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E.W.BARNES' APPROACH OF THE MULTIPLE GAMMA FUNCTIONS

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In this paper we provide a new proof of multiplication formulas for the simple and double gamma functions and also give some related asymptotic expansions.

1. E.W.Barnes' definition of multiple gamma functions

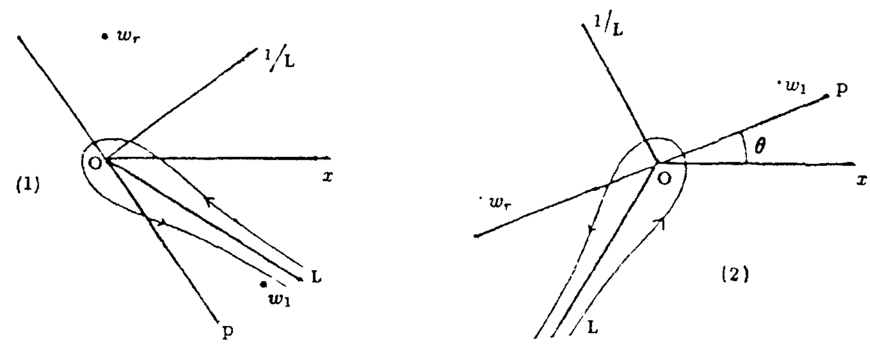
In [3] E. W. Barnes introduces the multiple Hurwitz ζ-function, for $\mathrm{Re}\, s > r$,

$$\zeta_r(s, a|w_1, \ldots, w_r) = \sum_{m_1, m_2, \ldots, m_r = 0}^{\infty} \frac{1}{(a+\Omega)^s}$$

where $\Omega = m_1 w_1 + m_2 w_2 + \cdots + m_r w_r$ and also represents the r-ple Hurwitz ζ-function by the contour integral

$$\zeta_r(s, a|w_1, w_2, \ldots, w_r) = \frac{i\Gamma(1-s)}{2\pi} \int_L \frac{e^{-az}(-z)^{s-1}}{\prod_{k=1}^{r}(1-e^{-w_k z})} dz$$

where the conditions for a and $w_1, \ldots, w_r$ are described in [3] and the possible contour L is given by



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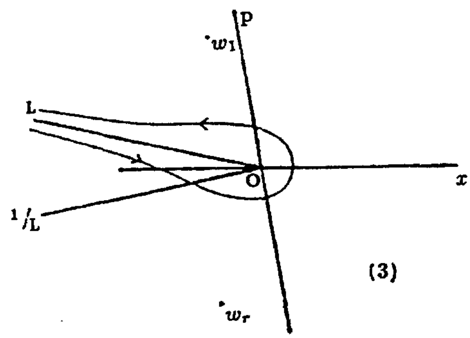
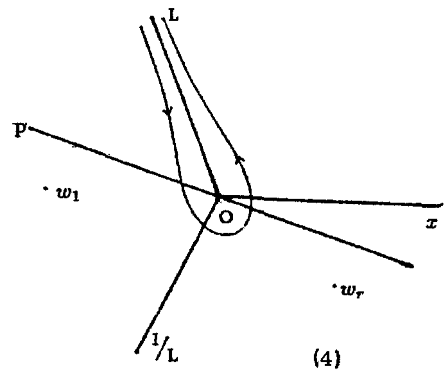


(3)



(4)

For our purpose we restrict these when $w_k = 1$, $k = 1, 2, \ldots, n$ and the contour C is the same as Fig. I in [4]. That is to say, $a > 0$, $\operatorname{Re} s > n$,

$$\zeta_n(s, a) = \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-s}.$$

Then $\zeta_n(s, a)$ can be continued to a meromorphic function with poles $s = 1, 2, \ldots, n$, $a > 0$, for by the contour integral representation

$$\zeta_n(s, a) = \frac{i\Gamma(1 - s)}{2\pi} \int_c \frac{e^{-az}(-z)^{s-1}}{(1 - e^{-z})^n} dz$$

the integral is valied for $a > 0$ and all s, so $\zeta_n(s, a)$ has possible poles only at the poles of $\Gamma(1 - s)$, i.e., $s = 1, 2, 3, \ldots$. But by the series definition $\zeta_n(s, a)$ is holomorphic for $\operatorname{Re} s > n$ [4]. In particular, when $n = 1$,

$$\zeta_1(s, a) = \sum_{k=0}^{\infty} (a + k)^{-s} = \zeta(s, a)$$

is the well-known Hurwitz ζ-function, which can be continued to a meromorphic function with only simple pole at $s = 1$ having its residue 1, by the contour integral representation [1], [4], [9].

