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NOTE ON A LOWER BOUND OF NIELSEN NUMBER

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I. Generalized Nielsen number $N_q(f)$

Let X be a topological space, and let $f : X \to X$ be a self map. A fixed point of f is a solution of the equation $x = f(x)$. The set of all fixed points of f will be denoted by Fix(f). Fixed point theory is a study of the nature of the fixed point set Fix(f) in relation to the space X and the map f, such as ; the number of fixed points #Fix(f); the behavior under homotopy; etc. But the actual number of fixed points of a self-map can be altered by arbitrarily small pertubation of the map. So, one proposes to determine the minimal number of fixed points in a homotopy class.

J. Nielsen introduced the concept of separating Fix(f) into fixed point classes and the number $N(f)$ of essential fixed point classes, now known as the Nielsen number. Nielsen theory is based on the theory of covering spaces. An alternative way is to consider nonempty fixed point classes only, and use paths instead of covering spaces to define them.

Although the Nielsen number plays an important role theoretically, its computation is no easy task. B. J. Jiang started with the concepts of liftings of self maps and used with success a subgroup of $\pi_1(X)$, now called Jiang subgroup denoted by $J(f)$.

He proved among other results the following (cf. [3], [5]) : Suppose $f_\pi(\pi_1(X)) \subset J(f)$. Then any two fixed point classes of f have the same index, and $N(f)$ can be computed by means of the Lefschetz number $L(f)$ and the Reidemeister number $R(f)$.

For a fiber map he found out some necessary and sufficient conditions that between f and the induced maps the naive product formular $N(f) = N(f_q) \cdot N(f_b)$ holds (cf. [3], [6]).

We will prove that, for a special fiber map f, $N(f)$ can be estimated by means of a certain lower bound of $N(f)$.

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DEFINITION 1.1. Let $q : X \to X_q$ be a map from a topological space X to a topological space X_q. Two fixed points x_0 and x_1 of $f : X \to X$ *belong to the same q-fixed point class* of f if there exists a path c from x_0 to x_1 such that $qfc \simeq qc$ (homotopic rel endpoints)

Note that if the map $q = 1_X : X \to X$ is the identity map then by a q-fixed point class we mean an ordinary fixed point class and if x_0, x_1 belong to the same ordinary fixed point class then x_0, x_1 belong to the same q-fixed point class for any map $q : X \to X_q$. We will use the notation F and F_q as an ordinary fixed point class and a q-fixed point class of f respectively.

LEMMA 1.1. *If $F_q \cap F \neq \phi$ then $F \subset F_q$.*

Proof. If $a_0 \in F \cap F_q$, $a_1 \in F$, then there exists a path c form a_0 to a_1 such that $c \simeq fc$(rel end points). Thus $qc \simeq qfc$.

Thus each fixed point class(empty or not) of f is contained in a unique q-fixed point class of f. Note that if F is empty, it is contained in every q-fixed point class of f in the set theoretical sense

LEMMA 1.2. *A q-fixed point class is consist of union of fixed point classes of f.*

Let $f : X \to X$ be a self map of a connected compact polyhedron. According to the definition of F and F_q, F_q is an isolated set of fixed points. Hence index (f, F_q) is defined and

$$\mathrm{index}(f, F_q) = \sum_{F \subset F_q} \mathrm{index}(f, F)$$

by the additivity property of index (cf. [3] [4], [6]).

F_q is said to be *essential* if index(f, F_q) is nonzero and is said to be *inessential* if index(f, F_q) is zero.

DEFINITION 1.2. The number of essential q-fixed point classes of f is called the *q-Nielsen number* of f, and denoted by $N_q(f)$.

Note that $N_{1_X}(f) = N(f)$.

THEOREM 1.1. *$N_q(f)$ is homotopy invariant.*

Proof. Since index$(f, F_q) = \sum_{F \subseteq F_q}$ index(f, F), by the following Lemma and Theorem 4.5 [3] we have that $N_q(f)$ is homotopy invariant.

