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THE GAUSS EQUATIONS AND ELLIPTICITY OF ISOMETRIC EMBEDDINGS[1]

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0. Introduction

In this paper we are concerned with the ellipticity of isometric embeddings of Riemannian manifolds into Euclidean spaces. The notion of ellipticity of embeddings was first defined by N. Tanaka [6] for the purpose of studying the rigidity of embeddings. He defined an isometric embedding F to be elliptic if for each normal direction the second fundamental form has two eigenvalues of the same sign. In [6] he showed that if F is an elliptic embedding, then the linear system of partial differential equations associated with infinitesimal deformations of F is elliptic, so that the space of infinitesimal deformations of an elliptic embedding of a compact Riemannian manifold is of finite dimension. It follows that an elliptic embedding F of a compact Riemannian manifold into a Euclidean space is rigid.

In §1 of this paper, we construct compatibility equations of isometric embedding equations by a method due to A. Finzi [1] and show that the classical equation of Gauss is a compatibility equation of this type. In §2 we show that under certain conditions these compatibility equations form an elliptic system. Theorem 4 proves that if an isometric embedding F is elliptic in Tanaka's sense then F is elliptic in our sense, in the case of codimension 1. (We do not know yet whether this is true for higher codimensions.) An immediate consequence is that if M is an analytic Riemannian manifold and F an elliptic embedding of M into a Euclidean space, then F is analytic provided F is twice continuously differentiable (Corollary 5). The authors thank the referee for his interest and for pointing out several mistakes.

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1. Compatibility equations of Finzi type

In this section we adopt the definitions and notations of Olver [5]: Let X be an open subset of $\mathbf{R}^p$ and for each $x \in X$ let $(x, u^{(n)})$ denote the n-jet of a function u, namely, u and all the partial derivatives of u up to order n at x. We denote by $X \times U^{(n)}$ the space of n-jets over X. Suppose we are considering a system of n-th order differential equations

$$\Delta_k(x, u^{(n)}) = 0, \quad k = 1, \ldots, l, \tag{1}$$

for unknown functions $u = (u^1, \ldots, u^q)$ of p independent variables $x = (x^1 \ldots, x^p)$, where Δ_k is a polynomial in $u^{(n)}$ with coefficients which are C^∞ functions of x.

By a *compatibility equation* for (1) we mean a partial differential equation which is satisfied by any solution $u = f(x)$ of (1). Suppose that there exist homogeneous m-th order differential operators

$$\mathcal{L}_k = \sum_{|J|=m} B_k^J(x, u^{(n)}) D_J, \quad k = 1, \ldots, l,$$

not all zero at any point $(x, u^{(n)})$ in the n-jet space $X \times U^{(n)}$, such that the combination $\sum_{k=1}^{l} \mathcal{L}_k \Delta_k$ depends only on derivatives of order at most $m + n - 1$, where D_J denotes total differentiation. Such operators exist when the principal part of each $\mathcal{L}_\nu \Delta_\nu$ cancels out in the process of summation. We then obtain equations which reveal the properties of solutions that are due to the lower order terms of (1). We shall call such an equation a compatibility equation of *Finzi type* for (1).

A sufficient condition on Δ for such a compatibility equation to exist is found in the following

THEOREM 1 (FINZI [1]). *Let $\Delta_k(x, u^{(n)}) = 0$, $k = 1, \ldots, q$ be an n-th order determined system of differential equations. Suppose that Δ has no noncharacteristic directions at $(x_0, u_0^{(n)})$. Then there exist homogeneous m-th order differential operators $\mathcal{L}_k = \sum_{|J|=m} B_k^J(x, u^{(n)}) D_J$, $k = 1, \ldots, q$, not all zero at $(x_0, u_0^{(n)})$, such that at $(x_0, u_0^{(n)})$ the combination $\sum_{k=1}^{l} \mathcal{L}_k \Delta_k$ depends only on derivatives of u of order at most $n + m - 1$.*

Moreover, if there are no noncharacteristic directions for Δ for all $(x, u^{(n)})$ in some relatively open subset $\{(x, u^{(n)}) \in X \times U^{(n)} : \Delta(x, u^{(n)}) = 0\} \cap V$, with V open in $X \times U^{(n)}$, then the differential operators $\mathcal{L}_k$ depend smoothly on $(x, u^{(n)})$.

