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A STRONG LARGE DEVIATIONS THEOREM FOR THE RATIO OF INDEPENDENT RANDOM VARIABLES

HAN SIK CHOI, CHI YONG KIM,
WOO CHUL KIM AND JONG WOO JEON

1. Introduction

Let $\{Y_n,\ n \geq 1\}$ be a sequence of random variables. One of the important problems in probability theory is to study the limit probability for large deviations, namely, $P(Y_n \geq y_n)$ where $\{y_n\}$ is a sequence of constants increasing without bound.

Since the original work of Cramèr (1938), there have been a great deal of investigations of large deviation problems, for instance, see Chernoff (1952), Bahadur (1960), Ellis (1984). Most of these results gave asymptotic expressions for log $P(Y_n \geq y_n)$, which was referred to as *weak large deviation* by N.R. Chaganty and J. Sethuraman (1987). Also Bahadur and Ranga Rao (1960), Chaganty and Sethuraman (1987) studied the large deviation problems, but their results gave asymptotic expressions for $P(Y_n \geq y_n)$, referred to as *strong large deviation.*

Bahadur and Ranga Rao (1960) proved a theorem giving an asymptotic expression for $P(Y_n \geq na)$ where $\{Y_n\}$ is a sequence of sums of independent and identically distributed random variables and a is a constant. Recently, Chaganty and Sethuraman (1987) obtained a generalization of Bahadur and Ranga Rao's theorem by considering $P(\frac{Y_n}{a_n} \geq r_n)$ where $\{Y_n,\ n \geq 1\}$ is an arbitrary sequence of random variables and $\{a_n\}$ is a sequence of constants without bound and $\{r_n\}$ is a sequence of appropriate constants.

In this paper, we prove a theorem, by the similar method to Chaganty and Sethuraman's, giving an asymptotic expression for $P(\frac{T_n}{S_n} \geq$

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r_n) where $\{T_n,\ n \geq 1\}$ and $\{S_n > 0,\ n \geq 1\}$ are two independent sequences of random variables. In case where $S_n = a_n$, our result reduces to that of Chaganty and Sethuraman (1987).

In Section 2, we introduce a local limit theorem given by Chaganty and Sethuraman (1987), which is requisite for the proof of our main theorem.

In Section 3, we state and prove the main theorem. Also we remark the relation between our result and that of Chaganty and Sethuraman (1987).

2. Preliminaries

In this section, we introduce two theorems which are given by Chaganty and Sethuraman (1987). These theorems play an important role in the proof of the main theorem.

THEOREM 2.1 (CHAGANTY AND SETHURAMAN).

Let $\{Y_n,\ n \geq 1\}$ be a sequence of nonlattice valued random variables which converges to Y in distribution. Let $\hat{f}_n$ be the characteristic function (ch. f.) of Y_n for $n \geq 1$ and $\hat{f}$ be the ch. f. of Y. Suppose that there are sequences $\{d_n\}$ and $\{b_n\}$ such that $d_n \to \infty$, $b_n \to \infty$ and $d_n = 0(b_n)$ as $n \to \infty$ and there is an integrable function $f^*(t)$ such that

$$(2\text{-}1) \qquad \sup_n |\hat{f}_n(t)| I(|t| < d_n) \leq f^*(t) \quad \text{for each} \quad t,$$

and

$$(2\text{-}2) \qquad \theta_n(\lambda) \overset{\text{def}}{=} \sup_{d_n \leq |t| \lambda b_n} |\hat{f}_n(t)| = o(\frac{1}{b_n}) \text{ for each } \lambda > 0, \quad \text{as } n \to \infty.$$

Then the random variable Y possesses a bounded and continuous probability density function (p.d.f.) f.

