J. Korean Math. Soc. 29 (1992), No. 1, pp. 175–190

LIPSCHITZ STABILITY AND EXPONENTIAL ASYMPTOTIC STABILITY IN PERTURBED SYSTEMS

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1. Introduction

Dannan and Elaydi [3] introduced a new notion of stability, which is called uniform Lipschitz stability (ULS), for systems of differential equations. This notion of ULS lies somewhere between uniform stability (US) on one side and the notions of asymptotic stability in variation (ASV) and uniform stability in variation (USV) on the other side. An important feature of ULS is that the linearized system inherits the property of ULS from the original nonlinear system [3, Theorem 3.4].

Also, Elaydi and Farran [5] introduced the notion of exponential asymptotic stability (EAS) which is a stronger notion than that of ULS. They investigated the properties of EAS dynamical systems on a compact Riemannian manifold, and gave some analytic criteria for an autonomous differential system and its perturbed systems to be EAS.

Athanassov [1] defined global exponential stability in variation (GESV) and then showed that the existence of Liapunov functions when the zero solution of a nonlinear system is GESV. The stronger notion than that of GESV is generalized exponential asymptotic stability in variation (GEASV) appeared in [6].

Taniguchi [9] obtained various stability theorems of perturbed differential systems. We use his technique to investigate ULS for linear perturbed systems.

In this paper we investigate the problems of ULS, EAS and GEASV for the following various perturbed differential systems of the nonlinear

Received April 12, 1991. Revised November 4, 1991.

*The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1990, Project No.BSRI-90-110.

differential system $x' = f(t,x)$:

$(P_1)\ y' = A(t)y + g(t,y)$,

$(P_2)\ y' = f(t,y) + g(t,y)$,

$(P_3)\ y' = f(t,y) + g(t,y,Ty)$,

$(P_4)\ y' = f(t,y) + g(t,y) + h(t,y,Ty)$.

2. Lipschitz stability

Let $\mathbf{R}^n$ and $\mathbf{R}^+$ be the n-dimensional Euclidean space and the set of all nonnegative real numbers, respectively. Let the symbol $|\cdot|$ denote any convenient norm on $\mathbf{R}^n$ and the corresponding norm for $n \times n$ real matrices. $C(X,Y)$ denotes the set of all continuous mappings from a topological space X to a topological space Y.

Consider the nonlinear differential system

$$\text{(N)} \qquad x' = f(t,x), \quad x(t_0) = x_0,$$

where $f \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R}^n)$ with $f(t,0) = 0$. If we assume that f has continuous partial derivatives $\partial f/\partial x$ on $\mathbf{R}^+ \times \mathbf{R}^n$ and the solution $x(t) = x(t,t_0,x_0)$ of (N) through $(t_0,x_0) \in \mathbf{R}^+ \times \mathbf{R}^n$ exists for $t \ge t_0 \ge 0$, then

$$\Phi(t,t_0,x_0) = \frac{\partial}{\partial x_0} x(t,t_0,x_0)$$

exists and is the solution of the variational system

$$(\mathrm{V}_1) \qquad z' = \frac{\partial}{\partial x} f(t, x(t,t_0,x_0))z$$

such that $\Phi(t_0,t_0,x_0)$ is the identity matrix [6].

DEFINITION 1. The zero solution $x = 0$ of (N) is said to be

(S) *stable* if for any $\varepsilon > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(t_0,\varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t,t_0,x_0)| < \varepsilon$ for all $t \ge t_0 \ge 0$,

(US) *uniformly stable* if the δ in (S) is independent of the time t_0,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t,t_0,x_0)| \le M|x_0|$ whenever $|x_0| < \delta$ and $t \ge t_0 \ge 0$,

(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t,t_0,x_0)| \le M$ for $|x_0| < \delta$ and $t \ge t_0 \ge 0$.

We recall Dannan and Elaydis' result [3, Theorem 2.1], emphasizing that ULS coincides with US in linear systems.

THEOREM 2.1. *For the linear system*

$$\text{(L)} \qquad x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is a continuous $n \times n$ matrix defined on $\mathbf{R}^+$, the following are equivalent:

(1) *the zero solution $x = 0$ of (L) is ULSV,*
(2) *the zero solution $x = 0$ of (L) is ULS,*
(3) *the zero solution $x = 0$ of (L) is US.*

REMARK. In definition 1.3 of [3], the word "in variation" was missing.

