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WELLPOSEDNESS OF THE CAUCHY PROBLEMS

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1. Introduction

Let $\mathcal{D}'_\omega(R^n)$ denote the space of ultradistributions on R^n defined by Beurling[1] and P(D) be a differential operator of order m with constant coefficients. In [10] we have shown the following statements are equivalent:

(α) P(D) is ω-hyperbolic with respect to the given vector N, that is, $P_m(N) \neq 0$ and there is a constant $c > 0$ such that $P(\xi + i\tau N) \neq 0$ for every $\xi \in R^n$ and $\tau < -c(1+\omega(\xi))$.

(β) P(D) has a fundamental solution in $\mathcal{D}'_\omega(R^n)$ whose support is contained in a proper cone of the half space generated by N.

In this paper we show that the wellposedness of Cauchy problem for P(D) in $\mathcal{E}_\omega$ and the above properties in $\mathcal{D}'_\omega(R^n)$ are equivalent for some limited class of ω's.

To show this result we denote by M(respectively, M_c) the set of all continuous real valued functions ω on R^n satisfying the following conditions (i) - (vi)(respectively, (i) - (iv))

$$(i)\quad 0 = \omega(0) \le \omega(\xi+\eta) \le \omega(\xi) + \omega(\eta), \qquad \xi, \eta \in R^n$$

$$(ii)\quad \int_{R^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} d\xi < \infty$$

$$(iii)\quad \omega(\xi) \ge a + b\log\,(1+|\xi|) \quad \text{for some constants a and} \quad b > 0$$

$$(iv)\quad \omega(\xi) = \Omega(|\xi|) \quad \text{for some even concave function} \quad \Omega \quad \text{on} \quad R$$

$$(v)\quad \log t = o(\Omega(t)) \quad \text{as} \quad t \to \infty$$

$$(vi)\quad \Phi : t \longmapsto \Omega(e^t) \quad \text{is a convex function on} \quad R.$$

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Beurling and Björck [2] defined the test function space as the set $\mathcal{D}_\omega(U)$ of all $\phi \in L^1(R^n)$ such that ϕ has compact support in the open set U in R^n and

$$\|\phi\|_\lambda = \int_{R^n} |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty \quad \text{for every} \quad \lambda > 0$$

and $\mathcal{E}_\omega(U)$ the set of all complex valued functions ψ on U such that $\phi\psi$ is in $\mathcal{D}_\omega(U)$ for every ϕ in $\mathcal{D}_\omega(U)$. In this case they only require that ω satisfies the property (i) - (iv). The reader can find the definition of other spaces and related properties in [2]. In this paper we add two more conditions (v) and (vi) for our purpose, which are introduced by Braun, Meise and Taylor[3].

With these conditions they proved the following representation of $\mathcal{D}_\omega(U)$:

LEMMA 1.1. *If $\omega \in M$ and U is an open set in R^n, then*

$$\mathcal{D}_\omega(U) = \{\phi \in C_c^\infty(U) | \ \forall k \in \mathrm{N},$$
$$\sup_{\alpha \in N_0^n} \sup_{x \in U} |\phi^{(\alpha)}(x)| exp(-k\Phi^*(\frac{|\alpha|}{k})) < \infty\}$$

where Φ^* denotes the Young's conjugate of the convex function $\Phi(t) = \Omega(e^t)$.

Using this representation we obtain the following lemma which we need later.

LEMMA 1.2. *If $\omega \in M$ and ϕ is in $\mathcal{D}_\omega(R^n)$ with $D_1^j\phi(0, x') = 0$ for all $j = 0, 1, 2, \ldots$ and $x' = (x_2, ..., x_n) \in R^{n-1}$, then the function ϕ_0 is in $\mathcal{D}_\omega(R^n)$, where ϕ_0 is given by*

$$\phi_0(x) = \begin{cases} \phi(x) & \text{if} \quad x_1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From the fact that $\phi_0 \in C^\infty(R^n)$ and

$$\sup_{\alpha \in N_0^n} \sup_{x \in R^n} |\phi_0^{(\alpha)}(x)| exp(-k\Phi^*(\frac{|\alpha|}{k}))$$
$$\leq \sup_{\alpha \in N_0^n} \sup_{x \in R^n} |\phi^{(\alpha)}(x)| exp(-k\Phi^*(\frac{|\alpha|}{k})) < \infty,$$

for every $k = 1, 2, ...$, it follows, due to lemma 1.1, $\phi_0 \in \mathcal{D}_\omega(R^n)$.