Let $h > 0$ and $y_n \to y^*$ as $n \to \infty$.
Define

$$(2\text{-}3) \qquad q_n(y) = \frac{b_n}{2h} P(|Y_n - y| < \frac{h}{b_n}).$$

Then

$$q_n(y_n) \to f(y^*) \quad \text{as} \quad n \to \infty. \tag{2-4}$$

Furthermore, there exist a finite constant M and an integer n_h such that

$$\sup_y [q_n(y)] \le M \quad \text{for} \quad n \ge n_h. \tag{2-5}$$

THEOREM 2.2 (CHAGANTY AND SETHURAMAN).
Let $\{Y_n,\ n \ge 1\}$ be a sequence of random variables with ch.f.'s $\{\hat{f}_n(t),\ n \ge 1\}$, and $\{d_n\}$ be a sequence of real numbers such that $d_n \to \infty$. Let $g_n(t) = \frac{1}{d_n^2}\log|\hat{f}_n(d_n t)|$ be finite and twice differentiable in a neighborhood of the origin which does not depend on n. Suppose that there exist $\delta > 0$, $\alpha > 0$ such that for $|t| < \delta$,

$$-g_n''(t) \ge \alpha \quad \text{for all} \quad n \ge 1. \tag{2-6}$$

Then condition (2-1) of Theorem 2.1 is satisfied with d_n replaced by δd_n.

3. Main Result

Before stating the main theorem, we introduce some notations. Let $\{T_n, n \ge 1\}$ be a sequence of random variables with absolutely continuous d.f.'s F_{1n} and $\{S_n,\ n \ge 1\}$ be a sequence of positive random variables with d.f.'s F_{2n}. We assume that two sequences are independent. Let $\{a_n,\ n \ge 1\}$ be a sequence of positive real numbers with $a_n \to \infty$ as $n \to \infty$. Define $\phi_{1n}(z) = E[\exp(zT_n)]$ and $\phi_{2n}(z) = E[\exp(zS_n)]$ for a complex number z. We assume that

$$\phi_{1n}(z) \text{ is analytic in } \Omega_1 = \{z \in \mathbf{C}; |\mathrm{Re}(z)| < A_1,\ A_1 > 0\}$$

and

$$\phi_{2n}(z) \text{ is analytic in } \Omega_2 = \{z \in \mathbf{C}; |\mathrm{Re}(z)| < A_2,\ A_2 > 0\}.$$

Define $\psi_{1n}(z) = \frac{1}{a_n}\log\phi_{1n}(z)$ for $z \in \Omega_1$ and $\psi_{2n}(z) = \frac{1}{a_n}\log\phi_{2n}(z)$ for $z \in \Omega_2$. Denote the interval $(-a_1, a_1)$ by J_1 and $(-a_2, a_2)$ by J_2 where $0 < a_1 < A_1$ and $0 < a_2 < A_2$. Let $\{r_n,\ n \ge 1\}$ be a sequence of real numbers such that $\sup|r_n| = r_0 < \infty$.

THEOREM 3.1.
Assume the following conditions;
(A) *There exist* $\beta_1 < \infty$ *and* $\beta_2 < \infty$ *such that*

$$|\psi_{1n}(z)| < \beta_1, \quad \text{for} \quad |z| < A_1, \ n \geq 1 \quad \text{and}$$
$$|\psi_{2n}(z)| < \beta_2, \quad \text{for} \quad |z| < A_2, \ n \geq 1.$$

(B) *There exists* $\tau_n \in J_1$ *such that for* $n \geq 1$

$$\tau_n > d \quad \text{for some positive number} \quad d,$$
$$r_n\tau_n \in J_2 \quad \text{and}$$
$$\psi'_{1n}(\tau_n) - r_n\psi'_{2n}(-r_n\tau_n) = 0.$$

(C) *There exists* $\alpha > 0$ *such that* $\psi''_{1n}(\tau_n) \geq \alpha$ *for* $n \geq 1$.
(D) *There exists* $\delta_0 > 0$ *such that*

$$\sup_{\delta \leq |t| \leq \lambda} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \cdot \frac{\phi_{2n}(-r_n\tau_n - ir_nt)}{\phi_{2n}(-r_n\tau_n)} \right| = o(\frac{1}{\sqrt{a_n}})$$
as $n \to \infty$, *for all* $0 < \delta < \delta_0$ *and for each* $\lambda > \delta_0$.

