$ and $f \in E^*$,*

$$fx = \min f(X) \quad implies \quad \inf f(Px) \leq \sup f(Qx);$$

and

(3) *for each $x \in X \backslash K$ and $f \in E^*$,*

$$fx = \min f[x, L] \quad implies \quad \inf f(Px) \leq \sup f(Qx).$$

Then there exists an $x \in X$ such that $Px \cap Qx \neq \emptyset$.

REMARKS. 1. Corollary 4.2 is due to Park [**36**, Theorem 4] and, for a paracompact X, the case (B) to Jiang [**25**, II, Theorem 2.2], which generalizes Jiang [**25**, I, Theorem 3.3], [**24**, Theorem 2.1], Ko and Tan [**28**, Lemma 1.2], and Fan [**17**, Theorems 9 and 10]. For $X = K$, Corollary 4.2(B) reduces to Simons [**47**, Theorem 3.1], where E is assumed to have the $w(E, E^*)$-topology, Browder [**9**, Theorems 3 and 5], Fan [**15**, Theorem 6], [**16**, Theorem 5], Lee and Tan [**31**, Theorem 4], and Granas and Liu [**20**, Theorem 10.3].

2. In [**25**], the quadruple (K, X, P, Q) is said to be *admissible* if (2) holds. Therefore, (3) is just the admissibility of $(X \backslash K, L, P, Q)$.

Theorem 4 is equivalent to the following existence theorem for critical points or zeros of multifunctions:

THEOREM 5. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. with topological dual E^*, and $B : E^* \to 2^{\hat{X}} \backslash \{\emptyset\}$ a multifunction with convex graph. Let R be a multifunction such that either*

(A) *E^* separates points of E and $R : X \to kc(E)$, or*

(B) *E is locally convex and $R : X \to cc(E)$.*

(I) *Suppose that for each $f \in E^*$,*

(5.1) *$X_f = \{x \in X : \inf f(Rx) \leq 0\}$ is compactly closed;*

(5.2) *$X_f \supset MBf \cap K$; and*

(5.3) *for each $x \in X \backslash K$ and $g \in Bf$,*

$$gx = \max g[x, L] \text{ implies } x \in X_f.$$

Then there exists an $x \in X$ such that $0 \in Rx$.

(II) *Suppose that for each $f \in E^*$,*

   (5.1)' *$X_f = \{x \in X : \sup f(Rx) \geq 0\}$ is compactly closed; and*
(5.2) *and* (5.3) *hold.*
   Then there exists an $x \in X$ such that $0 \in Rx$.

Proof. (I) Put $\overline{Px} = \{0\}$ and $\overline{Qx} = Rx$ for all $x \in X$ in Theorem 4.
(II) Put $\overline{Px} = Rx$ and $\overline{Qx} = \{0\}$ for all $x \in X$ in Theorem 4.

REMARKS. 1. Note that (5.1) is equivalent to (5.1)', and that, if R is u.h.c., then (5.1) holds automatically, but not conversely.

2. Theorems 4, 5(I), and 5(II) are equivalent. In fact, by putting $Rx := \overline{Qx} - \overline{Px}$ or $Rx := \overline{Px} - \overline{Qx}$, Theorems 5(I) or 5(II), resp., reduces to Theorem 4.

3. For $X = K$, Theorem 5(B) reduces to Bellenger and Simons [5, Corollary 3].

By putting $Bf = f \,|\, X$ for each $f \in E^*$ in Theorem 5, we have the following equivalent form of Corollary 4.1.

COROLLARY 5.1. *Let X, L, K, and E be as in Corollary 3.1. Let R be a function such that either*

   (A) *E^* separates points of E and $R : X \to kc(E)$, or*
   (B) *E is locally convex and $R : X \to cc(E)$.*
(I) *Suppose that, for each $f \in E^*$,*
   (1) *$X_f = \{x \in X : \inf f(Rx) \leq 0\}$ is compactly closed;*
   (2) *$X_f \supset M_f$; and*
   (3) *for each $x \in X \backslash K$, $fx = \max f[x, L]$ implies $x \in X_f$.*
   Then there exists an $x \in X$ such that $Rx \ni 0$.
(II) *Suppose that, for each $f \in E^*$,*
   (1)' *$X_f = \{x \in X : \sup f(Rx) \geq 0\}$ is compactly closed;*
   (2)' *$X_f \supset M_f$; and*
   (3)' *for each $x \in X \backslash K$, $fx = \max f[x, L]$ implies $x \in X_f$.*

   Then there exists an $x \in X$ such that $Rx \ni 0$.