2. Wellposedness of the Cauchy Problems

Let a be any real number. Consider the following Cauchy problem:

$$(*)\qquad \begin{aligned} &P(D)u = f && \text{for} && < x, N >> a \\ &D_N^j u = g_j && \text{for} && < x, N >= a \quad \text{and} \quad 0 \leq j < m, \end{aligned}$$

when $f \in \mathcal{E}_\omega(R^n)$ and $g_j \in \mathcal{E}_\omega(R^{n-1})$. Here $P(D)$ is a partial differential operator of order m and of constant coefficients, and D_N denotes the derivation along $N \in R^n - \{0\}$.

For a given real number a, the Cauchy problem $(*)$ is said to be ω-wellposed for P in the half-space $< x, N >\geq a$ if, for all $f \in \mathcal{E}_\omega(R^n)$ and all $g_j \in \mathcal{E}_\omega(R^{n-1})$, there exists a unique function u in $\mathcal{E}_\omega(R^n)$ such that $(*)$ holds. And the Cauchy problem is said to be ω-wellposed for P in the direction N if and only if it is ω-wellposed in every half-space $< x, N >\geq a$, $a \in R$. But we note that this is equivalent to the ω-wellposedness in the half space $< x, N >\geq 0$.

We may consider our problem for $N = (1, 0, ..., 0)$. Then we can wrtie

$$\begin{aligned} D &= (D_1, ..., D_n) = (D_1, D'), \\ \zeta &= (\zeta_1, \zeta') = (\xi_1 + i\eta_1, \xi' + i\eta'), \\ P(D) &= P(D_1, D') \quad \text{and} \quad P(\zeta) = P(\zeta_1, \zeta'). \end{aligned}$$

We now have the following lemma from the ω-hyperbolicity.

LEMMA 2.1. *Let $\omega \in M_c$, and $P(D)$ be of order m and ω-hyperbolic with respect to $N = (1, 0, ..., 0)$. Then, for $0 \leq k < m$ and $x_1 \in R$, there is a distribution $H_k(x_1) \in \mathcal{E}'_\omega(R^{n-1})$ such that*

(1) *$D_1^j H_k(x_1) \in \mathcal{E}'_\omega(R^{n-1})$ for every $j \geq 0$, $P(D_1, D')H_k(x_1) = 0, D_1^k H_k(0) = \delta$ and $D_1^j H_k(0) = 0$ when $k \neq j < m$.*

(2) *$\{(x_1^0, x') | x' \in suppH_k(x_1^0)\} \subset suppE \cap \{x | x_1 = x_1^0\}$ for $x_1^0 \geq 0$, when E is the fundamental solution of $P(D)$ given by the ω-hyperbolicity of $P(D)$ with respect to N.*

Proof. We write

$$P(\zeta) = P(\zeta_1, \zeta') = \sum_{j=0}^{m} \zeta_1^{m-j} q_j(\zeta')$$

and define

$$p_k(\zeta_1, \zeta') = \sum_{j=0}^{k} \zeta_1^{k-j} q_j(\zeta').$$

Let Γ be a simple, positively oriented curve which for fixed ζ' surrounds the zeros ζ_1 of $P(\zeta_1, \zeta')$. Then the function $\widehat{H}_k(x_1, \zeta')$, defined by

$$\widehat{H}_k(x_1, \zeta') = \frac{1}{2\pi i} \int_\Gamma e^{i\zeta_1 x_1} \frac{p_{m-1-k}(\zeta_1, \zeta')}{P(\zeta_1, \zeta')} d\zeta_1,$$

is an entire function of ζ' for every x_1 and every k by the continuity in ζ' of the solution curve. According to the ω-hyperbolicity of P(D), we have

$$|\zeta_1| \leq C(1 + |\zeta'|) \qquad \text{and} \qquad |\eta_1| \leq C(1 + |\eta'| + \Omega(|\xi'|) + \Omega(|\xi_1|))$$