We consider a perturbed system

$$(\mathrm{P}_1) \qquad y' = A(t)y + g(t, y), \; y(t_0) = y_0,$$

where $g \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R}^n)$ with $g(t,0) = 0$, of (L). Then the solution $y(t) = y(t, t_0, y_0)$ of (P_1) through $(t_0, y_0) \in \mathbf{R}^+ \times \mathbf{R}^n$ is given by

$$y(t) = \varphi(t, t_0)y_0 + \int_{t_0}^{t} \varphi(t, s)g(s, y(s))ds,$$

where $\varphi(t, t_0)$ is the fundamental matrix solution of (L), from the variation of constants formula.

To show that $y = 0$ of (P_1) is ULS, we need the following:

LEMMA 2.2 [6, THEOREM 5.1.1]. *Suppose that $h(t, u) \in C(\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+)$ is monotone nondecreasing in u for each fixed $t \geq t_0 \geq 0$ with the property that*

$$x(t) - \int_{t_0}^{t} h(s, x(s))ds < y(t) - \int_{t_0}^{t} h(s, y(s))ds, \quad t \geq t_0 \geq 0$$

for $x, y \in C(\mathbf{R}^+, \mathbf{R}^+)$. If $x(t_0) < y(t_0)$, then $x(t) < y(t)$ for all $t \geq t_0 \geq 0$.

THEOREM 2.3. *Assume that $x = 0$ of (L) is ULS. Let the following condition hold for* (P_1):

> $|g(t, y)| \le h(t, |y|)$, *where* $h(t, u) \in C(\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+)$ *is monotone nondecreasing in* u *for each fixed* $t \ge t_0 \ge 0$ *with* $h(t, 0) = 0$.

Consider the scalar differential equation

$$(S_1) \qquad u' = Mh(t, u), \; M \ge 1, \; u(t_0) = u_0 > 0$$

and suppose that $u = 0$ *of* (S_1) *is ULS.*

Then the solution $y = 0$ *of* (P_1) *is ULS.*

Proof. For a solution $y(t) = y(t, t_0, y_0)$, we have

$$|y(t)| \le |\varphi(t, t_0)|\,|y_0| + \int_{t_0}^{t} |\varphi(t, s)|\,|g(s, y(s))|ds.$$

Since $x = 0$ of (L) is ULS, it is ULSV by Theorem 2.1. Thus there exist $M > 0$ and $\delta_1 > 0$ such that $|\varphi(t, t_0)| \le M$ for $t \ge t_0 \ge 0$ and $|x_0| < \delta_1$. Therefore, by the assumption, we have

$$|y(t)| \le M|y_0| + M \int_{t_0}^{t} h(s, |y(s)|)ds.$$

It follows that

$$\begin{aligned} |y(t)| - M \int_{t_0}^{t} h(s, |y(s)|)ds &\le M|y_0| \\ &< u_0 \quad \text{if} \quad M|y_0| < u_0 \\ &= u(t) - M \int_{t_0}^{t} h(s, u(s))ds. \end{aligned}$$

Hence $|y(t)| < u(t)$ by Lemma 2.2. Since $u = 0$ of (S) is ULS, it easily follows that $y = 0$ of (P_1) is ULS.

COROLLARY 2.4. *Suppose that the solution $x = 0$ of (L) is ULS. Consider the scalar differential equation*

$$(S_2) \qquad u' = Ka(t)u, \quad u(t_0) = u_0,$$

where $u \geq 0$, $K \geq 1$ and $a \in C(\mathbf{R}^+)$ satisfying the conditions

(1) $|f(t,y)| \leq a(t)|y|$, *where $f(t,y)$ is in (P),*
(2) $\int_0^\infty a(s)ds < M$ *for some $M > 0$.*

Then the solution $y = 0$ of (P_1) is ULS.

Proof. Let $u(t) = u(t, t_0, u_0)$ be a solution of (S_1). Then $u(t) = u_0 e^{KM}$ by the condition (2). Thus we have

$$|u(t)| \leq |u_0|e^{KM} = L|u_0|,$$

where $L = e^{KM} > 0$. Therefore $u = 0$ of (S_2) is ULS. This implies that the solution $y = 0$ of (P_1) is ULS by Theorem 2.3.

