J. Korean Math. Soc. **29** (1992), No. 1, pp. 91–115

THE SUBHARMONIC BIFURCATION IN AREA-PRESERVING MAPS

YONG IN KIM* AND EOK KYUN LEE**

1. Introduction

Many authors [1, 2, 3] have studied two-dimensional area-preserving maps as typical discrete versions of nonintegrable autonomous Hamiltonian systems with two degrees of freedom or nonautonomous systems with one degree of freedom. In particular, Van der Weele [3] has performed a bifurcation analysis based on Meyer [4] to prove the appearance of n-cycles from the elliptic fixed point at each resonance values.

The purpose of this paper is to present mathematically more clear and general methods to analyze the pattern of n-cycles bifurcating from the origin by means of the theory of normal forms [5, 6, 7, 8, 9] and the Liapunov-Schmidt method [10, 11].

Our bifurcation analysis can be compared with that of Hopf-bifurcation [7], however, the assumptions on the linear part of a map are quite different from each other. In the case of Hopf bifurcation, the complex conjugate eigenvalues of the linear part of a map cross the unit circle transversally as a parameter varies through 0, whereas in our case those eigenvalues move along the unit circle due to the area-preserving property of the map.

The main point of our analysis is that even if the normal form of an area-preserving map may not be area-preserving, the orbits, especially the n-cycles of the area-preserving map are locally diffeomorphic to those of the normal form, because the nonlinear change of coordinates leading to the normal form is a μ-dependent local diffeomorphism.

Received February 21, 1991. Revised July 8, 1991.

This work was supported by NON DIRECTED RESEARCH FUND, Korea Reserch Foundation* and Korea Science and Engineering Foundation**.

2. The Normal form of an area-preserving map

Consider a two-dimensional area-preserving map $\mathbf{R}^2 \to \mathbf{R}^2$ of the following form [12, 13]

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2c & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(x) \\ 0 \end{bmatrix}, \tag{2.1}$$

where $f(x) = \sum_{k=2}^{\infty} a_k x^k$ is of class C^∞ and c is a real parameter. For any $c \in \mathbf{R}$, the origin is a fixed point of (2.1) and its stability is determined by the eigenvalues

$$\lambda_\pm = c \pm \sqrt{c^2 - 1}$$

of the linear part. Note that $\lambda_+ \cdot \lambda_- = 1$, in agreement with the area preserving condition of (2.1). For $|c| \leq 1$, the eigenvalues lie on the unit circle, complex conjugate to each other and the origin becomes an elliptic fixed point.

Introducing a new parameter $\mu \in \mathbf{R}$ by writing

$$c \pm i\sqrt{1 - c^2} = e^{\pm 2\pi i(\theta_0 + \mu)},$$

we have

$$c = c(\mu) = \cos 2\pi(\theta_0 + \mu), \tag{2.2}$$

where $\theta_0 = \dfrac{m}{n}$ with m and n relatively prime integers. Then, we can rewrite (2.1) in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = G_\mu(x, y) \equiv A_\mu \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(x) \\ 0 \end{bmatrix}, \tag{2.3}$$

where

$$A_\mu = D_{(x,y)} G_\mu(0,0) = \begin{bmatrix} 2\cos 2\pi(\theta_0 + \mu) & -1 \\ 1 & 0 \end{bmatrix} \in \mathbf{R}^{2\times 2}.$$

Let $\lambda(\mu)$ and $\bar{\lambda}(\mu)$ be the eigenvalues of A_μ and let $\lambda_0 = \lambda(0)$. Then we have

$$\lambda(\mu) = e^{2\pi i(\theta_0 + \mu)} = \lambda_0 e^{2\pi i \mu}, \qquad \lambda_0 = e^{2\pi i \theta_0}. \tag{2.4}$$

Notice that the eigenvalues move along the unit circle as μ varies through 0.