for some constant C when $P(\zeta_1, \zeta') = 0$. And from the conditions (i) and (ii) of ω we also have

$$|\xi_1| < C(1 + |\zeta'|) \quad \text{and} \quad |\eta_1| < C(1 + |\eta'| + \Omega(|\xi'|))$$

for some constant C when $P(\zeta_1, \zeta') = 0$. In order to estimate $D_1^j \widehat{H}_k(x_1, \zeta')$, we can then choose Γ as the rectangle

$$(**) \qquad |\xi_1| = C(1 + |\zeta'|); \qquad |\eta_1| = C(1 + |\eta'| + \Omega(|\xi'|)).$$

Since $|p_{m-1-k}(\zeta_1, \zeta')|$ is majorized by a constant mutiple of $(1+|\zeta'|)^{m-1-k}$, and both $|\zeta_1|$ and the length of Γ by constant multiples of $(1 + |\zeta'|)$, we get

$$(***) \qquad |D_1^j \widehat{H}_k(x_1, \zeta')| \leq C(1 + |\zeta'|)^{m-k+j} e^{C(|x_1|+1)[1+|\eta'|+\Omega(|\xi'|)]}$$

for some constant C. In particular,

$$|\widehat{H}_k(x_1, \zeta')| \leq C e^{C(|x_1|+1)|\eta'| + \epsilon|\eta'| + C(|x_1|+1)\Omega(|\xi'|)}$$

for all $\epsilon > 0$ and some constant C. Hence, by Paley-Wiener Theorem, $\widehat{H}_k(x_1, \zeta')$ is the Fourier-Laplace transform of an element $H_k(x_1)$ of $\mathcal{E}'_\omega(R^{n-1})$ given by

$$< H_k(x_1), \phi >= (2\pi)^{-n+1} \int_{R^{n-1}} \widehat{H}_k(x_1, \xi') \hat{\phi}(-\xi') d\xi'$$

when ϕ is in $\mathcal{D}_\omega(R^{n-1})$. We define $< D_1^j H_k(x_1),\phi >= D_1^j < H_k(x_1),\phi >$. Then our estimates imply

$$< D_1^j H_k(x_1), \phi >= (2\pi)^{-n+1} \int D_1^j \widehat{H}_k(x_1,\xi')\hat{\phi}(-\xi')d\xi'.$$

Thus $D_1^j H_k(x_1) \in \mathcal{E}'_\omega(R^{n-1})$ and $[D_1^j H_k(x_1)\widehat{]}(\zeta') = D_1^j \widehat{H}_k(x_1,\zeta')$. And

$$P(D_1,\xi')\widehat{H}_k(x_1,\xi') = (2\pi i)^{-1} \int_\Gamma e^{i\zeta_1 x_1} p_{m-1-k}(\zeta_1,\xi')d\zeta_1 = 0$$

since the integrand is analytic. So $P(D_1, D')H_k(x_1) = 0$. On the other hand, we have

$$D_1^j \widehat{H}_k(0,\zeta') = (2\pi i)^{-1} \int_\Gamma \zeta_1^j p_{m-1-k}(\zeta_1,\zeta')/P(\zeta_1,\zeta')d\zeta_1.$$

The integrand is

$$\begin{aligned}\zeta_1^j p_{m-1-k}(\zeta_1,\zeta')/P(\zeta_1,\zeta') =&\zeta_1^{j-k-1} + \zeta_1^{j-k-1}[\zeta_1^{k+1} p_{m-1-k}(\zeta_1,\zeta') \\ &- P(\zeta_1,\zeta')]/P(\zeta_1,\zeta').\end{aligned}$$

The degree of ζ_1 in the numerator of the second term is majorized by $j-k-1+k = j-1$, hence, by $m-2$ when $j < m$. Since the degree of ζ_1 in the denominator is m, we get

$$D_1^j \widehat{H}_k(0,\zeta') = (2\pi i)^{-1} \int_\Gamma \zeta_1^{j-k-1} d\zeta_1 \qquad \text{for} \quad 0 \le j < m.$$