The key to the proof is the observation of the fact that Δ has no noncharacteristic directions at $(x_0, u_0^{(n)})$ if and only if the determinant of the principal symbol matrix $\mathbf{M}(\xi)$ is identically zero for all $\xi \in \mathbf{R}^p$ at $(x_0, u_0^{(n)})$. So we can state the same existence theorem for overdetermined systems by replacing the condition of nonexistence of noncharacteristic directions with the one that the principal symbol matrix $\mathbf{M}(\xi)$ is not of maximal rank for any choice of ξ.

Now let (M, g) be an n-dimensional Riemannian manifold, and $F : M \to \mathbf{E}^{n+p}$ be an isometric embedding of M into a Euclidean space $\mathbf{E}^{n+p}$.

Let P be a point of M and let $\widetilde{M} = F(M)$. We may assume that $F(P) = O$, where O is the origin of $\mathbf{E}^{n+p}$. Let $(y^1, \ldots, y^{n+p})$ be the standard coordinates of $\mathbf{E}^{n+p}$ so that $T_O(\widetilde{M})$ is spanned by $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$, and $\frac{\partial}{\partial y^{n+1}}, \ldots, \frac{\partial}{\partial y^{n+p}}$ form a basis of $T_O(\widetilde{M})^\perp$. We write $F = (f^1, \ldots, f^{n+p})$ coordinatewise, and choose a coordinate system $(x^1, \ldots, x^n)$ on a neighborhood Ω of P such that $F_*(\frac{\partial}{\partial x^i}|_P) = \frac{\partial}{\partial y^i}|_O$. Then we have

$$\frac{\partial f^k}{\partial x^i}(P) = \begin{cases} 1 & \text{if } \; i = k \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

On Ω the Riemannian metric g is represented by a positive definite symmetric matrix $[g_{ij}]_{n \times n}$, where each g_{ij} is a real valued C^∞ function on Ω. Then the function $F = (f^1, \ldots, f^{n+p})$ satisfies the following system :

$$\sum_{k=1}^{n+p} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} = g_{ij}, \; i = 1, \ldots, n, \; j = i, \ldots, n, \tag{3}$$

which is called a *local isometric embedding equation.* Let

$$\Delta_{i,j}(x, u^{(1)}) \equiv \sum_{k=1}^{n+p} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} - g_{ij}(x), \tag{4}$$

then the isometric embedding equations are written as

$$\Delta_{i,j}(x, u^{(1)}) = 0, \quad i = 1, \dots, n,\ j = i, \dots, n. \tag{5}$$

This is a system of first order non-linear partial differential equations. From now on we assume $p \leq \frac{n(n-1)}{2}$, so that the system (5) is determined if $p = \frac{n(n-1)}{2}$ or overdetermined if $p < \frac{n(n-1)}{2}$. Furthermore, noting (2), we see that the last p columns of the symbol matrix $\mathbf{M}(\xi)$ of (5) at O are identically zero, so $\mathbf{M}(\xi)$ is not of maximal rank at $(P, F(P))$ for any $\xi \in \mathbf{R}^n$. Thus we can find Finzi type compatibility equations for (5).

THEOREM 2. *For each pair of integers $i < j = 2, \dots, n$, the equation*

$$-\frac{1}{2}\{D_{j,j}\Delta_{i,i} - 2D_{i,j}\Delta_{i,j} + D_{i,i}\Delta_{j,j}\} \equiv \mathbf{K}_{i,j} = 0, \tag{6}$$

where $D_{i,j}$ is the total differentiation with respect to x^i and x^j, is a compatibility equation of Finzi type for (5).