Now, we make a linear change of variables

(2.5)
$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \text{where}$$
$$P = \begin{bmatrix} 0 & 1 \\ -\sin 2\pi(\theta_0+\mu) & \cos 2\pi(\theta_0+\mu) \end{bmatrix},$$

to put (2.3) in the standard form,

(2.6)
$$\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = F_\mu(\xi,\eta) \equiv \begin{bmatrix} \cos 2\pi(\theta_0+\mu) & -\sin 2\pi(\theta_0+\mu) \\ \sin 2\pi(\theta_0+\mu) & \cos 2\pi(\theta_0+\mu) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + f(\eta) \begin{bmatrix} \cot 2\pi(\theta_0+\mu) \\ 1 \end{bmatrix},$$

where $F_\mu = P^{-1} \cdot G_\mu \cdot P$.

And again, by setting $z = \xi + i\eta$ and $\bar{z} = \xi - i\eta$ in (2.6), we can obtain the two-dimensional area-preserving map *in complex form*

$$z' = F_\mu(z) = \lambda(\mu) z + \frac{\lambda(\mu)}{Im\lambda(\mu)} \cdot f\left(\frac{z-\bar{z}}{2i}\right), \quad F_\mu : \mathbf{C} \to \mathbf{C}, \tag{2.7}$$

where $\lambda(\mu) = \lambda_0 e^{2\pi i\mu} = \lambda_0(1 + 2\pi i\mu + \mathcal{O}(|\mu|^2))$, and

$$f\left(\frac{z-\bar{z}}{2i}\right) = \sum_{k=2}^{\infty} a_k \left(\frac{z-\bar{z}}{2i}\right)^k. \tag{2.8}$$

Let us write (2.7) in the form

$$z' = F_\mu(z) = \lambda(\mu) z + R(\mu, z, \bar{z}), \tag{2.9}$$

where $R(\mu,z,\bar{z}) = R_2(\mu,z,\bar{z}) + R_3(\mu,z,\bar{z}) + \ldots$. with $R_l(\mu,z,\bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q, l \geq 2$. Then, from (2.8), the coefficients $c_{pq}(\mu)$ are given by

(2.10)
$$c_{20}(\mu) = -\frac{a_2}{4} m(\mu), \quad c_{11}(\mu) = \frac{a_2}{2} m(\mu), \quad c_{02}(\mu) = -\frac{a_2}{4} m(\mu),$$
$$c_{30}(\mu) = \frac{ia_3}{8} m(\mu), \quad c_{21}(\mu) = -\frac{3ia_3}{8} m(\mu), \quad c_{12}(\mu) = \frac{3ia_3}{8} m(\mu),$$
$$c_{03}(\mu) = -\frac{ia_3}{8} m(\mu), \quad \ldots$$

with $m(\mu) = \lambda(\mu)/\operatorname{Im}\lambda(\mu)$.

Finally, we put (2.9) in a normal form by means of a μ-dependent change of coordinates of the following form

$$z = w + \psi(\mu, w, \bar{w}) \equiv T_\mu(w), \tag{2.11}$$

where

$$\psi(\mu, w, \bar{w}) = \psi_2(\mu, w, \bar{w}) + \psi_3(\mu, w, \bar{w}) + \dots, \text{ and } \psi_l(\mu, w, \bar{w}) = \sum_{p+q=l} \psi_{pq}(\mu) w^p \bar{w}^q, \ell \geq 2,$$

with a suitable choice of the coefficients $\psi_{pq}(\mu)$.

Then the new map in w becomes

$$w' = \tilde{F}_\mu(w) = (T_\mu^{-1} \circ F_\mu \circ T_\mu)(w).$$

According to the theory of normal forms for maps [5, 6, 7, 8, 9], we can obtain the normal forms of $F_\mu(z)$ as given in the following.