Now we summarize some known propositions.

PROPOSITION 1.1. [7]. *Let* $\zeta(s, a) = \sum_{k=0}^{\infty} (k + a)^{-s}$ *be the Hurwitz*

ζ-function, where $a > 0$ and $Re\, s > 1$, then we have

$$\Gamma(a) = \frac{e^{\zeta'(0,a)}}{R_1}, \quad \textit{where } R_1 \textit{ is a contant and}$$
$$\zeta'(s,a) = \frac{\partial}{\partial s}\zeta(s,a).$$

Proof. As above, by the contour integral representation of $\zeta(s,a)$, $\zeta(s,a)$ is analytically continued for all $s \neq 1$, $(a > 0)$.

$$\begin{aligned}\zeta(s,a+1) &= \zeta(s,a) - a^{-s},\\ \zeta'(s,a+1) &= \zeta'(s+a) + a^{-s}\log a,\\ \zeta'(0,a+1) &= \zeta'(0,a) + \log a.\end{aligned}$$

Letting $G_1(a) = e^{\zeta'(0,a)}$, we have $G_1(a+1) = aG_1(a)$, $a > 0$, and

$$\frac{d^2}{da^2}\log G_1(a) = \frac{d^2}{da^2}\frac{d}{ds}\zeta(s,a)\big|_{s=0}.$$

Any by the analytic continuation of $\zeta(s,a)$ one sees that $G_1(a)$ is C^∞ on $\mathbf{R}^+$. So by the Bohr-Mollerup Theorem

$$G_1(a) = \Gamma(a)R_1, \ R_1 \quad \text{constant}.$$

Note that $R_1 = e^{\zeta'(0)}$ since $\zeta(s,1) = \zeta(s)$ and so

$$R_1 = G_1(1) = e^{\zeta'(0,1)}.$$

Now define

$$G_n(a) = e^{\zeta_n'(0,a)}, \text{ where } \zeta_n'(s,a) = \frac{\partial}{\partial s}\zeta_n(s,a).$$

The basic properties of $G_n(a)$ are now given by the following proposition.

PROPOSITION 1.2. [7].

(a) $G_{n+1}(a+1) = \dfrac{G_{n+1}(a)}{G_n(a)}$.

(b) $G_n(a)$ *can be continued a meromorphic function on* C *with poles at the negative integers and a simple pole at zero.*

(c) *Let* $R_n = \lim_{a\to 0} aG_n(a)$, *then* $G_n(1) = R_n/R_{n-1}$, *where* $R_0 = 1$.

In particular, when $n = 2$,

COROLLARY 1.3.

(a) $G_2(a+1) = G_2(a)/G_1(a)$.

(b) $G_2(a)$ *can be continued to a meromorphic function on* C *with poles at the negative integers and a simple pole at zero.*

Now we can get the relationship between multiple gamma functions and multiple Hurwitz ζ-functions.

PROPOSITION 1.4. [7].

$$\Gamma_n(a) = \Big(\prod_{m=1}^{n} R_{n-m+1}^{(-1)^m \binom{a}{m-1}} \Big) G_n(a).$$

Proof. See [4] and [7].

In particular, when $n = 2$.

COROLLARY 1.5.

$$\Gamma_2(a) = (R_2^{-1} R_1^a) G_2(a)$$

where $R_2 = e^{\zeta'(0)} e^{\zeta_2'(0,1)} = \lim_{a\to\infty} aG_2(a)$.

2. Multiplication formulas for Γ and Γ_2

In this section we provide other proofs of multiplication formulas for the simple and double gamma functions.

THEOREM 2.1.

$$\prod_{l=0}^{m-1}\Gamma(ka+\frac{kl}{m})=(2\pi)^{1/2m-1/2k}(\frac{k}{m})^{mak+1/2(mk-m-k)}\prod_{n=0}^{k-1}\Gamma(ma+\frac{mn}{k}),$$
$$k,m=1,2,3,\ldots.$$

Proof. Note that $\{i=0,1,2,\ldots\}=\{kj+n,\ 0\le n\le k-1,\ j=0,1,2,\ldots\}$.