Proof. (I) Put $Px := \{0\}$ and $Qx := Rx$ in Corollary 4.1.
(II) Put $Px := Rx$ and $Qx := \{0\}$ in Corollary 4.1.

REMARKS. 1. If $X = K$ and R is u.h.c., Theorem 5 reduces to Cornet [12, Theorem 3.1] or Aubin [2, Theorem 15.1.1], where a number of applications are given. Moreover, if E has the $w(E, E^*)$-topology,

Theorem 5 reduces to Simons [**47**, Theorem 3.1, Second and Third forms].

2. Note that, usually, (2) or (2)$'$ is called the boundary condition, and (3) or (3)$'$ the "coercivity" or "compactness" condition. Such conditions can be reformulated to more visualizable geometric forms as follows:

LEMMA 1 (JIANG [**25**, I, LEMMA 2.1]). *Let X be a nonempty subset of a t.v.s. E, L and K be nonempty subsets of X, and $R : X \to 2^E$ satisfy* (A) *or* (B) *in Corollary 5.1.*

Then, for any $x \in K$ and $f \in E^*$, the following are equivalent:
(i) $fx \geq \sup f(L)$ *implies* $fx \geq \inf f(Rx)$.
(i)$'$ $d_f(Rx, \overline{I}_L(x)) = 0$.
Moreover, for any $x \in K$ and $f \in E^*$, the following are equivalent:
(ii) $fx \geq \sup f(L)$ *implies* $\sup f(Rx) \geq fx$.
(ii)$'$ $d_f(Rx, \overline{O}_L(x)) = 0$.

From Lemma 1 and Corollary 5.1, we have the following fixed point and surjectivity theorem:

THEOREM 6. *Let X, L, K, E, and R be as in Corollary 5.1.*
(I) *Suppose that, for each $f \in E^*$,*
   (6.1) $\{x \in X : fx \geq \inf f(Rx)\}$ *is compactly closed;*
   (6.2) $d_f(Rx, \overline{I}_X(x)) = 0$ *for every $x \in K \cap \operatorname{Bd} X$; and*
   (6.3) $d_f(Rx, \overline{I}_L(x)) = 0$ *for every $x \in X \backslash K$.*
   Then R has a fixed point.
(II) *Suppose that, for each $f \in E^*$,*
   (6.1)$'$ $\{x \in X : \sup f(Rx) \geq fx\}$ *is compactly closed;*
   (6.2)$'$ $d_f(Rx, \overline{O}_X(x)) = 0$ *for every $x \in K \cap \operatorname{Bd} X$; and*
   (6.3)$'$ $d_f(Rx, \overline{O}_L(x)) = 0$ *for every $x \in X \backslash K$.*
   Then R has a fixed point. Further, if R is u.h.c., then $R(X) \supset X$.

Proof. (I) Put $Px := \{x\}$ and $Qx := Rx$ in Corollary 4.1; or consider $Rx - x$ instead of Rx in Corollary 5.1(I). Note that $I_X(x) = O_X(x) = E$ for $x \in \operatorname{Int} X$, and hence (6.2) is actually equivalent to that $d_f(Rx, \overline{I}_X(x)) = 0$ for all $x \in K$.

(II) Put $Px := Rx$ and $Qx := \{x\}$ in Corollary 4.1. For the surjectivity result, let $y \in X$. Consider $Rx - y$ instead of Rx and $L \cup \{y\}$

instead of L in Corollary 5.1(II). Then there exists an $x \in X$ such that $Rx \ni y$. This completes our proof.