Proof. This is a result of direct calculation of the left hand side of the equation. Substitute $\Delta_{i,j}$ in (6) by (4), then all the third order partial derivatives of u's cancel out and we have

$$\mathbf{K}_{i,j} = \sum_{k=1}^{n+p} \left\{ \frac{\partial^2 u^k}{(\partial x^i)^2} \frac{\partial^2 u^k}{(\partial x^j)^2} - \left[\frac{\partial^2 u^k}{\partial x^i \partial x^j} \right]^2 \right\} + \frac{1}{2} \left\{ \frac{\partial^2 g_{ii}}{(\partial x^j)^2} - 2\frac{\partial^2 g_{ij}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{jj}}{(\partial x^i)^2} \right\}.$$

From the compatibility equation $\mathbf{K}_{i,j} = 0$ we obtain $n(n-1)/2$ differential functions

$$\mathbf{I}_{i,j} \equiv \sum_{k=1}^{n+p} \left\{ \frac{\partial^2 u^k}{(\partial x^i)^2} \frac{\partial^2 u^k}{(\partial x^j)^2} - \left[\frac{\partial^2 u^k}{\partial x^i \partial x^j} \right]^2 \right\},$$

which are invariant under choice of solution u. Now we will show that (6) is equivalent to the Gauss curvature equations. These state that the sectional curvatures are given by

$$\kappa_M(X \wedge Y) = \frac{\langle \alpha(X,X), \alpha(Y,Y) \rangle - \langle \alpha(X,Y), \alpha(X,Y) \rangle}{\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle \langle X,Y \rangle}, \tag{7}$$

for any pair of linearly independent vectors X and Y of $T_x(M)$, $\forall x \in M$. Here α is the second fundamental form of M, κ_M denote the sectional curvature in M and $\langle \cdot, \cdot \rangle$ denote the usual Euclidean inner product. In local coordinates, (7) is represented by a system of second order partial differential equations. In particular at the reference point $P \in M$, the equation (7) with $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ becomes

$$\sum_{k=n+1}^{n+p} \left\{ \frac{\partial^2 u^k}{(\partial x^i)^2}(P) \frac{\partial^2 u^k}{(\partial x^k)^2}(P) - \left[\frac{\partial^2 u^k}{\partial x^i \partial x^j}(P) \right]^2 \right\} - \kappa_M \left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right)(P) = 0,$$

which turns out to be the expression at P of $\mathbf{K}_{i,j} = 0$. To see this, we define two real valued functions Φ and Ψ on Ω by

$$\Phi(x) = \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^j} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right\rangle,$$

$$\Psi(x) = \Phi(x) + \frac{1}{2} \left\{ \frac{\partial^2 g_{ii}}{(\partial x^j)^2} - 2 \frac{\partial^2 g_{ij}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{jj}}{(\partial x^i)^2} \right\},$$

where ∇ denotes the covariant differentiation of M. Then both Φ and Ψ describe quantities intrinsic to M, i.e., they are independent of embedding. Observe that

$$\sum_{k=1}^{n} \left\{ \frac{\partial^2 f^k}{(\partial x^i)^2}(P) \frac{\partial^2 f^k}{(\partial x^j)^2}(P) - \left[\frac{\partial^2 f^k}{\partial x^i \partial x^j}(P) \right]^2 \right\} = \Phi(P).$$

Then

$$\mathbf{K}_{i,j}[F](P) = \sum_{k=n+1}^{n+p} \left\{ \frac{\partial^2 f^k}{(\partial x^i)^2}(P) \frac{\partial^2 f^k}{(\partial x^j)^2}(P) - \left[\frac{\partial^2 f^k}{\partial x^i \partial x^j}(P) \right]^2 \right\} + \Psi(P).$$

Therefore the relation between the Finzi type invariants and the sectional curvatures in M is given by

$$\mathbf{I}_{i,j}(P) = \Phi(P) + \kappa_M \left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right)(P).$$

On the other hand, for each $i \le j = 1, \dots, n$, differentiating $\Delta_{i,j}$ with respect to x^i and x^j also produce compatibility equations for (5).