Then

$$P(\frac{T_n}{S_n} \geq r_n) \sim \frac{\exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n))]}{\tau_n\sqrt{2\pi a_n[\psi''_{1n}(\tau_n) + r_n^2\psi''_{2n}(-r_n\tau_n)]}} \tag{3-1}$$

where $g_n \sim f_n$ *means that* $\lim_{n\to\infty} \frac{g_n}{f_n} = 1$.

Proof. Let T_n^* be a random variable such that

$$F_{1n}^*(x) \stackrel{\text{def}}{=} P(T_n^* \leq x) = \int_{-\infty}^{x} \exp(u\tau_n - a_n\psi_{1n}(\tau_n))dF_{1n}(u) \tag{3-2}$$

and S_n^* be a random variable such that

$$\begin{aligned} F_{2n}^*(y) &\stackrel{\text{def}}{=} P(S_n^* \leq y) \\ &= \int_{0}^{y} \exp(-vr_n\tau_n - a_n\psi_{2n}(-r_n\tau_n))dF_{2n}(v). \end{aligned} \tag{3-3}$$

Hence we get

(3-4) $$dF_{1n}(x) = \exp(-x\tau_n + a_n\psi_{1n}(\tau_n))dF^*_{1n}(x) \text{ and}$$

(3-5) $$dF_{2n}(y) = \exp(yr_n\tau_n + a_n\psi_{2n}(-r_n\tau_n))dF^*_{2n}(y).$$

Let $T'_n = T^*_n - r_nS^*_n$.

Then, we have

(3-6)
$$\begin{aligned}
&P(\frac{T_n}{S_n} \geq r_n) \\
&= \int_0^\infty \int_{r_n y}^\infty dF_{1n}(x)dF_{2n}(y) \\
&= \int_0^\infty \int_{r_n y}^\infty \exp(-xr_n + a_n\psi_{1n}(\tau_n) + yr_n\tau_n \\
&\qquad + a_n\psi_{2n}(-r_n\tau_n))dF^*_{1n}(x)dF^*_{2n}(y) \\
&= \exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n))] \int_0^\infty \int_{r_n y}^\infty \exp(-xr_n + yr_n\tau_n) \\
&\qquad dF^*_{1n}(x)dF^*_{2n}(y) \\
&= \exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n))]E[\exp(-r_nT^*_n + r_n\tau_nS^*_n)I(\frac{T^*_n}{S^*_n} \geq r_n)] \\
&= \exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n))]E[\exp(-r_nT'_n)I(T'_n \geq 0)] \\
&= \exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n))]I_n \quad \text{(say)}.
\end{aligned}$$

We will later show that the following (3-7) and (3-8) hold.

(3-7)
$$\tau_n\sqrt{2\pi a_n[\psi''_{1n}(\tau_n) + r_n^2\psi''_{2n}(-r_n\tau_n)]}\, P((k-1)h \leq \tau_nT'_n < kh) \sim h$$

uniformly for bounded intervals of k, and there exist constants M and n_h such that for $n \geq n_h$

(3-8)
$$|\tau_n\sqrt{2\pi a_n[\psi''_{1n}(\tau_n) + r_n^2\psi''_{2n}(-r_n\tau_n)]}\, P((k-1)h \leq \tau_nT'_n < kh)| \leq Mh$$

for all $k \geq 1$.

We now write down lower and upper bounds for I_n;

(3-9)
$$\begin{aligned} I_n &= \sum_{k=1}^{\infty} E[\exp(-\tau_n T_n') I((k-1)h \le \tau_n T_n' < kh)] \\ &\ge \sum_{k=1}^{k_h} \exp(-kh) P((k-1)h \le \tau_n T_n' < kh), \end{aligned}$$

and

(3-10)
$$\begin{aligned} I_n \le & \sum_{k=1}^{k_h} \exp(-(k-1)h) P((k-1)h \le \tau_n T_n' < kh) \\ & + \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h) P((k-1)h \le \tau_n T_n' < kh) \end{aligned}$$