REMARKS. 1. If R is u.h.c., then (6.1) and (6.1)$'$ hold automatically, but not conversely. An example of R satisfying Theorem 6(I) but not u.h.c. is as follows: Let $X = [0, \infty)$, $L = (1, 2]$, and $K = [0, 1]$ in **R**. Define $R : X \to cc(\mathbf{R})$ by $Rx = (-\infty, -x]$ if $x \in X \setminus \{1/2\}$ and $R(1/2) = (-\infty, -2]$. Then R satisfies all requirements of (I), and has a fixed point 0. However, R is not u.h.c. For $f = 1_\mathbf{R}$, $\{x \in X : \sup f(Rx) \geq -3/2\} = [0, 1/2) \cup (1/2, 3/2]$ is not closed in X.

2. For an u.h.c. R, Theorem 6 is due to Park [36, Theorem 6] and, for a paracompact X, Theorem 6(B) to Jiang [25, II, Corollary 2.3]. In this case, Theorem 6 generalizes and unifies Jiang [25, I, Theorem 3.4], Shih and Tan [46, Theorems 4 and 5], and Fan [17, Corollary 2].

3. In the case $X = K$, Theorem 6(B) reduces to Simons [48, Theorem 2], which includes Glebov [18, Theorem], Cellina [10, Theorem 3], and others. In this case, for an u.h.c. R, Theorem 6 includes earlier well-known generalizations of the Brouwer or Kakutani type fixed point theorems due to Brouwer [7, Satz 4], Schauder [45, Satz 1], Tychonoff [50, Satz], Rothe [43], Kakutani [27, Theorem 1], Bohnenblust and Karlin [5], Glicksberg [19, Theorem], Fan [13], [14, Corollaire 2], [15, Theorem 3], [16, Theorem 6], [17, Corollary 3], Halpern [21], [23, Theorems 2, 3, and 5], Halpern and Bergman [22, Theorems 4.1 and 4.3], Browder [8, Theorems 1 and 2], [9, Theorems 3–5], Reich [40, Theorem 1.7], [41, Theorem 3.1], Cornet [12], Kaczynski [26, Théorème 1], Arino, Gautier, and Penot [1, Theorem 1], Granas and Liu [20, Theorems 10.4 and 10.5], and Park [32], [33], [37], [38].

4. The surjectivity $R(X) \supset X$ in (II) extends earlier works of Halpern [21], [23], Halpern and Bergman [22], Rogalski [42], Lasry and Robert [29, Théorème 10], and Cornet [11, Théorème 3.4] (See also [3, Theorems 6.4.14 and 6.4.15] and [2, Theorems 15.1.3 and 15.1.4].).

5. In our work [34], Theorem 6 is used to obtain generalizations of matching theorems concerning closed coverings of convex sets in a t.v.s. due to Fan [17], Shih and Tan [46], Park [36], and others. Also, Theorem 6 with $X = K$ is used in [35] to obtain fixed point theorems for certain condensing multifunctions with the inwardness condition.

For single–valued functions, Theorem 6 reduces to the following:

COROLLARY 6.1. *Let X, L, K, and E be as in Corollary 3.1 such that E^* separates points of E, and $q : X \to E$ a function.*
(I) *Suppose that, for each $f \in E^*$,*
   (1) *$\{x \in X : f(q(x)) \le fx\}$ is compactly closed;*
   (2) *$d_f(q(x), \overline{I}_X(x)) = 0$ for all $x \in K \cap \operatorname{Bd} X$; and*
   (3) *$d_f(q(x), \overline{I}_L(x)) = 0$ for all $x \in X \backslash K$.*
   Then q has a fixed point.
(II) *Suppose that, for each $f \in E^*$,*
   (1)' *$\{x \in X : f(q(x)) \ge fx\}$ is compactly closed;*
   (2)' *$d_f(q(x), \overline{O}_X(x)) = 0$ for all $x \in K \cap \operatorname{Bd} X$; and*
   (3)' *$d_f(q(x), \overline{O}_L(x)) = 0$ for all $x \in X \backslash K$.*
   Then q has a fixed point. Further, if q is continuous, then $q(X) \supset X$.

Proof. Immediate from Theorem 6 with $Rx := q(x)$.