LEMMA 1. *Let* $F_\mu(z) = \lambda(\mu)z + R_2(\mu, z, \bar{z}) + R_3(\mu, z, \bar{z}) + \dots$ *with*

$$R_l(\mu, z, \bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q, \quad \ell \geq 2 \quad \text{and}$$

$$\lambda(\mu) = \lambda_0 e^{2\pi i \mu}, \quad \text{where} \lambda_0 = e^{2\pi i \theta_0}.$$

Then there exists a μ*-dependent local diffeomorphism of the form (2.11) which transforms the map* $F_\mu(z)$ *to the following normal forms:*

(i) *when* $\theta_0 = \frac{1}{3}$

$$\tilde{F}_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + \eta_{21}(\mu)z^2\bar{z} + \eta_{40}(\mu)z^4 + \eta_{13}(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5)$$

If $c_{02}(0) = 0$, *the term* $\bar{z}^2$ *can be removed in the normal form.*

(ii) *when* $\theta_0 = \frac{1}{4}$

$$\tilde{F}_\mu(z) = \lambda(\mu)z + \eta_{21}(\mu)z^2\bar{z} + \eta_{03}(\mu)\bar{z}^3 + \eta_{05}(\mu)\bar{z}^5 + \eta_{14}(\mu)z\bar{z}^4 + \mathcal{O}(|z|^7)$$

(iii) *when* $\theta_0 = \frac{1}{n}$ $(n \geq 5)$

$$\tilde{F}_\mu(z) = \lambda(\mu)z + \eta_{21}(\mu)z^2\bar{z} + \eta_{0,n-1}(\mu)\bar{z}^{n-1} + \eta_{32}(\mu)z^3\bar{z}^2 + \mathcal{O}(|z|^7 + |z|^n)$$

The coefficients η_{ij} can be calculated from those of $F_\mu(z)$ as follows;

$$\eta_{21}(0) = c_{21}(0) + \frac{|c_{11}(0)|^2}{1-\bar\lambda_0} + \frac{2|c_{02}(0)|^2}{\lambda_0^2-\bar\lambda_0} + \frac{2\lambda_0-1}{\lambda_0(1-\lambda_0)}c_{11}(0)\cdot c_{20}(0)$$

$$\eta_{03}(0) = c_{03}(0) + \frac{c_{11}(0)c_{02}(0)}{\bar\lambda_0^2-\lambda_0} + \frac{2c_{02}(0)\cdot\bar c_{20}(0)}{\bar\lambda_0^2-\bar\lambda_0}.$$

Furthermore, writing $\tilde F_\mu(z) = \lambda(\mu)z + \tilde R(\mu, z, \bar z)$, $\tilde R(\mu, z, \bar z)$ satisfies

$$\tilde R(\mu, \lambda_0 z, \bar\lambda_0 \bar z) = \lambda_0 \tilde R(\mu, z, \bar z)$$

Proof. See [5], [6], [7].

As we mentioned in the introduction, the orbits of $F_\mu(z)$ are locally diffeomorphic to those of $\tilde F_\mu(z)$. Hence it is sufficient to examine the n-cycles of $\tilde F_\mu(z)$. From now on, we write $F_\mu(z)$ for $\tilde F_\mu(z)$ for notational simplicity.

3. The Liapunov-Schimdt method [7]

Assume that $\lambda_0^n = 1 (\theta_0 = \frac{1}{n})$ for $n \geq 3$. Let $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$ be a n-cycle of the map $F_\mu(z)$, that is

$$\begin{aligned} F_\mu(x_1) &= x_2 \\ F_\mu(x_2) &= x_3 \\ &\ldots \\ F_\mu(x_n) &= x_1 \end{aligned} \tag{3.1}$$

where $F_\mu(z)$ is in normal form as is given in Lemma 1. Let

$$\begin{aligned} \mathcal{F}_\mu(x) &= \begin{bmatrix} F_\mu(x_1) \\ \ldots \\ F_\mu(x_n) \end{bmatrix} = \begin{bmatrix} \lambda(\mu)x_1 + R(\mu, x_1, \bar x_1) \\ \ldots \\ \lambda(\mu)x_n + R(\mu, x_n, \bar x_n) \end{bmatrix} \\ &= \lambda(\mu)x + \mathcal{R}(\mu, x, \bar x), \end{aligned} \tag{3.2}$$

and

$$S = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & & & \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Then (3.1) can be written as

$$(3.3) \qquad Sx = \mathcal{F}_\mu(x), \qquad \left(\mathcal{F}_\mu \in C^\infty(\mathbf{C}^n, \mathbf{C}^n), \qquad S \in \mathbf{C}^{n\times n}\right).$$