where we choose $k_h = [\frac{1}{h^2}]$.
Using (3-7) and (3-8) we get

(3-11)
$$\begin{aligned} & \liminf_n \{\tau_n \sqrt{2\pi a_n [\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n\tau_n)]}\, I_n\} \\ \ge & h \sum_{k=1}^{k_h} \exp(-kh) \\ = & \frac{h[\exp(-h) - \exp(-(k_h+1)h)]}{1 - \exp(-h)} \end{aligned}$$

and

(3-12)
$$\begin{aligned} & \limsup_n \{\tau_n \sqrt{2\pi a_n [\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n\tau_n)]}\, I_n\} \\ \le & h \sum_{k=1}^{k_h} \exp(-(k-1)h) + Mh \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h) \\ = & \frac{h(1 - \exp(-k_h h))}{1 - \exp(-h)} + \frac{Mh \exp(-k_h h)}{1 - \exp(-h)}. \end{aligned}$$

Letting $h \to 0$, we get from (3-11) and (3-12)

$$I_n \sim \frac{1}{\tau_n\sqrt{2\pi a_n[\psi_{1n}''(\tau_n)+r_n^2\psi_{2n}''(-r_n\tau_n)]}}. \tag{3-13}$$

This completes the proof of Theorem 3.1.

Proof of (3-7) and (3-8).
The ch.f. of $T_n' = T_n^* - r_nS_n^*$ is given by

$$E[\exp(itT_n')] = \frac{\phi_{1n}(\tau_n+it)}{\phi_{1n}(\tau_n)}\cdot\frac{\phi_{2n}(-r_n\tau_n-ir_nt)}{\phi_{2n}(-r_n\tau_n)}. \tag{3-14}$$

Let $d_n = \sqrt{a_n[\psi_{1n}''(\tau_n)+r_n^2\psi_{2n}''(-r_n\tau_n)]}$ and $Y_n = \frac{T_n'}{d_n}$. Then the ch.f. of Y_n is given by

$$\hat{f}_n(t) = \frac{\phi_{1n}(\tau_n+i(t/d_n))}{\phi_{1n}(\tau_n)}\cdot\frac{\phi_{2n}(-r_n\tau_n-i(r_nt/d_n))}{\phi_{2n}(-r_n\tau_n)}. \tag{3-15}$$

By condition (A) and Cauchy's theorem for derivatives, we get

$$|\psi_{1n}^{(k)}(\tau_n+it)| \le \frac{k!\beta_1}{(A_1-a_1)^k} \text{ for } |t|<A_1-a_1, \quad k\ge 1 \tag{3-16}$$

and

$$|\psi_{2n}^{(k)}(-r_n\tau_n-i\tau_nt)| \le \frac{k!\beta_2}{(A_2-a_2)^k} \text{ for } |t|<A_2-a_2, \quad k\ge 1. \tag{3-17}$$

Using the Taylor series expansion, we can write

$$\psi_{1n}(\tau_n+it) = \psi_{1n}(\tau_n)+it\psi_{1n}'(\tau_n)+\frac{(it)^2}{2}\psi_{1n}''(\tau_n)+R_{1n}(\tau_n+it) \tag{3-18}$$

for $|t|<A_1-a_1$ where $|R_{1n}(\tau_n+it)| \le \frac{2\beta_1|t|^3}{(A_1-a_1)^3}$ and

$$\begin{aligned}\psi_{2n}(-r_n\tau_n-ir_nt) &= \psi_{2n}(-r_n\tau_n)+(-ir_nt)\psi_{2n}'(-r_n\tau_n)\\ &\quad+\frac{(-ir_nt)^2}{2}\psi_{2n}''(-r_n\tau_n)+R_{2n}(-r_n\tau_n-ir_nt)\end{aligned} \tag{3-19}$$

for $|t| < A_2 - a_2$ where $|R_{2n}(-r_n\tau_n - ir_nt)| \le \frac{2\beta_2|r_nt|^3}{(A_2-a_2)^3}$. Since $d_n \to \infty$ as $n \to \infty$, we obtain, from (3-15), (3-18), (3-19) and conditions (B) and (C), for any real t and for sufficiently large n, that