LEMMA 1.3. *Suppose* $\{h_t\} : X \to X$ *is a homotopy of* f_0 *to* f_1. *Let* F_0, F_0' *be fixed point classes of* $h_0 = f_0$ *and* F_1, F_1' *be those of* $f_1 = h_1$ *corresponding to* F_0, F_0' *by* $\{h_t\}$. *Then* F_0, F_0' *belong to the same* q*-fixed point class of* f_0 *if and only if* F_1, F_1' *belong to the same* q*-fixed point class of* f_1.

Proof. Let $x_1 \in F_1 \subset F_q^1$, $x_1' \in F_1'$. Since F_0 corresponds to F_1 via $\{h_t\}$, by Theorem 2.9 [3], there is a path $c = \{c_t\}$ in X from $c_0 = x_0 \in F_0$ to $c_1 = x_1$ such that $\{h_t c_t\} \simeq \{c_t\}$ with end points fixed.

Note that $\{f_1 c_t\} \simeq \{c_t\} \simeq \{h_t c_t\}$ with point x_1 fixed. Indeed, if we define $H' : I \times I \to X$ by $H'(t,s) = h_{(1-t)s+t} c_t$, then $H'(t,0) = h_t c_t$, $H'(t,1) = h_1 c_t = f_1 c_t$, $H'(1,s) = h_1 c_1 = x_1$.

Similarly we can have a path $d = \{d_t\}$ from $d_0 = x_0'$ to $d_1 = x_1'$ such that $\{h_t d_t\} \simeq \{d_t\}$ with end points fixed. Also we can have $\{f_0 d_t\} \simeq \{h_t d_t\} \simeq \{d_t\}$ with point x_0 fixed.

On the other hand, x_0, $x_0' \in F_q^0$ implies that there is a path e from x_0 to x_0' such that $\{q f_0 e_t\} \simeq \{q e_t\}$ with end point fixed. Consider the path $\{g_t\} = \{c_t^{-1} * e_t * d_t\}$, then

$$\begin{aligned}
\{q f_1 g_t\} &= \{q f_1 c_t^{-1} * q f_1 e_t * q f_1 d_t\} \\
&\simeq \{q c_t^{-1} * q f_0 e_t * q d_t\} \quad \text{(with end point fixed)} \\
&\simeq \{q c_t^{-1} * q e_t * q d_t\} \quad \text{(with end point fixed)} \\
&\simeq \{q(c_t^{-1} * e_t * d_t)\} \\
&= \{q g_t\}.
\end{aligned}$$

Thus $x_1, x_1' \in F_q^1$. This completes the proof.

The next theorems follow directly from the definition and the properties of the fixed point index.

THEOREM 1.2. *Let* $f : X \to X$ *be a self map and* $q : X \to X_q$ *be a map. Then we have*

(1) $N_q(f) \leq N(f) \leq Min\{\#Fix(g) \,|\, g \simeq f\}$.
(2) $L(f) = \sum_{F_q} index(f, F_q)$.

THEOREM 1.3. *Let* $q : X \to X_q$, $q' : X \to X_{q'}$ *be maps and* $f : X \to X$ *a self map. If there exists a map* $r : X_q \to X_{q'}$ *such that* $rq = q'$, *then* $N_q(f) \geq N_{q'}(f)$.

THEOREM 1.4. *If $f : X \to X$ is a self map, then $N_f(f) = N(f)$.*

Proof. It suffices to show that $F_f \cap F \neq \emptyset$ implies $F = F_f$. Let $x_0, x_1 \in F_f$, then there is a path c from x_0 to x_1 such that $ffc \simeq fc$ (homotopy rel and point). Let d be the path fc. Then we have $fd \simeq d$. This means that $x_1, x_0 \in F$.