REMARK. Dannan and Elaydi [3, Theorem 2.14] showed ULS for (P_1) under the assumption that

$$|f(t,y)| \leq \gamma(t)|y| \quad \text{and} \quad \int_\theta^\infty \gamma(t)dt < \infty \quad \text{for all} \quad \theta \geq 0.$$

For the perturbation

$$(P_2) \qquad y' = f(t,y) + g(t,y)$$

of (N), Dannan and Elaydi [3, Theorem 2.14] investigated ULS by using the fundamental matrix $\Phi(t, t_0, x_0)$ of (V_1).

Now we consider the perturbation

$$(P_3) \qquad y' = f(t,y) + g(t,y,Ty), \; y(t_0) = y_0,$$

where $g \in C(\mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n, \; \mathbf{R}^n)$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous operator, of (N).

First, we need an integral inequality which is a generalization of Pachpatte's inequality [8].

LEMMA 2.5. *If* $u(t)$, $a(t)$, $b(t)$ *and* $c(t)$ *are elements of* $C(\mathbf{R}^+)$ *with the property that*

$$u(t) \le u_0 + \int_{t_0}^{t} a(s)u(s)ds + \int_{t_0}^{t} b(s) \int_{t_0}^{s} c(\tau)u(\tau)d\tau\, ds,$$

then

$$u(t) \le u_0 \exp\left\{ \int_{t_0}^{t} [a(s) + b(s) \int_{t_0}^{s} c(\tau)d\tau]ds \right\},\ 0 \le t_0 \le t < \infty.$$

Proof. Let

$$U(t) = u_0 + \int_{t_0}^{t} a(s)u(s)ds + \int_{t_0}^{t} b(s) \int_{t_0}^{s} c(\tau)u(\tau)d\tau\, ds.$$

Then $u(t) \le U(t)$ and thus

$$\begin{aligned} U'(t) &= a(t)u(t) + b(t) \int_{t_0}^{t} c(s)u(s)ds \\ &\le a(t)U(t) + b(t) \int_{t_0}^{t} c(s)U(s)ds \\ &\le U(t)[a(t) + b(t) \int_{t_0}^{t} c(s)ds] \end{aligned}$$

since $U(t)$ is nondecreasing. Integrating both sides of the above inequality we can obtain the result.

THEOREM 2.6. *For the perturbed system* (P_3)*, we assume that*

$$|g(t, y, Ty)| \le a(t)|y(t)| + b(t) \int_{t_0}^{t} c(s)|y(s)|ds, \tag{1}$$

where $a, b, c \in C(\mathbf{R}^+)$,

$$\int_{t_0}^{t} [a(s) + b(s) \int_{t_0}^{s} c(\tau)d\tau]ds < \infty. \tag{2}$$

Then the zero solution of (P_3) is ULS whenever the zero solution of (N) is ULSV.

Proof. By the nonlinear variation of constants formula of Alekseev, we have

$$\begin{aligned} y(t) &= y(t, t_0, y_0) \\ &= x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) g(t, y(s), Ty(s)) ds \end{aligned}$$

[6, Theorem 2.6.3]. Since there are $M > 0$ and $\delta > 0$ with $|\Phi(t, t_0, x_0)| \leq M$ for all $t \geq t_0 \geq 0$ and $|x_0| < \delta$, we have

$$\begin{aligned} |y(t)| \leq M|y_0| + M \int_{t_0}^{t} a(s)|y(s)|ds \\ + M \int_{t_0}^{t} b(s) \int_{t_0}^{s} c(\tau)|y(\tau)|d\tau \, ds \end{aligned}$$

by the assumption (1). Hence

$$|y(t)| < M|y_0| \exp \int_{t_0}^{t} [a(s) + b(s) \int_{t_0}^{s} c(\tau) d\tau] ds$$

by Lemma 2.5. In view of (2), we have $|y(t)| \leq L|y_0|$ for some $L > 0$ whenever $|y_0| < \delta$.

3. Exponential asymptotic stability

We recall some notions of stability.

DEFINITION 2. The zero solution of (N) is said to be

(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$ such that

$$|x(t, t_0, x_0)| \leq K|x_0|e^{-c(t-t_0)}$$

for $t \geq t_0 \geq 0$,

(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0,\ c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq Ke^{-c(t-t_0)}$$

for $t \geq t_0 \geq 0$,

(GEASV) *generalized exponentially asymptotically stable in variation* if

$$|\Phi(t, t_0, x_0)| \leq K(t)e^{p(t_0)-p(t)}$$

for $t \geq t_0 \geq 0$ where $K > 0$ is continuous on $\mathbf{R}^+$, $p \in \kappa$ and $p(t) \to \infty$ as $t \to \infty$. Here $p \in \kappa$ means $p \in C(\mathbf{R}^+)$, $p(0) = 0$, and $p(t)$ is strictly increasing in $t \in \mathbf{R}^+$.