(3-20)
$$\begin{aligned}
&\log \hat{f}_n(t)\\
=&a_n[\psi_{1n}(\tau_n + i(t/d_n)) - \psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n - i(r_nt/d_n))\\
&\quad - \psi_{2n}(-r_n\tau_n)]\\
=&-\frac{t^2}{2} + a_nR_{1n}(\tau_n + i(t/d_n)) + a_nR_{2n}(-r_n\tau_n - i(r_nt/d_n))
\end{aligned}$$

and

$$|a_nR_{1n}(\tau_n + i(t/d_n))| \le \frac{2a_n\beta_1|t|^3}{d_n^3(A_1 - a_1)^3} \to 0 \text{ as } n \to \infty \tag{3-21}$$

and

$$|a_nR_{2n}(-r_n\tau_n - i(r_nt/d_n))| \le \frac{2a_n\beta_2|r_nt|^3}{d_n^3(A_2 - a_2)^3} \to 0 \text{ as } n \to \infty. \tag{3-22}$$

Hence $Y_n = \frac{T_n'}{d_n}$ converges in distribution to the standard normal random variable. We now proceed to verify conditions (2-1) and (2-2) of Theorem 2.1.

Let

(3-23)
$$\begin{aligned}
g_n(t) =& \frac{1}{d_n^2}\log|\hat{f}_n(d_nt)|\\
=&\frac{a_n}{d_n^2}Re[\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n - ir_nt) - \psi_{2n}(-r_n\tau_n)]\\
=&\frac{Re[\psi_{1n}(\tau_n + it) - \psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n - ir_nt) - \psi_{2n}(-r_n\tau_n)]}{\psi_{1n}''(\tau_n) + r_n^2\psi_{2n}''(-r_n\tau_n)}.
\end{aligned}$$

Since $\psi_{2n}''(t) \ge 0$ for $t \in J_2$, we get, from conditions (A) and (C), that

$$|g_n(t)| < \infty. \tag{3-24}$$

Also we get

(3-25)

$$\begin{aligned} g_n''(t) &= \frac{-Re[\psi_{1n}''(\tau_n + it) + r_n^2 \psi_{2n}''(-r_n\tau_n - ir_n t)]}{\psi_{1n}''(\tau_n) + r_n^2\psi_{2n}''(-r_n\tau_n)} \\ &= \frac{-Re[\psi_{1n}''(\tau_n) + it\xi_{1n} + r_n^2\psi_{2n}''(-r_n\tau_n) + ir_n^3 t\xi_{2n}]}{\psi_{1n}''(\tau_n) + r_n^2\psi_{2n}''(-r_n\tau_n)} \\ &= -1 + Re\left[\frac{it\xi_{1n} + ir_n^3 t\xi_{2n}}{\psi_{1n}''(\tau_n) + r_n^2\psi_{2n}''(-r_n\tau_n)}\right] \\ &\le -1 + \frac{|t|[|\xi_{1n}| + r_0^3|\xi_{2n}|]}{\alpha} \end{aligned}$$

where ξ_{1n} and ξ_{2n} are appropriate complex numbers depending on the third derivatives of ψ_{1n} and ψ_{2n} respectively, satisfying

$$|\xi_{1n}| \le \frac{3!\beta_1}{(A_1 - a_1)^3} \quad \text{for} \quad n \ge 1 \tag{3-26}$$

and

$$|\xi_{2n}| \le \frac{3!\beta_2}{(A_2 - a_2)^3} \quad \text{for} \quad n \ge 1. \tag{3-27}$$

Therefore we can find $\delta > 0$ such that for $|t| < \delta$

$$g_n''(t) \le -\frac{1}{2} \quad \text{for all} \quad n \ge 1. \tag{3-28}$$

This verifies condition (2-6) of Theorem 2.2 and hence condition (2-1) of Theorem 2.1 with d_n replaced by δd_n.