$$\sum_{k=1}^{n+p} \left\{ \frac{\partial u^k}{\partial x^j} \frac{\partial^2 u^k}{(\partial x^i)^2} + \frac{\partial u^k}{\partial x^i} \frac{\partial^2 u^k}{\partial x^i \partial x^j} \right\} - \frac{\partial g_{ij}}{\partial x^i} \equiv \mathbf{H}_{i,j} = 0,$$

$$\sum_{k=1}^{n+p} \left\{ \frac{\partial u^k}{\partial x^i} \frac{\partial^2 u^k}{(\partial x^j)^2} + \frac{\partial u^k}{\partial x^j} \frac{\partial^2 u^k}{\partial x^i \partial x^j} \right\} - \frac{\partial g_{ij}}{\partial x^j} \equiv \mathbf{H}_{j,i} = 0.$$

Consequently we obtain a system of compatibility equations for (5):

$$\begin{aligned} &\mathbf{H}_{i,j}(x, u^{(2)}) = 0, \quad i = 1, \dots, n, \ j = 1, \dots, n, \\ &\mathbf{K}_{i,j}(x, u^{(2)}) = 0, \quad i = 1, \dots, n, \ j = i+1, \dots, n. \end{aligned} \tag{8}$$

2. The ellipticity of isometric embeddings

The isometric embedding equation (3) is not elliptic if $p \ge 1$. However, the system (8) of compatibility equations is elliptic under certain conditions. In [2], it is shown that in the cases of embeddings of codimension 1, the first equation of (8) and the Gauss curvature equations (7) form an elliptic system if the second fundamental form has three nonzero eigenvalues. In this section we generalize this result :

Let (M, g) be an n-dimensional Riemannian manifold and $F : M \to \mathbf{E}^{n+p}$ be an isometric embedding with $\widetilde{M} = F(M)$. For any vector N normal to $\widetilde{M}$ at $x \in \widetilde{M}$, we define a symmetric bilinear form Θ_N on $T_x(\widetilde{M})$ by

$$\Theta_N(X, Y) = \langle N, \nabla_X \nabla_Y F \rangle, \quad \text{for} \quad X, Y \in T_x(\widetilde{M}).$$

This symmetric bilinear form Θ_N is called the second fundamental form of F corresponding to the normal vector N. Our main result is the following

THEOREM 3. *Suppose that (M, g) is a Riemannian manifold of dimension n and $F : M \to \mathbf{E}^{n+p}$ $(p < n)$ is a local isometric embedding. Suppose that for any vector N normal to $\widetilde{M}$ at $O = F(P) \in \widetilde{M}$, all* *eigenvalues of Θ_N are of the same sign or zero and at least two of them are nonzero. Then the system (8) of compatibility equations is elliptic at F.*

Proof. Consider the principal symbol matrix $\mathbf{M}(O,\xi)$ of (8) at P, which is of size $\{n^2 + \frac{1}{2}n(n-1)\} \times (n+p)$. We decompose this into $n+1$ blocks as

$$\mathbf{M}(O,\xi) = \begin{bmatrix} \mathbf{M}_1(O,\xi) \\ \vdots \\ \mathbf{M}_n(O,\xi) \\ \mathbf{M}_{n+1}(O,\xi) \end{bmatrix},$$

where $\mathbf{M}_i(O,\xi)$, $i = 1,\dots,n$ is the principal symbol of the system consisting of n equations, $\mathbf{H}_{i,j} = 0$, $j = 1,\dots,n$, and $\mathbf{M}_{n+1}(O,\xi)$ is that of the system $\mathbf{K}_{i,j} = 0$, $i = 1,\dots,n$, $j = i+1,\dots,n$. Then, for each $i = 1,\dots,n$,

$$\mathbf{M}_i(O,\xi) = \begin{bmatrix} \xi_i^2 & 0 & \cdots & 0 & \xi_1\xi_i & 0\cdots0 & 0 & 0\cdots0 \\ 0 & \xi_i^2 & \cdots & 0 & \xi_2\xi_i & 0\cdots0 & 0 & 0\cdots0 \\ & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & \xi_n\xi_i & 0\cdots0 & \xi_i^2 & 0\cdots0 \end{bmatrix}_{n\times(n+p)}$$

$$\begin{matrix} \uparrow & \quad & \uparrow \\ i\text{-th column} & & n\text{-th column} \end{matrix}$$