Let $\overline{p} : \overline{X} \to X$ be a regular covering of X. For $x_0 \in X$ and $\overline{x}_0 \in \overline{p}^{-1}(x_0)$, the subgroup $K(x_0) = \overline{p}_\pi \pi_1(\overline{X}, \overline{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$. For any path w from x_0 to x_1, $w_* : \pi_1(X, x_1) \to \pi_1(X, x_0)$ sends $K(x_1)$ onto $K(x_0)$. Hence, as in the case of the universal covering space, the base point is not of much concern. Regular covering spaces are, up to isomorphism, in $1-1$ correspondence with normal subgroups of $\pi_1(x)$, corresponding the universal covering space to the trivial subgroup. Given a subgroup K, with $\pi_1(X)$ identified with the group of covering translations on the universal covering $p : \tilde{X} \to X$, we may consider the quotient space $\tilde{X}/K$ and obtain a commutative triangle of covering maps

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{q_K} & \tilde{X}/K \\ {}_p\searrow & & \swarrow_{p_K} \\ & X & \end{array}$$

We will take $p_K : \tilde{X}/K \to X$ as the model of the covering corresponding to K. The group D_K of covering translations on this regular covering space is the quotient group $\pi_1(X)/K$.

The only obstacle to developing a theory of fixed point classes with respect to a regular covering is that not every map $f : X \to X$ can be lifted to $\tilde{f}_K : \tilde{X}/K \to \tilde{X}/K$. We know from the covering space theory that such a lifting exists iff $f_\pi(K) \subset K$. So, we restrict our attention to maps f with $f_\pi(K) \subset K$.

LEMMA 1.4.[3]. *The fixed point set Fix(f) splits into a disjoint union of mod K fixed point classes. Two fixed points x_0 and x_1 belong to the same mod K fixed point class iff there is a path c from x_0 to x_1 such that $c(fc)^{-1} \in K$.*

THEOREM 1.5. *If $q : X \to X_q$ satisfies the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X_q & \xrightarrow{\ f_q\ } & X_q \end{array},$$

then our q-fixed point classes coincide with the mod $K(= \ker q_\pi(q_\pi :$
$\pi_1(X)$
$\to \pi_1(X_q))$ fixed point classes.

The following Theorems are well known when $\ker q_\pi$ is K and can be proved with little modification.

THEOREM 1.6. *Let $f : X \to X$ be a self map with $f_\pi(\ker q_\pi) \subset \ker q_\pi$. Then the number of ordinary fixed point classes in any given q-fixed point class is greater than or equal to $\#\eta \circ \theta(\ker q_\pi)$, where $\eta \circ \theta : \pi_1(X) \to \mathrm{Coker}(1 - f_{1*} : H_1(X) \to H_1(X))$. If f is eventually commutative, then equality holds.*

THEOREM 1.7. *Suppose $f_\pi(\ker q_\pi) \subseteq J(f)$. Then any two ordinary fixed point classes in a given q-fixed point class have the same index.*

II. Nielsen numbers and fibre maps

A map $q : E \to B$ is a *fibration* if it has the homotopy lifting property. The subspace $q^{-1}(b)$ of E is called *fibre* at $b \in B$, written $F_b = q^{-1}(b)$. Let $q : E \to B$ be a fibration with E, B and all fibers compact connected polyhedra, so that we can talk about the fixed point index in E, B and fibers. Let $p_E : \tilde{E} \to E$ and $p_B : \tilde{B} \to B$ be the universal covering of E and B respectively. Let $\tilde{q} : \tilde{E} \to \tilde{B}$ be a lifting of q. Then $qp_E = p_B\tilde{q}$.

DEFINITION 2.1. Let $q : E \to B$ be a fibration. A map $f : E \to E$ is called a *fiber map* if there is an *induced map* $f_q : B \to B$ such that $qf = f_q q$. For $b \in \mathrm{Fix}(f_q)$ the map $f\,|_{F_b} : F_b \to F_b$ is well defined and will be denoted by f_b.