Now, with $b_n = \tau_n d_n$ and using condition (D) for fixed $\lambda > 0$, we get that

(3-29)

$$\begin{aligned}\sup_{\delta d_n \le |t| \le \lambda b_n} |\hat{f}_n(t)| &= \sup_{\delta < |t| \le \lambda r_n} \left|\frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \cdot \frac{\phi_{2n}(-r_n\tau_n - ir_n t)}{\phi_{2n}(-r_n\tau_n)}\right| \\ &= o(\frac{1}{\sqrt{a_n}}) \\ &= o(\frac{1}{\tau_n d_n}) \\ &= o(\frac{1}{b_n}) \quad \text{as} \quad n \to \infty,\end{aligned}$$

since $0 < d < \tau_n < a_1$ and $d_n = 0(\sqrt{a_n})$. This verifies condition (2-2) of Theorem 2.1 with $b_n = \tau_n d_n$. Now, by replacing h by $\frac{h}{2}$ and taking $y_n = \frac{(2k-1)h}{2b_n}$, we get that

$$q_n(y_n) = \frac{\tau_n d_n}{h} P((k-1)h < \tau_n T_n' < kh) \tag{3-30}$$

since $b_n = \tau_n d_n$ and $Y_n = \frac{T_n'}{d_n}$.
Thus, the assertions (3-7) and (3-8) follow from (2-4) and (2-5) of Theorem 2.1.

REMARK 3.1.
With $S_n = a_n$, $\phi_{2n}(z) = \exp(za_n)$ is analytic in the whole complex plane and $\psi_{2n}(z) = z$, $\psi_{2n}'(z) = 1$ and $\psi_{2n}''(z) = 0$ and for $n \ge 1$. Then the condition $r_n\tau_n \in J_2$ in condition (B) may be deleted and the condition $\psi_{1n}'(\tau_n) - r_n\psi_{2n}'(-r_n\tau_n) = 0$ for $n \ge 1$ is reduced to $\psi_{1n}'(\tau_n) = r_n$ for $n \ge 1$. Also condition (D) is identical with condition (B) of Chaganty and Sethuraman's (1987) Theorem 3.1. And the claimed asymptotic expression (3-1) becomes

$$P(\frac{T_n}{a_n} \ge r_n) \sim \frac{\exp[a_n(\psi_{1n}(\tau_n) - r_n\tau_n)]}{\tau_n\sqrt{2\pi a_n \psi_{1n}''(\tau_n)}}. \tag{3-31}$$

This is exactly the same as that obtained by Chaganty and Sethuraman (1987), with $m_n = r_n$.

REMARK 3.2.

In condition (B), we assume that $\tau_n > d$ for some positive constant d and for $n \geq 1$. Now consider the case where $\tau_n \to 0$ and $\tau_n \sqrt{a_n} \to \infty$ as $n \to \infty$. Then, by the same reason of Chaganty and Sethuraman (1987)'s Theorem 3.5, we can get the stronger result that the conclusion of Theorem 3.1 holds without condition (D), that is, if the conditions (A), (B) and (C) of Theorem 3.1 hold, then

$$P(\frac{T_n}{S_n} \geq r_n) \sim \frac{\exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n))]}{\tau_n\sqrt{2\pi a_n[\psi''_{1n}(\tau_n) + r_n^2\psi''_{2n}(-r_n\tau_n)]}}. \tag{3-32}$$

REMARK 3.3.

Let T_n be a normal random variable with mean 0 and variance n. Let $S_n = n$ and $r_n = c$ where c is a positive constant. Then our result becomes

$$P(\frac{T_n}{n} \geq c) \sim \frac{\exp(-nc^2/2)}{c\sqrt{2\pi n}} \tag{3-33}$$

which coincides with the fact that as $x \to \infty$

$$1 - \Phi(x) \sim \frac{1}{x}\phi(x)$$

where Φ and ϕ are the d.f. and the p.d.f. of a standard normal random variable, respectively.

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Department of Computer Science and Statistics
Seoul National University
Seoul 151–742, Korea