Thus $D_1^k H_k(0) = \delta$ and $D_1^j H_k(0) = 0$ when $k \neq j < m$. Finally, we localize the support of $H_k(x_1^0)$. Let ϕ be in $\mathcal{D}_\omega(R^{n-1})$ with $\{(x_1^0, x')|x' \in supp\phi\} \cap suppE = \emptyset$ and take $\psi \in \mathcal{D}_\omega(R)$ satisfying $supp\psi \subset [-1,1]$ and $\int \psi dx = 1$. We set

$$\chi_\epsilon(x_1,\dots,x_n) = \chi_\epsilon(x_1,x') = \epsilon^{-1}\psi(\epsilon^{-1}(x_1-x_1^0))\phi(x').$$

Then $\hat{\chi}_\epsilon(\zeta) = \hat{\chi}_\epsilon(\zeta_1,\zeta') = exp[-i\zeta_1 x_1^0]\hat{\psi}(\epsilon\zeta_1)\hat{\phi}(\zeta')$ and, for small $\epsilon > 0$, supp $\chi_\epsilon \cap$supp E $= \emptyset$. Hence, for all small $\epsilon > 0$,

$$\begin{aligned}0 &= E(p_{m-1-k}(-D_1,-D')\chi_\epsilon) \\ &= (2\pi)^{-n} \int_{\sigma(N,t)} e^{i\zeta_1 x_1^0} p_{m-1-k}(\zeta_1,\zeta')\hat{\psi}(-\epsilon\zeta_1)\hat{\phi}(-\zeta')/P(\zeta_1,\zeta')d\zeta\end{aligned}$$

where $\sigma(N,t)$ is the surface $(\xi_1+it(1+\Omega(|\xi_1|)+\Omega(|\xi'|)),\xi_2,...,\xi_n)$ with $t\leq -C(N)$, where C(N) is the ω-hyperbolicity constant of P(D) with respect to N. From Paley-Wiener Theorem, we have, for all $\lambda>0$, and some C_λ,

$$|e^{i\zeta_1 x_1^0}\hat{\psi}(-\epsilon\zeta_1)|\leq C_\lambda e^{-\eta_1 x_1^0+\epsilon|\eta_1|-\lambda\Omega(\epsilon|\xi_1|)}.$$

Integrating first with respect to ξ_1 for fixed ξ', this estimate and the analyticity of the integrand show that the integration path can be deformed to a positively oriented circle Γ surrounding the zeros ζ_1 of $P(\zeta_1,\zeta')$ when $0<\epsilon<x_1^0$. Hence, letting $\epsilon\longrightarrow 0$, we have

$$\begin{aligned}0&=(2\pi)^{-n}\int_{R^{n-1}}\int_\Gamma e^{i\zeta_1 x_1^0}\frac{p_{m-1-k}(\zeta_1,\xi')\hat{\phi}(-\xi')}{P(\zeta_1,\xi')}d\zeta_1 d\xi'\\&=i<H_k(x_1^0),\phi> \qquad \text{for} \quad x_1^0>0.\end{aligned}$$

Thus, $(x_1^0, suppH_k(x_1^0))$ is contained in supp E and in $\{x|x_1=x_1^0\}$ when $x_1^0>0$. Since this is trivial for $x_1^0=0$, the proof of the existence is complete.

We remark that $<H_k(x_1),\phi>\in\mathcal{E}_\omega(R)$.

THEOREM 2.2. *Let $\omega\in M$. Let P(D) be of order m which is ω-hyperbolic with respect to $N=(1,0,...,0)$. Then the Cauchy problem*

$$\begin{aligned}&P(D_1,D')\phi(x_1,x')=f(x_1,x') \qquad \text{for} \quad x_1>a\\&D_1^j\phi(a,x')=g_j(x') \qquad \text{for} \quad 0\leq j<m\end{aligned}$$

is ω-wellposed in the direction N.