It is easy to see that for any nonzero $\xi \in \mathbf{R}^n$ the first n columns of $\mathbf{M}(O,\xi)$ are linearly independent. The point of the proof is to show that the last p columns of $\mathbf{M}_{n+1}(O,\xi)$ are linearly independent. Let $\mathbf{M}_{n+1}^j$ be the $(n+j)$-th column of $\mathbf{M}_{n+1}(O,\xi)$ and for a given nonzero p-tuple $(a_1,\dots,a_p)$, consider linear combination $\mathbf{A} = \sum_{j=1}^n a_j\mathbf{M}_{n+1}^j$.

Then **A** can be rewritten as

$$\begin{bmatrix} \Theta_{N_a}(\xi_1\frac{\partial}{\partial x^2}-\xi_2\frac{\partial}{\partial x^1},\xi_1\frac{\partial}{\partial x^2}-\xi_2\frac{\partial}{\partial x^1}) \\ \Theta_{N_a}(\xi_1\frac{\partial}{\partial x^3}-\xi_3\frac{\partial}{\partial x^1},\xi_1\frac{\partial}{\partial x^3}-\xi_3\frac{\partial}{\partial x^1}) \\ \vdots \\ \Theta_{N_a}(\xi_1\frac{\partial}{\partial x^n}-\xi_n\frac{\partial}{\partial x^1},\xi_1\frac{\partial}{\partial x^n}-\xi_n\frac{\partial}{\partial x^1}) \\ \Theta_{N_a}(\xi_2\frac{\partial}{\partial x^3}-\xi_3\frac{\partial}{\partial x^2},\xi_2\frac{\partial}{\partial x^3}-\xi_3\frac{\partial}{\partial x^2}) \\ \vdots \\ \Theta_{N_a}(\xi_2\frac{\partial}{\partial x^n}-\xi_n\frac{\partial}{\partial x^2},\xi_2\frac{\partial}{\partial x^n}-\xi_n\frac{\partial}{\partial x^2}) \\ \vdots \\ \Theta_{N_a}(\xi_{n-1}\frac{\partial}{\partial x^n}-\xi_n\frac{\partial}{\partial x^{n-1}},\xi_{n-1}\frac{\partial}{\partial x^n}-\xi_n\frac{\partial}{\partial x^{n-1}}) \end{bmatrix}_{\frac{n(n-1)}{2}\times 1},$$

where $N_a = a_1\frac{\partial}{\partial y^{n+1}} + \cdots + a_p\frac{\partial}{\partial y^{n+p}} \in T_O(\widetilde{M})^{\perp}$. Let $\lambda_1,\ldots,\lambda_n$ be eigenvalues of Θ_{N_a} and let the vectors $\mathbf{Z}_1,\ldots,\mathbf{Z}_n \in T_O(\widetilde{M})$ be corresponding orthonormal eigenvectors. Then there is an orthogonal matrix $\mathbf{B} = [\mathbf{b}_{ij}]_{n\times n}$ such that

$$\frac{\partial}{\partial x^i}|_P = \sum_{j=1}^{n}\mathbf{b}_{ij}\mathbf{Z}_j, \quad i = 1,\ldots,n,$$

and **A** is written by

$$\mathbf{A} = \begin{bmatrix} \sum_{k=1}^{n}(\xi_1\mathbf{b}_{2k}-\xi_2\mathbf{b}_{1k})^2\lambda_k \\ \sum_{k=1}^{n}(\xi_1\mathbf{b}_{3k}-\xi_3\mathbf{b}_{1k})^2\lambda_k \\ \vdots \\ \sum_{k=1}^{n}(\xi_1\mathbf{b}_{nk}-\xi_n\mathbf{b}_{1k})^2\lambda_k \\ \sum_{k=1}^{n}(\xi_2\mathbf{b}_{3k}-\xi_3\mathbf{b}_{2k})^2\lambda_k \\ \vdots \\ \sum_{k=1}^{n}(\xi_2\mathbf{b}_{nk}-\xi_n\mathbf{b}_{2k})^2\lambda_k \\ \vdots \\ \sum_{k=1}^{n}(\xi_{n-1}\mathbf{b}_{nk}-\xi_n\mathbf{b}_{(n-1)k})^2\lambda_k \end{bmatrix}$$