Note that GEASV becomes EASV when $K(t) = K > 0$ and $p(t) = \alpha t$, $\alpha > 0$.

DEFINITION 3. The zero solution of (N) is said to be

(ASV) *asymptotically stable in variation* if there exists $M > 0$ such that

$$\int_{t_0}^{t} |\Phi(t, s)| ds \leq M$$

for every $t \geq t_0 \geq 0$, where $\Phi(t, t_0)$ is the fundamental matrix solution of the variational system

$$(\mathrm{V}_2) \qquad y' = f_x(t, 0)y, \quad y(t_0) = y_0,$$

with $\Phi(t_0, t_0)$ the unit matrix.

THEOREM 3.1. *The solution $x = 0$ of (N) is ULS if it is EAS.*

Proof. The solution $y = 0$ of (V_2) is EAS. Then there are $K > 0$ and $c > 0$ such that

$$|y(t, t_0, y_0)| = |\Phi(t, t_0)y_0| \leq K|y_0|e^{-c(t-t_0)}, \quad t \geq t_0 \geq 0.$$

Thus we have

$$\int_{t_0}^{t} |\Phi(t, s)| ds \leq K \int_{t_0}^{t} e^{-c(s-t_0)} ds = -K[e^{-c(t-t_0)} - 1]/c < K/c = M.$$

It follows that the solution $x = 0$ is ASV. Hence it is ULS by Theorem 2.8 in [3].

EXAMPLE. EAS is not implied by ULS. Consider the scalar differential equation $x' = -x^3$, $x(t_0) = x_0$, whose general solution is

$$x(t) = x_0[1 + 2x_0^2(t - t_0)]^{-1/2}, \quad t \geq t_0 \geq 0.$$

Since

$$\Phi(t, t_0, x_0) = [1 + 2x_0^2(t - t_0)]^{-3/2}, \quad t \geq t_0 \geq 0,$$

we have $|\Phi(t, t_0, x_0)| \leq 1$ for all $t \geq t_0 \geq 0$. Therefore $x = 0$ is ULSV and so it is ULS [3, Theorem 3.3]. However it is not EAS [7, Example 3].

REMARK. In [3], Figure 1 illustrated the possible known implications among various types of stability notions. It is very useful to investigate various stabilities for differential systems.

Brauer [2, Theorem 2] examined EAS for the trivial solution of (P_2) and obtained EAS for the trivial solution of

$$(P_3) \qquad y' = f(t, y) + g(t, y) + h(t, y),$$

where $h \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R}^n)$ with $h(t, 0) = 0$, as a corollary of his Theorem 2. We obtain an asymptotic behavior of solutions of (P_3).

THEOREM 3.2. *For the system (P_3) we assume the following conditions:*

(1) $|g(t, y)| = o(|y|)$ *as* $|y| \to 0$ *uniformly in* t,

(2) *there exists an* $\alpha > 0$ *such that* $|x| < \alpha$ *and* $t \in \mathbf{R}^+$ *imply* $|h(t, x)| \leq \gamma(t)$, *where* $\gamma \in C(\mathbf{R}^+)$ *with* $\int_0^\infty \gamma(t)dt < \infty$.

If the solution $x = 0$ *of (N) is EAS, then there exist* $T_0 \geq 0$ *and* $\delta > 0$ *such that* $t_0 \geq T_0$ *and* $|x_0| < \delta$ *imply every solution* $y(t)$ *of (P_3) tends to zero as* $t \to \infty$.

Proof. By the assumption we have $|\Phi(t, t_0, x_0) \leq Ke^{-c(t-t_0)}$ for some $K > 0$ and $c > 0$. We choose $T \geq 1$ and $\delta \leq \varepsilon$. Let $T_0 \geq T$ be so large that $t \geq T_0$ implies

$$\int_1^t \exp[-(c - K\varepsilon)(t - s)]\gamma(s)ds < \delta/2K = \delta_1.$$

This is possible by the fact

$$\lim_{t\to\infty} e^{-ct}\int_1^t e^{cs}\gamma(s)ds = 0$$

[6, Theorem 2.14.6].