Proof. By translation invariance it is enough to consider the hyperplane $<x,N>=0$. Because of the ω-hyperbolicity with respect to N and $-$N, P(D) has a unique fundamental solution E_1 with support in $\{x|x_1\geq 0\}$, and a unique fundamental solution E_2 with support in $\{x|x_1\leq 0\}$. Write $f=f_1+f_2$ where supp f_1 is in $\{x|x_1\geq -1\}$, and supp f_2 is in $\{x|x_1\leq 1\}$ and f_1,f_2 are in $\mathcal{E}_\omega(R^n)$. Set

$$(E_1*f_1)(x_1,x')+(E_2*f_2)(x_1,x')=V(x_1,x').$$

We apply Lemma 2.1 and the notations there. Writing

$$<H_k(x_1),\psi>=\int_{R^{n-1}}H_k(x_1,x')\psi(x')dx'$$

the function

$$\phi(x_1, x') = \sum_{k=0}^{m} \int_{R^{n-1}} H_k(x_1, y')[g_k(x'-y') - D^k V(0, x'-y')]dy' + V(x_1, x')$$

belongs to $\mathcal{E}_\omega(R^n)$ and solves the given problem. Indeed, from Lemma 2.2 in [10], we have $V \in \mathcal{E}_\omega(R^n)$. When $\beta \in \mathcal{D}_\omega(R^{n-1})$, we get

$$\begin{aligned}[H_k(x_1) * \beta](x') &= < H_k(x_1)(y'), \beta(x'-y') > \\ &= (2\pi)^{-n+1} \int \widehat{H}_k(x_1, \xi') e^{i<x', \xi'>} \hat{\beta}(\xi') d\xi'.\end{aligned}$$

Let $\alpha(x)$ be in $\mathcal{D}_\omega(R^n)$, and put $u(x_1, x') = (H_k(x_1) * \beta)(x')\alpha(x)$. Then, for all μ in R^n, we get

$$\begin{aligned}\hat{u}(\mu) = (2\pi i)^{-n} \int_{R^n} \int_{R^{n-1}} \int_\Gamma & e^{-i<x, \mu - (\zeta_1, \xi')>} \\ & \alpha(x)\hat{\beta}(\xi') p_{m-1-k}(\zeta_1, \xi') d\zeta_1 d\xi' dx.\end{aligned}$$

Hence, by Fubini's Theorem, we have

$$\begin{aligned}|\hat{u}(\mu)| &\le C' \int_{R^{n-1}} \int_\Gamma |\hat{\alpha}(\mu - (\zeta_1, \xi'))||\hat{\beta}(\xi')||d\zeta_1|d\xi' \\ &\le C_\lambda \int_{R^{n-1}} e^{-\lambda\Omega(|\mu - (\xi_1, \xi')|)} e^{A|\eta_1|} |\hat{\beta}(\xi')| d\xi'\end{aligned}$$

and $|\xi_1|, |\eta_1| \le C(1 + \Omega(|\xi'|)$ for some constants C', C_λ, and C. Thus we get

$$|\hat{u}(\mu)| \le C_\lambda e^{-\lambda\Omega(|\mu|)} \int_{R^{n-1}} e^{B(1+\Omega(|\xi'|))} |\hat{\beta}(\xi')| d\xi'$$

for some constants C_λ and B. Since β is in $\mathcal{D}_\omega(R^{n-1})$, we conclude that u is in $\mathcal{D}_\omega(R^n)$. We also have $H_k(x_1) * \beta \in \mathcal{E}_\omega(R^n)$ when $\beta \in \mathcal{E}_\omega(R^{n-1})$, using a local unit. Consequently, $\phi(x_1, x') \in \mathcal{E}_\omega(R^n)$ since

$V(0,x') \in \mathcal{E}_\omega(R^{n-1})$. And we have

$$\begin{aligned} P(D_1,D')\phi(x_1,x') &= P(D_1,D')\sum_{k=0}^{m-1}\int_{R^{n-1}} H_k(x_1,y')[g_k(x'-y') \\ &\quad - D_1^k V(0,x'-y')]dy' + P(D_1,D')V(x_1,x') \\ &= \sum_{k=o}^{m-1} P(D)H_k(x_1) * (g_k - D_1^k V(0)) + f_1 + f_2 \\ &= f, \end{aligned}$$

and

$$\begin{aligned} D_1^j\phi(0,x') &= \sum_{k=o}^{m-1} D_1^j H_k(0) * (g_k - D_1^k V(0)) + D_1^j V(0,x') \\ &= D_1^j H_j(0) * (g_j - D_1^j V(0)) + D_1^j V(0,x') \\ &= g_j(x') - D_1^j V(0) + D_1^j V(0) \\ &= g_j(x') \qquad \text{for} \quad 0 \le j < m. \end{aligned}$$