REMARKS. 1. For a locally convex space E, e.g., $d_f(q(x), \overline{I}_X(x)) = 0$ for all $f \in E^*$ if and only if $q(x) \in \overline{I}_X(x)$.

2. For a continuous function q, (1) and (1)' hold automatically. In this case, for $X = K$, Corollary 6.1 reduces to Simons [47, Corollary 3.3].

3. Note that the Brouwer fixed point theorem easily follows from the following consequence of Corollary 6.1:

COROLLARY 6.2. *Let X be a compact convex subset in a t.v.s. E on which E^* separates points, and $q : X \to X$ a function such that $\{x \in X : f(q(x)) \le fx\}$ is closed for each $f \in E^*$. Then q has a fixed point.*

REMARK. Even for Euclidean spaces, the following example shows that Corollary 6.2 properly generalizes the Brouwer theorem: Let $X = [0,1]$ in **R**, $q(x) = x$ for $x \in X \backslash (1/3, 2/3)$, and $q(x) = 1$ for $x \in (1/3, 2/3)$. Note that q is not u.h.c., and has fixed points.

In Theorem 6(I), if we replace (6.2) by a more restrictive condition ("strongly inward") in Cornet [11], then we obtain another surjectivity result.

THEOREM 7. *Let X, L, K, E, and R be as in Corollary 5.1. Suppose that, for each $f \in E^*$, the conditions (6.1), (6.3), and the following hold:*

(7.2) for any $x \in M_f$, we have $\inf f(X) \geq \inf f(Rx)$.

Then R has a fixed point. Further, if R is u.h.c. and $X = K$, then $R(X) \supset X$.

Proof. Clearly (7.2) implies (6.2) by Lemma 1, and hence R has a fixed point by Theorem 6. For the second part, let $y \in X = K$. Then the function $x \mapsto Rx - y$ satisfies all of the requirements of Corollary 5.1(I). In fact, since R is u.h.c., (1) holds. For any $f \in E^*$ and $x \in M_f$, we have $\inf f(Rx - y) = \inf f(Rx) - fy \leq \inf f(Rx) - \inf f(X) \leq 0$ by (7.2), and hence (2) holds. Moreover, (3) holds trivially. Therefore, by Corollary 5.1(I), there exists a solution $\bar{x}$ to the inclusion $0 \in R\bar{x} - y$. Consequently, we have $R(X) \supset X$.

REMARKS. For $X = K$ and an u.h.c. R, Theorem 7 reduces to a result of Cornet [**11**, Théorème 3.2] (See also [**2**, Theorem 15.1.4]). A number of consequences and applications of this result are given in [**2**].

The following consequence of Corollary 5.1 is useful to obtain generalizations of the fixed point theorems for weakly inward (outward) single-valued functions.

THEOREM 8. *Let X, L, K, and E be as in Corollary 3.1 such that E^* separates points of E, and $r : X \to E$ nonvanishing.*
(I) *Suppose that, for each $f \in E^*$,*
    (8.1) $X_f = \{x \in X : f(r(x)) \leq 0\}$ *is compactly closed; and*
    (8.2) *for each $x \in X \backslash K$, $fx \geq \sup f(L)$ implies $x \in X_f$.*
    Then there exist an $f \in E^*\backslash\{0\}$ and an $x \in M_f$ such that

$$f(r(x)) > 0.$$

(II) *Suppose that, for each $f \in E^*$,*
    (8.1)' $X_f = \{x \in X : f(r(x)) \geq 0\}$ *is compactly closed; and*
    (8.2)' *for each $x \in X \backslash K$, $fx \geq \sup f(L)$ implies $x \in X_f$.*
    Then there exist an $f \in E^*\backslash\{0\}$ and an $x \in M_f$ such that

$$f(r(x)) < 0.$$

Proof. Immediate from Corollary 5.1(I) and (II).

REMARKS. 1. If r is continuous, then (8.1) and (8.1)$'$ hold automatically, but not conversely. For example, let X be a closed convex subset of $\mathbf{R}$ and $r : X \to \mathbf{R}$ a function such that $r(x) > 0$ for all $x \in X$. Then r satisfies (8.1) and (8.1)$'$, but r does not need to be continuous.