Let $t_0 \ge T_0$ and $|y_0| < \delta/2K = \delta_1 < \delta$. Then we have

$$\begin{aligned}|y(t)| &\le |\Phi(t,t_0,x_0)||y_0| + \int_{t_0}^t |\Phi(t,s,y(s))||g(s,y(s)) \\ &\quad + h(s,y(s))|ds \\ &\le K|y_0|e^{-c(t-t_0)} + \int_{t_0}^t Ke^{-c(t-s)}[\varepsilon y(s)+\gamma(s)]ds\end{aligned}$$

by Theorem 2.6.3 in [6]. Thus

$$\begin{aligned}|y(t)|e^{ct} &\le K|y_0|\exp(ct_0)\exp[K\varepsilon(t-t_0)] \\ &\quad + \int_{t_0}^t Ke^{cs}\gamma(s)\exp[K\varepsilon(t-s)]ds\end{aligned}$$

by the Gronwall's inequality. In other words, we have

$$\begin{aligned}|y(t)| &\le K\delta_1\exp[-(c-K\varepsilon)(t-t_0)] \\ &\quad + K\int_1^t \exp[-(c-K\varepsilon)(t-s)]\gamma(s)ds.\end{aligned}$$

This inequality yields

$$\begin{aligned}|y(t)| &\le K\delta_1 + K\int_1^t \exp[-(c-K\varepsilon(t-s)]\gamma(s)ds \\ &\le K\delta_1 + \delta/2 < \delta,\end{aligned}$$

i.e., $|y(t)| < \varepsilon$ holds on $[t_0,\infty)$. This implies that the above inequality is true for $t \ge t_0$. Hence $y(t) \to 0$ as $t \to \infty$.

Now, we consider the perturbed system

$$(\mathrm{P}_4) \qquad y' = f(t,y) + g(t,y) + h(t,y,Ty),$$

where $h \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R}^n)$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous operator. Pachpatte [8, Theorem 2] obtained an asymptotic behavior of the system

$$y' = f(t, y) + g(t, y, Ty).$$

As an adaptation of this, an asymptotic behavior of (P_4) can be obtained. To do this we need an integral inequality which is similar to that of Lemma 2.5.

LEMMA 3.3. *If $u(t)$, $a(t)$, $b(t)$ and $c(t)$ are nonnegative continuous functions on $\mathbf{R}^+$ with the property that*

$$u(t) \le u_0 + \int_{t_0}^t u(s)ds + \int_{t_0}^t a(s)u(s)ds + \int_{t_0}^t b(s) \int_{t_0}^t c(\tau)d\tau\, ds,$$

where u_0 is a nonnegative constant, then

$$u(t) \le u_0 \left\{ \int_{t_0}^t [1 + a(s) + b(s) \int_{t_0}^t c(\tau)d\tau]ds \right\}.$$

THEOREM 3.4. *For the system (P_4), we assume that*

(1) *$g(t, y) = o(|y|)$ as $y \to 0$ uniformly in t,*
(2) *$h(t, y, Ty) \le \lambda(t)(|y| + |Ty|)$, where $\lambda \in C(\mathbf{R}^+, \mathbf{R})$ with*

$$\int_{t_0}^\infty \lambda(s)ds < \infty.$$

(3) *$|Ty(t)| \le e^{-ct} \int_{t_0}^t \mu(s)|y(s)|ds$, where $\mu \in C(\mathbf{R}^+, \mathbf{R})$, with*

$$\int_{t_0}^\infty \mu(s)ds < \infty.$$

Then every solution $y(t)$ of (P_4) approaches to zero as $t \to \infty$ whenever $x = 0$ of (N) is EAS.

Proof. Note that

$$\begin{aligned} y(t) &= y(t, t_0, y_0) \\ &= x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)[g(t, y(s) + h(t, y(s), Ty(s))]ds \end{aligned}$$

by the nonlinear variation of constants formula of Alekseev. Thus

$$\begin{aligned}|y(t)| &\le K|y_0|e^{-c(t-t_0)} \\ &\quad + Ke^{-c(t-s)}|g(t,y(s)) + h(t,y(s),Ty(s))|ds \\ &\le K|y_0|e^{-c(t-t_0)} + \varepsilon K\int_{t_0}^{t} e^{-c(t-s)}|y(s)|ds \\ &\quad + K\int_{t_0}^{t} e^{-c(t-s)}\lambda(s)[|y(s)| + e^{-cs}\int_{t_0}^{s}\mu(\tau)|y(\tau)|d\tau]ds.\end{aligned}$$