In order to prove the uniqueness, let

$$\begin{aligned} P(D_1,D')\phi(x_1,x') &= 0 \qquad \text{for} \quad x_1 > 0 \\ D_1^j\phi(0,x') &= 0, \qquad \text{for} \quad 0 \le j < m, \end{aligned}$$

where $\phi \in \mathcal{E}_\omega(R^n)$. Since $P_m(N) = 0$, this implies that $D_1^j\phi(0,x') = 0$ for every integer $j \ge 0$ and $x' \in R^{n-1}$. Hence, applying Lemma 1.2, we can write $\phi = g_1 + g_2$ where supp g_1 is in $\{x | x_1 \ge 0\}$ and supp g_2 is in $\{x | x_1 \le 0\}$, and g_1, g_2 belong to $\mathcal{E}_\omega(R^n)$. And then we get

$$g_1 = g_1 * \delta = g_1 * P(D)E_2 = P(D)g_1 * E_2 = 0.$$

We have the following converse:

THEOREM 2.3. *Let $\omega \in M$. If the Cauchy problem is ω-wellposed in the direction N = (1, 0, ..., 0), then P is ω-hyperbolic with respect to N.*

Proof. Assuming $P_m(N) \neq 0$, we first prove that for any h in $\mathcal{E}_\omega(H)$, the set of all functions in $\mathcal{E}_\omega(R^n)$ with supprots in H = $\{x | <x, N> \ge$

$0\}$, there is a unique function v in $\mathcal{E}_\omega(H)$ such that P(D)v = h in R^n. Let h be any function in $\mathcal{E}_\omega(H)$. Then, from the hypothesis, there is a function ϕ in $\mathcal{E}_\omega(R^n)$ such that $P(D)\phi = h$ for $x_1 > 0$ and $D_1^j\phi = 0$ for $x_1 = 0, 0 \le j < m$. Since $P_m(N) = 0$, this implies that $D_1^{m+j}\phi = 0$ for $x_1 = 0$ and for all $j \ge 0$. Putting

$$v = \phi_0 = \begin{cases} \phi & if \quad x_1 \ge 0 \\ 0 & if \quad x_1 \le 0 \end{cases}$$

we then have $v \in \mathcal{E}_\omega(H)$ and P(D)v = h in R^n by Lemma 1.2. The uniqueness follows from the ω-wellposedness in the halfspace $x_1 \ge 0$.

We now prove that $P_m(N) \ne 0$. Suppose that $P_m(N) = 0$. Let ξ be a fixed non-zero vector in R^n for which $P_m(\xi) = 0$ and consider

$$P(sN + t\xi) = 0, \qquad s, t \in C.$$

Using Puiseux's Theorem, we have that for some positive integer p the solution of this equation is

$$t(s) = s\sum_{j=1}^{\infty} c_j(s^{-\frac{1}{p}})^j$$

which is analytic for $|s^{\frac{1}{p}}| > M$, for some constant M. Therefore, we have

$$|t(s)| \le C|s|^{1-\frac{1}{p}} \qquad \text{if} \quad |s| > M, \quad M \quad \text{suitable.}$$

Now we choose a number ρ such that $1 - \frac{1}{p} < \rho < 1$ and set, with $\tau > M$,

$$u(x) = \int_{i\tau-\infty}^{i\tau+\infty} e^{i<x, sN+t(s)\xi>} e^{-(\frac{s}{i})^\rho} ds.$$