Let us assume that λ_1 and λ_2 are nonzero. Since all the eigenvalues of Θ_{N_a} are of the same sign or zero, $\sum_{j=1}^{p} a_j M_{n+1}^j = 0$ implies

$$\xi_i\mathbf{b}_{j1} - \xi_j\mathbf{b}_{i1} = 0,$$
$$\xi_i\mathbf{b}_{j2} - \xi_j\mathbf{b}_{i2} = 0,$$

for $i = 1, \ldots, n$, $j = i + 1, \ldots, n$. It follows from the independence of the first two columns of $\mathbf{B}$ that $\xi = 0$. In other words, the last p columns of $\mathbf{M}_{n+1}(O, \xi)$ are linearly independent unless $\xi = 0$. It then follows that the principal symbol matrix is of maximal rank.

If $p = 1$, we have the following

THEOREM 4. *Suppose that (M, g) is a Riemannian manifold of dimension n and $F : M \to \mathbf{E}^{n+1}$ is an isometric embedding with $\widetilde{M} = F(M)$. Suppose that the submanifold $\widetilde{M}$ has two nonzero principal curvatures of the same sign at $O = F(P)$. Then (8) is elliptic at F.*

Proof. It suffices to show that for any nonzero $\xi \in \mathbf{R}^n$, the last column of the principal symbol matrix $\mathbf{M}_{n+1}(O, \xi)$ never vanishes under the condition. Let N be the unit normal vector field of $\widetilde{M}$ with $N|_O = -\frac{\partial}{\partial y^{n+1}}|_O$. (Recall that we are assuming $\widetilde{M}$ is tangent to $y^{n+1} = 0$ at O.) The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the second fundamental form Θ_{N_O} of F corresponding to N_O are called the principal curvatures at O. Since the eigenvectors corresponding to these eigenvalues are known to be orthonormal, we may assume that $\frac{\partial}{\partial x^j}|_P$ is the eigenvector corresponding to λ_j, so that

$$\frac{\partial^2 f^{n+1}}{\partial x^i \partial x^j}(P) = \begin{cases} \lambda_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Then using (2) and (9), the last column of $\mathbf{M}_{n+1}(O, \xi)$ is

$$\begin{bmatrix} \lambda_2 \xi_1^2 + \lambda_1 \xi_2^2 \\ \lambda_3 \xi_1^2 + \lambda_1 \xi_3^2 \\ \vdots \\ \lambda_n \xi_1^2 + \lambda_1 \xi_n^2 \\ \lambda_3 \xi_2^2 + \lambda_2 \xi_3^2 \\ \lambda_4 \xi_2^2 + \lambda_2 \xi_4^2 \\ \vdots \\ \lambda_n \xi_2^2 + \lambda_2 \xi_n^2 \\ \vdots \\ \lambda_n \xi_{n-1}^2 + \lambda_{n-1} \xi_n^2 \end{bmatrix},$$

which is a non-zero vector unless $\xi = 0$. (Here, the condition on the principal curvatures is needed). Therefore the principal symbol matrix $\mathbf{M}(O, \xi)$ is of maximal rank. This completes the proof.

From the theory on the regularity of elliptic partial differential equations (cf. [4]), it follows the following

COROLLARY 5. *Suppose that M is a real analytic Riemannian manifold of dimension n and $F : M \to \mathbf{E}^{n+p}$ is an isometric embedding. If F is twice continuously differentiable and satisfies the conditions of Theorem 3 or Theorem 4, then F is real analytic.*

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