$$\begin{aligned}\sum_{l=0}^{m-1}\zeta(s,ka+\frac{kl}{m}) &= \sum_{l=0}^{m-1}\sum_{i=0}^{\infty}(ka+\frac{kl}{m}+i)^{-s}\\ &=(\frac{m}{k})^s\sum_{i=0}^{\infty}\sum_{l=0}^{m-1}\frac{1}{(ma+l+\frac{m}{k}i)^s}\\ &=(\frac{m}{k})^s\sum_{l=0}^{m-1}\sum_{j=0}^{\infty}\sum_{n=0}^{k-1}\frac{1}{(ma+l+\frac{m}{k}(kj+n))^s}\\ &=(\frac{m}{k})^s\sum_{n=0}^{k-1}\sum_{j=0}^{\infty}\sum_{l=0}^{m-1}\frac{1}{(ma+\frac{mn}{k}+mj+l)^s}\\ &=(\frac{m}{k})^s\sum_{n=0}^{k-1}\sum_{j=0}^{\infty}\frac{1}{(ma+\frac{mn}{k}+j)^s}\\ &=(\frac{m}{k})^s\sum_{n=0}^{k-1}\zeta(s,ma+\frac{mn}{k}).\end{aligned}$$

$$\sum_{l=0}^{m-1}\zeta(s,ka+\frac{kl}{m})=(\frac{m}{k})^s\sum_{n=0}^{k-1}\zeta(s,ma+\frac{mn}{k}).$$

Now we have

$$\sum_{l=0}^{m-1}\zeta'(s,ka+\frac{kl}{m})$$
$$=(\frac{m}{k})^s\log(\frac{m}{k})\sum_{n=0}^{k-1}\zeta(s,ma+\frac{mn}{k})+(\frac{m}{k})^s\sum_{n=0}^{k-1}\zeta'(s,ma+\frac{mn}{k}),$$

where the accent' denotes the differentiation with respect to s. We have

$$\sum_{j=0}^{m-1} \zeta'(0, ka + \frac{kl}{m}) = \log(\frac{m}{k})\Big(\sum_{n=0}^{k-1} \zeta(0, ma + \frac{mn}{k})\Big)$$
$$+ \sum_{n=0}^{k-1} \zeta'(0, ma + \frac{mn}{k}).$$

Since $\zeta(0, a) = \frac{1}{2} - a$, we have

$$\sum_{n=0}^{k-1} \zeta(0, ma + \frac{mn}{k}) = \sum_{n=0}^{k-1}(1/2 - ma - \frac{mn}{k})$$
$$= (1/2 - ma)k - \frac{m}{2}(k-1).$$

Thus

$$\prod_{l=0}^{m-1} e^{\zeta'(0, ka + \frac{kl}{m})} = (\frac{m}{k})^{(1/2-ma)k - \frac{m}{2}(k-1)} \prod_{n=0}^{k-1} e^{\zeta'(0, ma + \frac{mn}{k})}.$$

By Proposition 1.1, $e^{\zeta'(0,a)} = e^{\zeta'(0)}\Gamma(a)$, so we have

$$e^{m\zeta'(0)} \prod_{l=0}^{m-1} \Gamma(ma + \frac{kl}{m}) = (\frac{m}{k})^{(1/2-ma)k - \frac{m}{2}(k-1)} e^{k\zeta'(0)} \prod_{n=0}^{k-1} \Gamma(ma + \frac{mn}{k}).$$

$$\prod_{l=0}^{m-1} \Gamma(ka + \frac{kl}{m}) = e^{(k-m)\zeta'(0)} (\frac{m}{k})^{(1/2-ma)k - \frac{m}{2}(k-1)} \prod_{n=0}^{k-1} \Gamma(ma + \frac{ma}{k}).$$

Note that

$$\zeta'(0) = -1/2 \log(2\pi),$$
$$e^{(k-m)\zeta'(0)} = (2\pi)^{1/2(m-k)},$$
$$(\frac{m}{k})^{(1/2-ma)k - \frac{m}{2}(k-1)} = (\frac{k}{m})^{mak + 1/2(mk-m-k)}.$$

This completes the proof of Theorem 2.1.

Now we get the classical Gauss' multiplication formula as the special case of Theorem 2.1.

COROLLARY 2.2.

$$\prod_{l=0}^{m-1} \Gamma(a+\frac{1}{m}) = (2\pi)^{1/2m-1/2} m^{1/2-ma}\Gamma(ma),\ m=2,3,4,\ldots.$$

Proof. Plug $k=1$, $m=2,3,4,\ldots$ in the formula of Theorem 2.1.

Finally we provide another proof of the multiplication formula for Γ_2.