To diagonalize S, we make a linear change of coordinates

$$(3.4) \qquad y = Px, \qquad y = (y_1, \dots, y_n) \in \mathbf{C}^n,$$

where

$$P = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \lambda_0 & \lambda_0^2 & \dots & \lambda_0^{n-1} \\ \dots & \dots & & & \\ 1 & \lambda_0^{n-1} & \lambda_0^{2(n-1)} & \dots & \lambda_0^{(n-1)(n-1)} \end{bmatrix}.$$

Then (3.3) is reduced to the equation

$$(3.5) \qquad \Lambda y = P\mathcal{F}_\mu(P^{-1}y),$$

where $\Lambda = \operatorname{diag}(1, \bar{\lambda}_0, \dots, \bar{\lambda}_0^{n-1})$.

If we define the map $\Phi : \mathbf{C}^n \times \mathbf{R} \to \mathbf{C}^n$ by

$$(3.6) \qquad \begin{aligned} \Phi(y, \mu) &= P\mathcal{F}_\mu(P^{-1}y) - \Lambda y \\ &= [\lambda(\mu)I - \Lambda]y + P\mathcal{R}(\mu, P^{-1}y, \overline{P^{-1}y}), \end{aligned}$$

and let

$$(3.7) \quad L \equiv D_y\Phi(0,0) = \lambda_0 I - \Lambda = \operatorname{diag}(\lambda_0 - 1, \lambda_0 - \bar{\lambda}_0, \dots, \lambda_0 - \bar{\lambda}_0^{n-1}),$$

then, since $\lambda_0 - \bar{\lambda}_0^{n-1} = 0$, L has rank $n-1$, and $\ker L$ is a one dimensional subspace of $\mathbf{C}^n$ as

$$(3.8) \qquad \ker L = \{y_n v_n | y_n \in \mathbf{C}, v_n = (0, \dots, 0, 1)^\top \in \mathbf{C}^n\}.$$

By writing

$$\mathbf{C}^n = \ker L \oplus (\ker L)^\perp,$$

any $y \in \mathbf{C}^n$ can be written as

$$(3.9) \qquad y = y_n v_n + w, \text{ where } v_n \in \ker L \text{ and } w \in (\ker L)^\perp = \operatorname{Im} L.$$

Let $E : \mathbf{C}^n \to (\ker L)^\perp$ be the projection. Then, $I - E : \mathbf{C}^n \to \ker L$, $Ey = (y_1, \dots, y_{n-1}, 0)^\top = w$, and $(I-E)y = y_n v_n$. Also, we can easily notice that $E, I-E$ and L commute each other. Consequently, the equation $\Phi(y, \mu) = 0$ is equivalent to the following pair of equations

$$(3.10) \qquad \begin{cases} E\Phi(y_n v_n + w, \mu) = 0 & \text{(a)} \\ (I-E)\Phi(y_n v_n + w, \mu) = 0 & \text{(b)} \end{cases}.$$

Notice that (3.10) (a) is uniquely solvable for w as a function of (y_n, μ) near (0, 0) by the implicit function theorem. Denoting $w = w^*(y_n, \mu)$, we can easily verify that [7]

$$(3.11) \qquad \begin{cases} w^*(y_n, \mu) = \mathcal{O}(|\mu||y_n| + |y_n|^2) \\ w^*(\lambda_0 y_n, \mu) = \Lambda w^*(y_n, \mu) \end{cases}.$$

After substituting $w^*(y_n, \mu)$ into (3.10) (b), we define a function $\gamma : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$ by

$$(3.12) \qquad \gamma(y_n, \mu) = \langle (I-E)\Phi(y_n v_n + w^*(y_n, \mu), \mu), v_n \rangle .$$