Since $|\Phi(t,t_0,x_0)| \le Ke^{-cs(t-t_0)}$ for all $t \ge t_0 \ge 0$ and for any $\varepsilon > 0$, $|g(t,y)| < \varepsilon|y|$ as long as $|y| < \delta$ for some $\delta > 0$, by letting $u(t) = |y(t)|e^{ct}$, we have

$$\begin{aligned}u(t) &\le K|y_0|e^{ct_0} + \varepsilon K\int_{t_0}^{t} e^{cs}|y(s)|ds \\ &\quad + K\int_{t_0}^{t} e^{cs}\lambda(s)\{|y(s)| + e^{-cs}\int_{t_0}^{s}\mu(s)e^{c\tau}|y(\tau)|d\tau\}ds \\ &= K|y_0|e^{ct_0} + \varepsilon K\int_{t_0}^{t} e^{cs}|y(s)ds \\ &\quad + K\int_{t_0}^{t} e^{cs}\lambda(s)|y(s)|ds + K\int_{t_0}^{t}\lambda(s)\int_{t_0}^{s}\mu(\tau)e^{c\tau}|y(\tau)|d\tau ds \\ &= Ku_0 + \varepsilon K\int_{t_0}^{t} u(s)ds + K\int_{t_0}^{t}\lambda(s)u(s)ds \\ &\quad + K\int_{t_0}^{t}\lambda(s)\int_{t_0}^{s}\mu(\tau)u(\tau)d\tau\, ds.\end{aligned}$$

In view of Lemma 3.3, we have

$$u(t) \le Ku_0 \exp\int_{t_0}^{t}[\varepsilon K + K\lambda(s) + K\lambda(s)\int_{t_0}^{s}\mu(\tau)d\tau]ds.$$

It follows that

$$|y(t)| \le K|y_0|e^{-c(t-t_0)} \exp\int_{t_0}^{t}[\varepsilon K + K\lambda(s) + K\lambda(s)\int_{t_0}^{s}\mu(\tau)d\tau]ds.$$

Therefore the right-hand side of this inequality approaches to zero if K and $|y_0|$ are small enough.

Corresponding to the function $V \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R})$ we define the total derivative V' with respect to (N) by

$$V'_{(N)}(t,x) = \limsup_{h\to 0+} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)]$$

and if $x(t)$ is a solution on (N) we denote by $V'(t, x(t))$ the upper right-hand Dini derivative of $V(t, x(t))$, i.e.,

$$V'(t,x(t)) = \limsup_{h\to 0+} \frac{1}{h}[V(t+h, x(t+h)) - V(t,x)].$$

It is well-known that $V'_{(N)}(t,x) = V'(t,x(t))$ if V is Lipschitzian with respect to x.

Athanassov [1] proved Massera type converse theorem for the kind of EASV by constructing a suitable Liapunov function. We can obtain a converse theorem for GEASV.

THEOREM 3.5. *Assume that $x = 0$ of (N) is GEASV. If $p'(t)$ exists and is continuous on $\mathbf{R}$, then there is a function $V \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R})$ satisfying*

(1) $|x| \le V(t,x) \le K(t)|x|$ *for all* $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^n$,
(2) $|V(t,x) - V(t,y)| \le K(t)|x-y|$ *for all* $(t,x),(t,y) \in \mathbf{R}^+ \times \mathbf{R}^n$,
(3) $V'_{(N)}(t,x) \le -p'(t)V(t,x)$ *for all* $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^n$.

Proof. We define

$$V(t,x) = \sup_{\tau\ge 0} |x(t+\tau, t, x)| e^{p(t+\tau)-p(t)}.$$

Then the proof is the same as in [1, Theorem 2.2] and [6, Theorem 3.6.1].

Finally, we can obtain the GEASV for (P_2) when $x = 0$ of (N) is GEASV by using the following two basic comparison lemmas.

LEMMA 3.6. *Assume that $V \in C(\mathbf{R}^+ \times \mathbf{R}^n, \mathbf{R})$ is Lipschitzian in x with Lipschitz constant L. If $x(s)$ and $y(s)$ are differentiable functions defined for $s \geq t$ with $x(t) = y(t) = x$, then*

$$V'(t, y(t)) \leq V'(t, x(t)) + L|y'(t) - x'(t)|.$$

Proof. It is straightforward.