THEOREM 2.3. [2],[4].

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1} \Gamma_2(x+\frac{i+j}{n}) = K(2\pi)^{-n(n-1)\frac{x}{2}} n^{\frac{n^2x^2}{2}-nx}\Gamma_2(nx)$$

where

$$K = A^{n^2-1} e^{\frac{1-n^2}{12}} (2\pi)^{\frac{(n-1)}{2}} n^{\frac{5}{12}}.$$

LEMMA.

$$\zeta_2(s,x) = \zeta(s-1,x) + (1-x)\zeta(s,x).$$

In particular, $\zeta_2(s,1) = \zeta(s-1)$ *and so* $\zeta_2'(0,1) = \zeta'(-1)$.

Note that the classical result is (See Chapter 1 in [4])

$$\zeta(-m,x) = -\frac{B_{m+1}(x)}{m+1}, \quad m=0,1,2,\cdots$$
$$B_0(x) = 1,$$
$$B_1(x) = x - \frac{1}{2},$$
$$B_2(x) = x^2 - x + \frac{1}{6},$$
$$\vdots$$

Proof of Lemma. Note that

$$\zeta_n(s,x) = \sum_{k_1,\ldots,k_n=0}^{\infty} (x+k_1+k_2+\cdots+k_n)^{-s} = \sum_{k=0}^{\infty} \frac{\binom{k+n-1}{n-1}}{(x+k)^s}.$$

This is because the number of solutions of $k_1 + k_2 + \cdots + k_n = k$, $k = 0, 1, 2, \ldots$, $(k_1, k_2, \ldots, k_n) \in \mathrm{N}^n$ is equal to the coefficient of x^k in the expansion of the Maclaurin series of $(1-x)^{-n}$,

$$\text{i.e.,}\quad \binom{k+n-1}{n-1}.$$

In particular,

$$\begin{aligned}
\zeta_2(s,x) &= \sum_{k_1,k_2=0}^{\infty} (x+k_1+k_2)^{-s} \\
&= \sum_{k=0}^{\infty} \frac{k+1}{(x+k)^s} \\
&= \sum_{k=0}^{\infty} \frac{(x+k)}{(x+k)^s} + \sum_{k=0}^{\infty} \frac{1-x}{(x+k)^s} \\
&= \zeta(s-1,x) + (1-x)\zeta(s,x).
\end{aligned}$$

Thus we have $\zeta_2(s,x) = \zeta(s-1,x) + (1-x)\zeta(s,x)$.

Proof of Theorem 2.3. Consider

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \zeta_2(s, x + \frac{i+j}{n}) \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k_1,k_2=0}^{\infty} (x + \frac{i+j}{n} + k_1 + k_2)^{-s} \\
&= n^s \sum_{k_1,k_2=0}^{\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (nx+i+j+nk_1+nk_2)^{-s} \\
&= n^s \sum_{k_1,k_2=0}^{\infty} (nx+k_1+k_2)^{-s} \\
&= n^s \zeta_2(s,nx).
\end{aligned}$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \zeta_2'(s, x + \frac{i+j}{n}) = (\log n) n^s \zeta_2(s,nx) + n^s \zeta_2'(s,nx),$$

where the accent' denotes the differentiation with respect to s. Therefore we have

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\zeta_2'(0,x+\frac{i+j}{n})=(\log n)\zeta_2(0,nx)+\zeta_2'(0,nx).$$

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}e^{\zeta_2'(0,x+\frac{i+j}{n})}=n^{\zeta_2(0,nx)}e^{\zeta_2'(0,nx)}.$$

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}G_2(x+\frac{i+j}{n})=n^{\zeta_2(0,nx)}G_2(nx).$$

Note that $G_2(x)=R_2R_1^{-a}\Gamma_2(x)$, by Corollary 1.5.

$$R_2^{n^2}\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}\Gamma_2(x+\frac{i+j}{n})R_1^{-(x+\frac{i+j}{n})}=n^{\zeta_2(0,nx)}R_2R_1^{-nx}\Gamma_2(nx).$$

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}R_1^{-(x+\frac{i+j}{n})}=R_1^{-\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(x+\frac{i+j}{n})}$$
$$=R_1^{-(n^2x+n^2-n)}.$$

Thus we have

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}\Gamma_2(x+\frac{i+j}{n})=F(x,n)\Gamma_2(nx)$$

where $F(x,n)=n^{\zeta_2(0,nx)}R_2^{1-n^2}R_1^{n^2x-nx+n^2-n}$.