Then, solutions (y, μ) of $\Phi(y, \mu) = 0$ are locally one-to-one correspondence to the solutions (y_n, μ) of $\gamma(y_n, \mu) = 0$ via the relation

$$(3.13) \qquad y = y_n v_n + w^*(y_n, \mu).$$

From (3.6), we have

$$(3.14) \qquad \begin{aligned} \gamma(y_n, \mu) &= \langle \Phi(y_n v_n + w^*(y_n, \mu), \mu), v_n \rangle \\ &= (\lambda(\mu) - \lambda_0) y_n + \left\langle P\mathcal{R}(\mu, P^{-1}y, \overline{P^{-1}}y), v_n \right\rangle , \end{aligned}$$

where y is given in (3.13).

LEMMA 2. *Let $z = \frac{1}{n} y_n$. Then the equation $\gamma(y_n, \mu) = 0$ is equivalent to the following equation in $\mathbf{C}$:*

$$(3.15) \qquad \lambda_0 z = F_\mu(z) = \lambda(\mu) z + R(\mu, z, \bar{z}),$$

where $F_\mu(z)$ is in normal form.

Proof. Letting $z = \frac{1}{n}y_n$ in (3.14), $\gamma(y_n, \mu) = 0$ becomes

$$\lambda_0 z = \lambda(\mu)z + \frac{1}{n}\left\langle P\mathcal{R}(\mu, P^{-1}y, \overline{P^{-1}}y), v_n\right\rangle.$$

Recall that $F_\mu(z) = \lambda(\mu)z + R(\mu, z, \bar{z})$, where $R(\mu, z, \bar{z})\Big(= \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1})\Big)$ is in normal form up to order r and hence $\tilde{R}(\mu, \lambda_0 z, \bar{\lambda}_0\bar{z}) = \lambda_0\tilde{R}(\mu, z, \bar{z})$. From (3.2) and (3.4) we have

$$\begin{aligned}\left\langle P\mathcal{R}(\mu, P^{-1}y, \overline{P^{-1}}y), v_n\right\rangle &= R(\mu, x_1, \bar{x_1}) + \bar{\lambda}_0 R(\mu, x_2, \bar{x_2}) + \dots \\ &\quad + \bar{\lambda}_0^{n-1}R(\mu, x_n, \bar{x}_n),\end{aligned}$$

where $\{x_1, \dots, x_n\}$ is the n-cycle for the system (3.1) given by

$$\begin{aligned} x_k = (P^{-1}y)_k &= [P^{-1}(y_n v_n + w^*(y_n, \mu))]_k \\ &= \frac{1}{n}\sum_{j=1}^{n-1}\bar{\lambda}_0^{(k-1)(j-1)}w_j^*(y_n, \mu) + \frac{1}{n}\bar{\lambda}_0^{(k-1)(n-1)}y_n \\ &= \frac{1}{n}\sum_{j=1}^{n-1}\bar{\lambda}_0^{(k-1)(j-1)}w_j^*(nz, \mu) + \lambda_0^{k-1}z \qquad (k = 1, 2, \dots, n).\end{aligned}$$

Note that if we write

$$x_1 = \varphi_\mu(z) \equiv z + \frac{1}{n}\sum_{j=1}^{n-1}w_j^*(nz, \mu) = z + \mathcal{O}(|\mu||z| + |z|^2), \tag{3.16}$$

then the other n-periodic points $x_2, \dots, x_n$ can be obtained from the property (3.11) as

$$x_2 = \varphi_\mu(\lambda_0 z), x_3 = \varphi_\mu(\lambda_0^2 z), \dots, x_n = \varphi_\mu(\lambda_0^{n-1}z). \tag{3.17}$$

Then we have

$$\begin{aligned} R(\mu, x_1, \bar{x_1}) &= R(\mu, \varphi_\mu(z), \bar{\varphi_\mu}(z)) \\ &= \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1}) \\ \bar{\lambda}_0 R(\mu, x_2, \bar{x}_2) &= \bar{\lambda}_0 R(\mu, \varphi_\mu(\lambda_0 z), \bar{\varphi}_\mu(\lambda_0 z)) \\ &= \bar{\lambda}_0 R(\mu, \lambda_0 z + \mathcal{O}(|\mu||z| + |z|^2), \bar{\lambda}_0\bar{z} + \mathcal{O}(|\mu||z| + |z|^2)) \\ &= \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1}).\end{aligned}$$