LEMMA 2.1.[3]. *Let $q : E \to B$ be a fibration and $\tilde{q} : \tilde{E} \to \tilde{B}$ a lifting. Let $f : E \to E$ be a fiber map inducing $f_q : B \to B$. Then every lifting $\tilde{f} : \tilde{E} \to \tilde{E}$ of f is a fiber map inducing a lifting $\tilde{f}_q : \tilde{B} \to \tilde{B}$ of f_q. Every fixed point class of f is mapped by q into a fixed point class of f_q, namely $q_{FPC}[\tilde{f}] = [\tilde{f}_q]$ and $q(p_E\text{Fix}(\tilde{f})) \subset p_B\text{Fix}(\tilde{f}_q)$. Furthermore, for every fixed point class of f_q there exists a fixed point class of f that is mapped by q into it.*

In fact $q_{FPC}^{-1}[\tilde{f}_q] = \{[\tilde{f}] \mid [\tilde{f}]$ is fixed point class contained in the q-fixed point class $[\tilde{f}]_q\}$, i.e., $q_{FPC_K}[\tilde{f}]_q = [\tilde{f}_q]$ where $k = \ker(q_\pi : \pi_1(E) \to \pi_1(B))$.

The following lemma is very useful in computing the fixed point index in fiber spaces.

LEMMA 2.2.[3]. *Let $q : E \to B$ be a fibration and let $f : E \to E$ be a fiber map inducing $f_q : B \to B$. Let $\tilde{f} : \tilde{E} \to \tilde{E}$ be a lifting of f inducing $\tilde{f}_q : \tilde{B} \to \tilde{B}$. Then any $b \in p_B\text{Fix}(\tilde{f}_q)$, we have*

$$\text{index}(f, p_E\text{Fix}(\tilde{f})) = \text{index}(f_q, p_B\text{Fix}(\tilde{f}_q)) \cdot \text{index}(f_b, F_b \cap p_E\text{Fix}(\tilde{f})).$$

THEOREM 2.1.[3]. *A fixed point class of f in E is essential iff its projection in B is an essential fixed point class of f_q and its intersection with an invariant fiber F_b consists of essential fixed point classes of f_b.*

THEOREM 2.2. *Let $q : E \to B$ be a fibration with E, B and all fibers compact connected polyhedra. Let $f : E \to E$ be a fiber map inducing $f_q : B \to B$. Then $N_q(f) \leq N(f_q) \leq N(f)$. Moreover, if $f_\pi(\ker q_\pi) \subset J(f)$, then $N_q(f) = N(f_q)$.*

Proof. Let F_i', $i = 1, 2, \ldots, N(f_q)$, be essential fixed point classes of f_q. According to Theorem 2.1, each essential fixed point class of f projects to an essential fixed point class of f_q. Let $F_{i1}, \ldots, F_{ic_i}$ be essential fixed point classes of f lying above F_i'. Note that these F_{ij}, $j = 1, \ldots, c_i$ are contained in the same q-fixed point class F_q^i by Lemma 2.1. Thus we have

$$N(f) = \sum_{i=1}^{N(f_q)} c_i \geq \sum_{i=1}^{N(f_q)} 1 = N(f_q).$$

Moreover since

$$\begin{aligned}\operatorname{index}(f, F_q^i) &= \sum_{j=1}^{c_i} \operatorname{index}(f, F_{ij}) \\ &= \sum_{j=1}^{c_i} \operatorname{index}(f_q, F_i') \cdot \operatorname{index}(f_b, F_b \cap F_{ij}),\end{aligned}$$

for any essential q-fixed point class, there exists at least one essential fixed point class F_{ij}. Thus F_i' is also essential fixed point class of f_q. This implies $N_q(f) \leq N(f_q)$.

Now we will prove the last paragraph. Since $f_\pi(\ker q_\pi) \subseteq J(f)$, by Theorem 1.7, any two ordinary fixed point classes in a given q-fixed point class have the same index. If F_i' is essential, then by Theorem 2.1, we have an essential fixed point class F_{ij} so that the q-fixed point class F_q^i containing F_{ij} is also essential. Thus we have $N(f_q) \leq N_q(f)$. This completes the proof.