2. For $X = K$ and a continuous function r, Theorem 8 reduces to Simons [47, Corollary 3.2]. Note that Corollary 6.1 also follows from Theorem 8.

For a normed vector space E, let d denote the metric induced by the norm. The following is useful:

LEMMA 2 (JIANG [25, I, COROLLARY 3.6]). *In a normed vector space E, for any $A, B \in cc(E)$,*

$$d(A, B) = 0 \quad \text{iff} \quad d_f(A, B) = 0 \quad \text{for all} \quad f \in E^*.$$

In view of Lemma 2, Theorem 6 reduces to the following:

THEOREM 9. *Let X be a nonempty convex subset of a real normed vector space E, L a c-compact subset of X, K a nonempty compact subset of X, and $R : X \to cc(E)$.*
(I) *Suppose that*
  (9.1) *for each $f \in E^*$, $\{x \in X : fx \geq \inf f(Rx)\}$ is compactly closed;*
  (9.2) $d(Rx, \overline{I}_X(x)) = 0$ *for every $x \in K \cap \operatorname{Bd} X$; and*
  (9.3) $d(Rx, \overline{I}_L(x)) = 0$ *for every $x \in X \backslash K$.*
  Then R has a fixed point.
(II) *Suppose that*
  (9.1)$'$ *for each $f \in E^*$, $\{x \in X : \sup f(Rx) \geq fx\}$ is compactly closed;*
  (9.2)$'$ $d(Rx, \overline{O}_X(x)) = 0$ *for every $x \in K \cap \operatorname{Bd} X$; and*
  (9.3)$'$ $d(Rx, \overline{O}_L(x)) = 0$ *for every $x \in X \backslash K$.*
  Then R has a fixed point. Further, if R is u.h.c., then $R(X) \supset X$.

REMARKS. 1. Note that if R is u.h.c., then (9.1) and (9.1)$'$ are fulfilled automatically. Hence, Theorem 9 includes Jiang [25, II, Corollary 2.4; I, Corollary 3.6] and Park [36, Corollary 6.1].

2. If $R = r$ is single-valued, then, e.g., $d(r(x), \overline{I}_X(x)) = 0$ iff $r(x) \in \overline{I}_X(x)$.

Finally, the major particular forms of Theorem 6 can be adequately summarized by the following enlarged version of the diagrams given in [32], [38].

E	$f : K \to K$			$F : K \to 2^K$		
I	Brouwer	1910	[7]	Kakutani	1941	[27]
II	Schauder	1930	[45]	Bohnenblust and Karlin	1950	[6]
III	Tychonoff	1935	[50]	Fan	1952	[13]
				Glicksberg	1952	[19]
IV	Fan	1964	[14]	Granas and Liu	1986	[20]

	$f : K \to E$			$F : K \to 2^E$		
II	Rothe	1937	[43]			
	Halpern	1965	[21]			
III	Fan	1969	[15]	Browder	1968	[9]
	Reich	1972	[40]	Fan	1969	[15]
				Glebov	1969	[18]
				Halpern	1970	[23]
				Cellina	1970	[10]
				Reich	1972	[40]
					1978	[41]
				Cornet	1975	[11]
				Simons	1986	[47] [48]
IV	Halpern and Bergman	1968	[22]	Granas and Liu	1986	[20]
	Kaczynski	1983	[26]	Park	1988	[32]

		$F : X \to 2^E$		
III		Fan	1984	[17]
		Shih and Tan	1987	[46]
		Jiang	1988	[25]
IV		Park	1990	[36]

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex spaces, and IV for

topological vector spaces having sufficiently many linear functionals. Moreover, K stands for a nonempty compact convex subset of a space E, and X for a nonempty convex subset of E satisfying a certain coercivity condition with respect to a multifunction $F : X \to 2^E$.

In fact, Theorem 6 contains all of the results in the diagram.

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Department of Mathematics, Seoul National University
and
The Mathematical Sciences Research Institute of Korea
Seoul 151–742, Korea