Similarly, $\bar{\lambda}_0^{n-1} R(\mu, x_n, \bar{x}_n) = \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1})$. Therefore

$$\frac{1}{n}\left\langle P\mathcal{R}(\mu, P^{-1}y, \overline{P^{-1}y}), v_n \right\rangle = \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1}).$$

Consequently, $g(y_n, \mu) = 0$ is equivalent to

$$\lambda_0 z = \lambda(\mu) z + \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1}) = F_\mu(z)$$

Thus, our problem to find the n-periodic fixed points for the area-preserving map $F_\mu(z)$ written in the normal form has been reduced to solving the scalar equation (3.15) and the coordinates of the n-periodic fixed points are given by (3.16) and (3.17).

4. Bifurcation Analysis of n-cycles [11]

(i) The case $n = 3$ and $a_2 \neq 0$

In this case, $\lambda_0 = e^{2\pi i/3}$ and from the Lemma 1, we have,

$$F_\mu(z) = \lambda(\mu) z + c_{02}(\mu)\bar{z}^2 + \mathcal{O}(|z|^3), \tag{4.1}$$

where $\lambda(\mu) = \lambda_0(1 + \mu\lambda_1 + \mathcal{O}(|\mu|^2))$ with $\lambda_1 = 2\pi i$ and

$$c_{02}(0) = -\frac{a_2}{2\sqrt{3}}\lambda_0. \tag{4.2}$$

Then (3.15) becomes

$$\mu\lambda_1 z + \bar{\lambda}_0 c_{02}(0)\bar{z}^2 + \mathcal{O}(|\mu|^2|z| + |\mu||z|^2 + |z|^3) = 0.$$

Let $z = re^{2\pi i\varphi}$. Then we have,

$$\mu\lambda_1 r e^{2\pi i\varphi} + \bar{\lambda}_0 c_{02}(0) r^2 e^{-4\pi i\varphi} + g_1(\mu, re^{2\pi i\varphi}, re^{-2\pi i\varphi}) = 0,$$

where $g_1(\mu, re^{2\pi i\varphi}, re^{-2\pi i\varphi}) = \mathcal{O}(|\mu|^2 r + |\mu| r^2 + r^3)$.

Separating the trivial solution $r = 0$, we have

$$\mu\lambda_1 + \bar{\lambda}_0 c_{02}(0) r e^{-6\pi i\varphi} + g(\mu, r, \varphi) = 0,$$

or

$$2\pi i\mu - \frac{a_2}{2\sqrt{3}} e^{-6\pi i\varphi} r + g(\mu, r, \varphi) = 0, \tag{4.3}$$

where

$$\begin{aligned} g(\mu, r, \varphi) &= r^{-1} e^{-2\pi i\varphi} g_1(\mu, re^{2\pi i\varphi}, re^{-2\pi i\varphi}) \\ &= \mathcal{O}(|\mu|^2 + |\mu|r + r^2). \end{aligned}$$

Note that $g(\mu, r, \varphi)$ has the following property

$$g(\mu, r, \varphi + \frac{1}{3}) = g(\mu, r, \varphi).$$

Now, if we set

$$\begin{cases} r = 4\pi\sqrt{3}|\frac{\mu}{a_2}|(1 + r_1) \\ \varphi = \varphi_0 + \varphi_1, \qquad \varphi_0 = -\frac{1}{6\pi} arg(\frac{i\mu}{a_2}) \qquad (\text{mod } \frac{1}{3}), \end{cases} \tag{4.4}$$

substituting (4.4) in (4.3) and simplifying (4.3), we have

$$\mu - \mu e^{-6\pi i\varphi_1}(1 + r_1) + g(\mu, 4\pi\sqrt{3}|\frac{\mu}{a_2}|(1 + r_1), \varphi_0 + \varphi_1) = 0.$$

Set

$$h(\mu, r_1, \varphi_1) = 1 - e^{-6\pi i\varphi_1}(1 + r_1) + g_2(\mu, r_1, \varphi_1),$$

where $g_2(\mu, r_1, \varphi_1) = \mu^{-1} g(\mu, 4\pi\sqrt{3}|\frac{\mu}{a_2}|(1 + r_1), \varphi_0 + \varphi_1) = \mathcal{O}(|\mu|)$.