$$R_1=e^{\zeta'(0)}=e^{-1/2\log(2\pi)}=(2\pi)^{-1/2}.$$
$$R_2=e^{\zeta'(0)}e^{\zeta_2'(0,1)}=(2\pi)^{-1/2}e^{\zeta_2'(-1)}\text{ (Lemma)}$$
$$=(2\pi)^{-1/2}A^{-1}e^{\frac{1}{12}}\text{ (see Chapter 2 in [4]).}$$

$$\begin{aligned}\zeta_2(0,nx) &= \zeta(-1,nx)+(1-nx)\zeta(0,nx)\\ &= -1/2n^2x^2+1/2nx-\frac{1}{12}+(1-nx)(1/2-nx)\\ &= 1/2n^2x^2-nx+\frac{5}{12}.\end{aligned}$$

Then we have

$$\begin{aligned}F(x,n) &= n^{1/2n^2x^2-nx+\frac{5}{12}}(2\pi)^{-1/2(1-n^2)}A^{n^2-1}e^{\frac{1}{12}(1-n^2)}\\ &\times (2\pi)^{-1/2(n^2x-nx)-1/2(n^2-n)}\\ &= K(2\pi)^{-n(n-1)\frac{x}{2}}n^{\frac{n^2x^2}{2}-nx}\end{aligned}$$

where $K=A^{n^2-1}e^{\frac{1-n^2}{12}}(2\pi)^{\frac{n-1}{2}}n^{\frac{5}{12}}$.

Therefore we have

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}\Gamma_2(x+\frac{i+j}{n})=K(2\pi)^{-n(n-1)\frac{x}{2}}n^{\frac{n^2x^2}{2}-nx}\Gamma_2(nx).$$

We make $x=0$ in the formula just obtained.

COROLLARY 2.4.

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}{}'\,\Gamma_2(\frac{i+j}{n})=\frac{K}{n}$$

where the accent' denotes that we remove the case $i=0,\ j=0$.

Proof. Note that

$$\lim_{x\to 0}\frac{\Gamma(nx)}{\Gamma(x)}=\frac{1}{n}.$$

From the first expression on $\Gamma_2^{-1}(x+1)$,

$$\frac{\Gamma(x)}{\Gamma_2(x)}=\frac{1}{\Gamma_2(x+1)}=A(x)$$

where

$$A(x)=(2\pi)^{\frac{x}{2}}e^{-\frac{x(x+1)}{2}-\gamma\frac{x^2}{2}}\prod_{k=1}^{\infty}\left((1+\frac{x}{k})^k e^{-x+\frac{x^2}{2k}}\right).$$

Then we have

$$\Gamma(x) = A(x)\Gamma_2(x), \ \lim_{x\to 0} A(x) = 1 = \lim_{x\to 0} A(nx).$$

$$\lim_{x\to 0}\frac{\Gamma_2(nx)}{\Gamma_2(x)} = \lim_{x\to 0}\frac{\Gamma(nx)}{A(nx)}\frac{A(x)}{\Gamma(x)} = \lim_{x\to 0}\frac{\Gamma(nx)}{\Gamma(x)}.$$

$$\lim_{x\to 0}\frac{xe^{\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})e^{-\frac{x}{k}}}{nxe^{\gamma nx}\prod_{k=1}^{\infty}(1+\frac{nx}{k})e^{-\frac{nx}{k}}} = \frac{1}{n}.$$

$$\prod_{i=0}^{n-1}\prod_{j=0}^{n-1}\Gamma_2(x+\frac{i+j}{n}) = \Gamma_2(x)\prod_{i=0}^{n-1}\prod_{j=0}^{n-1'}\Gamma_2(x+\frac{i+j}{n}).$$

$$\lim_{x\to 0}\prod_{i=0}^{n-1}\prod_{j=0}^{n-1'}\Gamma_2(x+\frac{i+j}{n}) = K\lim_{x\to 0}\frac{\Gamma_2(nx)}{\Gamma_2(x)} = \frac{K}{n}.$$

3. Some related asymptotic expansions

PROPOSITION 3.1.
(a) $\frac{\partial}{\partial s}\zeta(s,a)|_{s=0} = \log\Gamma(a) - 1/2\log(2\pi)$
(b) *For real* $a > 0$,

$$\frac{\partial}{\partial s}\zeta(s,a)|_{s=0} = (a-1/2)\log a - a + \frac{\theta(a)}{12a},$$

where $0 < \theta(a) < 1$.