LEMMA 3.7 [1, LEMMA 3.1]. *Let $x(t) = x(t, t_0, x_0)$ be a solution of (N) existing for $t \geq t_0$. Suppose that $V \in C(\mathbf{R}^+ \times \mathbf{R}^n,\ \mathbf{R}^n)$, $V(t, x)$ is Lipschitzian in x and $V'(t, x)$ satisfies for all $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^n$,*

$$V'(t, x) \leq w(t, V(t, x)),$$

where $w \in C(\mathbf{R}^+ \times \mathbf{R}, \mathbf{R})$. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of the scalar differential equation

$$u' = w(t, u), \quad u(t_0) = u_0 \geq 0$$

existing for $t \geq t_0$. Then, for $t \geq t_0$,

$$V(t, x(t)) \leq r(t)$$

whenever $V(t_0, x_0) \leq u_0$.

THEOREM 3.8. *Let $x = 0$ of (N) be GEASV. Assume that in (P_2) the perturbing term $g(t, y)$ satisfies*

$$|g(t, y)| \leq \varphi(t, |y|), \quad t \geq t_0 \geq 0,\ |y| < \infty,$$

where $\varphi \in C(\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R})$ is increasing in x for $t \in \mathbf{R}^+$. If the maximal solution of the scalar differential equation

$$(S_3) \qquad u' = [-p'(t) + \lambda(t)]K(t)u,\ u(t_0) = u_0 \geq 0,$$

where $p(t)$ and $K(t)$ are the functions from the definition of GEASV, is GEASV, then every solution of (P_2) is GEASV.

Proof. By the assumption, we have

$$|x(t)| \leq K(t_0)e^{p(t_0) - p(t)}, \quad t \geq t_0 \geq 0,\ |x_0| < \infty$$

for some continuous function $K(t) > 0$ and $p(t) \in \kappa$ with $p(t) \to \infty$ as $t \to \infty$. Then, by Lemma 3.6, we have

$$\begin{aligned} V'_{(P_1)}(t,x) &\le V'_{(N)}(t,x) + K(t)|g(t,x)| \\ &\le -p'(t)V(t,x) + K(t)\varphi(t,|x|) \\ &\le -p'(t)K(t)|x| + K(t)\varphi(t,|x|) \end{aligned}$$

for all $t \ge t_0 \ge 0$ and $|x| < \infty$.

Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of (S_3) existing for $t \ge t_0 \ge 0$ with $|u_0| \le K(t)|y_0|$, $|y_0| < \infty$. Let $y(t)$ be any solution of (P_2) existing for $t \ge t_0 \ge 0$. Then $V(t_0, y_0) \le K(t)|y_0| \le u_0$. In view of Lemma 3.7, we have $V(t,y) \le r(t)$. Thus we have

$$|y(t)| \le V(t,y) \le r(t).$$

Consequently the result follows from the assumption that $r(t)$ is GEASV.

References

1. Zh. S. Athanassov, *Perturbation theorems for nonlinear systems of ordinary differential equations*, J. Math. Anal. Appl. **86**(1982), 194–207.
2. F. Brauer, *Perturbations of nonlinear systems of differential equations* II, J. Math. Anal. Appl. **17**(1967), 418–434.
3. F. M. Dannan and S. Elaydi, *Lipschitz stability of nonlinear systems of differential equations*, J. Math. Anal, Appl. **113**(1986), 562–577.
4. ———, *Lipschitz stability of nonlinear systems of differential equations, II, Liapunov function*, J. Math. Anal. Appl. **143**(1989), 517–529.
5. S. Elaydi and H. R. Farran, *Exponentially asymptotically stable dynamical systems*, Appl. Anal. **25**(1987), 243–252.
6. V. Lakshmikantham and S. Leela, *Differential and integral inequalities, theory and applications*, Vol. I: Ordinary differential equations (Academic Press, New York, 1969).
7. J. L. Massera, *Constributions to stability theory*, Ann. of Math. **64**(1956), 182–206.
8. B. G. Pachpatte, *Stability and asymptotic behavior of perturbrd nonlinear system*, J. Differential Equations, **16**(1974), 14–25.
9. T. Taniguchi, *Stability theorems of perturbed linear ordinary differential equations*, J. Math. Anal. Appl. **149**(1990), 583–598.

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