Here we define $(\frac{s}{i})^\rho$ so that it is real and positive when s is on the positive imaginary axis, and we choose a fixed branch of $s^{\frac{1}{p}}$ in the upper half plane. Then we can prove that P(D)u = 0 and $v_1(x) = u(-x)$ is in $C^\infty(H)$ and by Theorem 1.7.3 in [2] the function $v_1 * \phi$ is in $\mathcal{E}_\omega(H)$ for all $\phi \in \mathcal{D}_\omega(H)$. We can choose $\phi_0 \in \mathcal{D}_\omega(H)$ for which $v_1 * \phi_0$ does not vanish identically. For if $\phi \in \mathcal{D}_\omega(H)$ and $\int \phi dx = 1$ with supp

$\phi \subset \overline{B(0,1)}$, then the function $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$ belongs to $\mathcal{D}_\omega(H)$. But $v_1 * \phi_\epsilon \longrightarrow v_1$ in R^n. Since v_1 does not vanish identically, this implies the assertion. Now the function $v(x) = v_1 * \phi_0$ is in $\mathcal{E}_\omega(H)$ and v does not vanish identically with P(D)v = 0, which contradicts the ω-wellposedness of the Cauchy problem. We have proved that $P_m(N) \neq 0$ and for all $h \in \mathcal{E}_\omega(H)$, there is a unique function v in $\mathcal{E}_\omega(H)$ such that P(D)v = h. This implies that the mapping $\phi \longmapsto P(D)\phi$ is a bijection from $\mathcal{E}_\omega(H)$ onto itself. Hence, by Theorems 2.1 and 2.2 in [10], P(D) is ω-hyperbolic with respect to N. The proof is complete.

REMARK. Because of the condition (v), $\omega(\xi) = log(1+|\xi|)$ can not be contained in M. But the results in Lemma 1.2 still holds in this case. So our previous results hold when $\omega(\xi) = log(1+|\xi|)$, which is the result of Gårding[5]. And the results of Larsson[8] for $\omega(\xi) = |\xi|^{\frac{1}{d}}, d > 1$, is included in our results.

We now give an example referring to [9]. Let $a_1, ..., a_n$ be n fixed real numbers such that $a_n \neq 0$. Let P(D) be a differential operator defined by

$$P(D) = a_1\frac{\partial}{\partial x_1} + ... + a_{n-1}\frac{\partial}{\partial x_{n-1}} - a_n\frac{\partial^2}{\partial x_n^2}.$$

And let $\omega(\xi) = |\xi|^{\frac{1}{2}} = (\xi_1^2+...+\xi_n^2)^{\frac{1}{4}}$. Then, by Example in [10], P(D) is ω-hyperbolic with respect to N = (0, ..., 0, 1), but not hyperbolic with respect to N in the distribution spaces provided that $a_n \neq 0$ and $a_k \neq 0$ for some $1 \leq k \leq n-1$. Hence the Cauchy problem is ω-wellposed in the direction N but not C^∞-wellposed in the direction N if $a_n \neq 0$ and $a_k \neq 0$ for some $1 \leq k \leq n-1$.

References

1. A. Beurling, *On quasi-analyticity and general distributions*, Lectures 4 and 5. A.M.S. Summer Institute, Stanford (1961).(mimeographed).
2. G. Björck, *Linear partial differential operators and generalized distributions*, Arkiv. Math. 6 (1966), 351–407.
3. R. W. Braun, R. Meise and B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, manuscript.
4. J. Chazarain and A. Piriou, *Introduction to the theory of linear partial differential equations*, Tonbridge, England (1981), 325–378.

5. L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*, Acta Math. **85** (1950), 1–62.
6. L. Hörmander, *Linear partial differential operators*, Band **116**, Berlin, Germany (1969), 114–155.
7. ———, *The analysis of linear partial differential operators II*, Grundlehren der Mathematischen Wissenschaften 257, Berlin, Germany (1983), 112–181.
8. E. Larsson, *Generalized hyperbolicity*, Arkiv för Mathematik, Band 7 nr 2, Inst. of Mathematics, Lund, Sweden (1967), 11–32.
9. S. Mizohata, *On the Cauchy problem*, **3**, Notes and Reports in Mathematics in Science and Engineering (1985), 60–62.
10. D. H. Pahk and B. H. Kang, *Hyperbolicity in the Beurling's generalized distribution spaces*, Journal of Korean Math. Soc. **27** (1990), 33–45.
11. M. C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Ec. Norm. Sup. **77**, France (1960), 41–121.

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