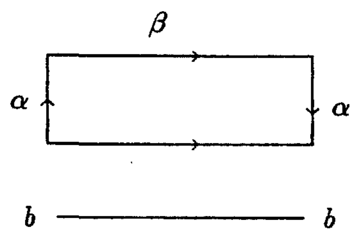
COROLLARY 2.3. *Let $q : E \to B$ be a fibration with E, B and all fibers compact connected polyhedra. Let $f : E \to E$ be a fiber map inducing $f_q : B \to B$. Let the base space B be a Jiang space. Then $N_q(f) \leq \#Coker(1 - f_{q_1^*}) \leq N(f)$. Moreover if $f_\pi(\ker q_\pi) \subseteq J(f)$, then $N_q(f) = \#Coker(1 - f_{q_1^*})$.*

DEFINITION 2.2. A fibration $q : E \to B$ is said to be (homotopically) *orientable* if for any two paths w, w' in B with same endpoints, the fiber translations $\tau_w \simeq \tau_{w'} : F_{w(0)} \to F_{w(1)}$.

COROLLARY 2.4. *Let $q : E \to B$ be orientable. Let $f : E \to E$ be a fiber map such that f and $f_b : F_b \to F_b$ are eventually commutative for $b \in Fix(f_q)$. Let $P(f)$ be the Pak number [3] of the square*

$$\begin{array}{ccc} F_b & \xrightarrow{f_b} & F_b \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & E \end{array}$$

Then $P(f) \leq N(f_b)$.



EXAMPLE. Let K^2 be the Klein bottle. A fibration $q : K^2 \to S^1$ is shown in the figure, where K^2 is obtained by identifying the opposite side of a rectangle as indicated and α, β are generators of $\pi_1(K^2)$ with $\alpha\beta\alpha = \beta$.

Let the fibre map $f : K^2 \to K^2$ induce the homomorphism $f_\pi(\alpha) = 1$, $f_\pi(\beta) = \beta^d$. Then the induced map $f_q : S^1 \to S^1$ has degree d, and

$$N_q(f) \leq N(f_q) = |1 - d| \leq N(f).$$

Since $N(f) \leq R(f)$, if we show that $R(f) = |1 - d|$, then we have $N(f) = |1 - d|$. Recall that in order to compute $R(f)$ we must find, given $\mu, \nu \in \pi_1(K^2)$, necessary and sufficient conditions for he existence of $\gamma \in \pi_1(K^2)$ such that $\mu = \gamma\nu f_\pi(\gamma^{-1})$. Since $f_\pi(\pi_1(K^2))$ is generated by β^d, $f_\pi(\gamma^{-1}) = \beta^{kd}$ for some integer k. Hence $\mu = \gamma\nu\beta^{kd}$. Let $w \in \pi_1(K^2)$ by any element represented by

$$w = \alpha^{p(1)}\beta^{p(2)} \cdots \alpha^{p(2r-1)}\beta^{p(2r)}$$

for some $p(j) \in Z$, $r \geq 1$. Define $|w| = \sum_{q=1}^{r} p(2q)$. Then $|w|$ is independent of any representation of w. Note that $|w^{-1}| = -|w|$ and $f_\pi = \beta^{d|w|}$. Applying f_π to both sides of equation $\mu = \gamma\nu\beta^{dk}$ gives $\beta^{d|\mu|} = \beta^{-d_\beta d|\nu|_\beta d(dk)}$ or $\beta^{|\mu|-|\nu|} = \beta^{k(d-1)}$. We have proved that $\mu = \gamma\nu f_\pi(\gamma^{-1})$ for some γ iff $|\mu| \equiv |\nu| \ (mod\ |1-d|)$, and $R(f) = |1-d|$. Moreover since $f_\pi(\ker q_\pi) = \{1\} \subseteq J(f)$, we have $N_q(f) = N(f_q) = N(f) = |1 - d|$.

REMARK. Brown [1] proved that in above example $N(f) = R(f) = |1 - d|$ if d is even, using the fact $N(f) = R(f)$ if $f_\pi(\pi_1(K^2)) \subset Z(\pi_1(K^2))$.

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