Since

$$h(0,0,0) = 0, \quad \frac{\partial h}{\partial r_1}(0,0,0) = -1, \quad \frac{\partial h}{\partial \varphi_1}(0,0,0) = 6\pi i,$$

by the implicit function theorem, we have

$$r_1 = r_1(\mu), \quad r_1(0) = 0, \quad \varphi_1 = \varphi_1(\mu), \quad \varphi_1(0) = 0.$$

Consequently, we have

$$\begin{cases} r = 4\pi\sqrt{3}|\frac{\mu}{a_2}| + \mathcal{O}(|\mu|^2) \\ \varphi = -\frac{1}{6\pi} arg(\frac{i\mu}{a_2}) + \mathcal{O}(|\mu|) \quad (\text{mod } \frac{1}{3}). \end{cases} \tag{4.5}$$

and the coordinate of the 3-periodic points for the area-preserving map $F_\mu(z)$ in normal form is given, from (3.16), (3.17), (4.5), by

$$\text{(4.6)}\qquad \begin{cases} x_1 = \varphi_\mu(z) \equiv z(\mu) + \mathcal{O}(|\mu||z| + |z|^2) \\ \quad = r(\mu)e^{2\pi i \varphi(\mu)} + \mathcal{O}(|\mu|^2) \\ \quad = 4\pi\sqrt{3}|\frac{\mu}{a_2}|e^{2\pi i \varphi_0} + \mathcal{O}(|\mu|^2), \\ x_2 = \varphi_\mu(\lambda_0 z) \\ x_3 = \varphi_\mu(\lambda_0^2 z), \end{cases}$$

where $\varphi_0 = -\frac{1}{6\pi}arg(\frac{i\mu}{a_2})$ and $\lambda_0 = e^{2\pi i/3}$.

Notice that as μ varies from $\mu < 0$ to $\mu > 0, \arg(\dfrac{i\mu}{a_2})$ changes by π, and hence the orientation of the 3-cycle is reversed as μ crosses 0 (Fig. 1).

Also note that the 3-cycle of the original area-preserving map (2.9) is given in the same form as in (4.6), since, for μ near 0, the original map (2.9) is transformed to the normal forms via the near-identity transformation of the form (2.11).

To examine the stability of the 3-cycle for the map

$$F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + \mathcal{O}(|z|^3),$$

we consider the map

$$F_\mu^3(z) = [1 + 3\mu\lambda_1 + \mathcal{O}(|\mu|^2)]z + 3c_{02}(0)\bar{\lambda}_0\bar{z}^2 + \mathcal{O}(|\mu||z|^2 + |z|^3).$$

Then, we can easily see that one of the eigenvalues of the Jacobian $\partial(F_\mu^3(z), \bar{F}_\mu^3(z))/\partial(z, \bar{z})$ at one of the 3 fixed points in (4.6) is outside the unit circle and the other inside it, so the 3-cycle is hyperbolic(saddle) on both sides of $\mu = 0$.

We can summarize the above results in the following theorem.