Proof.
(a) We know that $\Gamma(a) = e^{\zeta'(0,a)}/R_1$, from Proposition 1.1, where $R_1 = e^{\zeta'(0)}$. Thus

$$e^{\zeta'(0,a)} = e^{\zeta'(0)}\Gamma(a) = (2\pi)^{-1/2}\Gamma(a) \text{ [4],[5]}.$$

$$\zeta'(0,a) = \frac{\partial}{\partial s}\zeta(s,a)|_{s=0} = \log\Gamma(a) - 1/2\log(2\pi).$$

(b) It is known [1] that for real $x > 0$,

$$\Gamma(x) = \sqrt{2\pi}\,x^{x-1/2}e^{-x}e^{\frac{\theta(x)}{12x}} \quad \text{with} \quad 0 < \theta(x) < 1.$$

Then $\log\Gamma(a) = \frac{1}{2}\log(2\pi) + (a - \frac{1}{2})\log a - a + \frac{\theta(a)}{12a}$.

Therefore, as in (a),

$$\zeta'(0, a) = (a - 1/2)\log a - a + \frac{\theta(a)}{12a},$$

for real $a > 0$, where $0 < \theta(a) < 1$.

PROPOSITION 3.2.

(a) $\frac{\partial}{\partial s}\zeta_2(s, x)|_{s=0} = \log\Gamma_2(x) + 1/2(x - 1)\log(2\pi) + \frac{1}{12} - \log A$ *where A is the Kinkelin's constant.*

(b) *For real $x > 0$,*

$$\frac{\partial}{\partial s}\zeta_2(s, x)|_{s=0} = \frac{3x^2}{4} - x - (\frac{x^2}{2} - x + \frac{5}{12})\log x + O(\frac{1}{x}), \ x \to \infty.$$

Proof. (a) We know that

$$\Gamma_2(x) = (R_2^{-1} R_1^x) e^{\zeta'(0,x)},$$

where $R_1 = e^{\zeta'(0)}$, $R_2 = e^{\zeta'(0)} e^{\zeta_2'(0,1)}$, Corollary 1.5.

Then we have

$$\begin{aligned}\zeta_2'(0, x) &= \frac{\partial}{\partial s}\zeta_2(s, x)|_{s=0} \\ &= \log\Gamma_2(x) + (\log R_2 - x\log R_1) \\ &= \log\Gamma_2(x) + (1 - x)\zeta'(0) + \zeta_2'(0, 1) \\ &= \log\Gamma_2(x) + 1/2(x - 1)\log(2\pi) + \zeta'(-1),\end{aligned}$$

since $\zeta_2'(0, 1) = \zeta'(-1)$, $\zeta'(0) = -1/2\log(2\pi)$.

$$\zeta_2'(0, x) = \log\Gamma_2(x) + 1/2(x - 1)\log(2\pi) + 1/12 - \log A$$

since

$$A = e^{1/12} - \zeta'(-1) \text{ and so } \zeta'(-1) = 1/12 - \log A.$$

(b) From Stirling's formula, for real $x > 0$,

$$\log \Gamma_2(x+1)^{-1} = \frac{x}{2}\log 2\pi - \log A + \frac{1}{12} - \frac{3x^2}{4} + (\frac{x^2}{2} - \frac{1}{12})\log x + O(\frac{1}{x}),\ x \to \infty.$$

We know that $\Gamma_2^{-1}(x+1) = \Gamma_2^{-1}(x)\Gamma(x)$.
Thus $\log \Gamma_2^{-1}(x) = \log \Gamma_2^{-1}(x+1) - \log \Gamma(x)$.
In the course of proof of (b), Proposition 3.1.

$$\log \Gamma(x) = 1/2\log(2\pi) + (x - 1/2)\log x - x + O(\frac{1}{x}),\ x \to \infty,\ x > 0.$$

Therefore we have

$$\log \Gamma_2^{-1}(x) = (\frac{x}{2} - 1/2)\log(2\pi) + (\frac{x^2}{2} - x + \frac{5}{12})\log x - \frac{3}{4}x^2 + x - \log A + \frac{1}{12} + O(\frac{1}{x}),\quad x \to \infty.$$

From (a),

$$\partial/\partial s\zeta_2(s,x)|_{s=0} = \frac{3}{4}x^2 - x - (\frac{x^2}{2} - x + \frac{5}{12})\log x + O(\frac{1}{x}),\ x \to \infty.$$

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