THEOREM 1. *Let $F_\mu : \mathbf{C} \to \mathbf{C}$ be the map in complex form given in (2.7) and assume that $\lambda_0^3 = 1 (\lambda_0 \neq 1)$ and $a_2 \neq 0$. Then, a one-parameter family of 3-periodic fixed points $\{(x_1(\mu), x_2(\mu), x_3(\mu)) | \mu \in \mathbf{R}\}$ undergoes transcritical bifurcation from the origin (elliptic fixed* *point) and reverses the orientation as μ crosses 0. The 3-periodic points are given by*

$$\begin{aligned} x_1(\mu) &= r(\mu)e^{2\pi i\varphi(\mu)} + \mathcal{O}(|\mu|^2) \\ x_2(\mu) &= r(\mu)e^{2\pi i(\varphi(\mu)+\frac{1}{3})} + \mathcal{O}(|\mu|^2) \\ x_3(\mu) &= r(\mu)e^{2\pi i(\varphi(\mu)+\frac{2}{3})} + \mathcal{O}(|\mu|^2), \end{aligned}$$

where

$$\begin{aligned} r(\mu) &= 4\pi\sqrt{3}|\frac{\mu}{a_2}| + \mathcal{O}(|\mu|^2) \\ \varphi(\mu) &= -\frac{1}{6\pi}arg(\frac{i\mu}{a_2}) + \mathcal{O}(|\mu|) \quad (\text{mod } \frac{1}{3}) \end{aligned}$$

and they are hyperbolic (saddle) on both sides of $\mu = 0$.

(ii) The case $n = 3$ and $a_2 = 0$

In this case, $c_{pq}(\mu) = 0$ for all p, q with $p+q = 2$ and from Lemma 1, we have the normal form,

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5), \tag{4.7}$$

where the coefficients $\alpha_0 \equiv \alpha(0), \beta_0 \equiv \beta(0)$ and $\gamma_0 \equiv \gamma(0)$ are given by

$$\begin{cases} \alpha_0 = c_{21}(0) = -\dfrac{3ia_3}{8}\dfrac{\lambda_0}{\operatorname{Im}\lambda_0} = -\dfrac{\sqrt{3}i}{4}a_3\lambda_0 \\ \beta_0 = c_{40}(0) = \dfrac{a_4}{16}\dfrac{\lambda_0}{\operatorname{Im}\lambda_0} = \dfrac{a_4}{8\sqrt{3}}\lambda_0 \\ \gamma_0 = c_{13}(0) = -\dfrac{a_4}{4}\dfrac{\lambda_0}{\operatorname{Im}\lambda_0} = -\dfrac{a_4}{2\sqrt{3}}\lambda_0 \end{cases} \tag{4.8}$$

Eq. (3.5) becomes

$$\begin{aligned} &\mu\lambda_1 z + \bar{\lambda}_0\alpha_0 z^2\bar{z} + \bar{\lambda}_0\beta_0 z^4 + \bar{\lambda}_0\gamma_0 z\bar{z}^3 \\ &\qquad + \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0. \end{aligned} \tag{4.9}$$

Let $z = re^{2\pi i\varphi}$. Then,

$$\mu\lambda_1 re^{2\pi i\varphi} + \bar{\lambda}_0\alpha_0 r^3 e^{2\pi i\varphi} + \bar{\lambda}_0\beta_0 r^4 e^{8\pi i\varphi} + \bar{\lambda}_0\gamma_0 r^4 e^{-4\pi i\varphi} + \mathcal{O}(|\mu|^2 r + |\mu| r^3 + |\mu| r^4 + r^5) = 0.$$

Separating the trivial solution $r = 0$,

$$\mu\lambda_1 + \bar{\lambda}_0\alpha_0 r^2 + \bar{\lambda}_0\beta_0 r^4 e^{6\pi i\varphi} + \bar{\lambda}_0\gamma_0 r^3 e^{-6\pi i\varphi} + \mathcal{O}(|\mu|^2 + |\mu| r^2 + |\mu| r^3 + r^4) = 0.$$

or

$$2\pi i\mu - \frac{\sqrt{3}i}{4} a_3 r^2 + \frac{a_4}{8\sqrt{3}} r^3 e^{6\pi i\varphi} - \frac{a_4}{2\sqrt{3}} r^3 e^{-6\pi i\varphi} + \mathcal{O}(|\mu|^2 + |\mu| r^2 + |\mu| r^3 + r^4) = 0. \tag{4.10}$$

Set

$$\begin{cases} \mu = \mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3 \\ \varphi = \varphi_0 + \varphi_1 \end{cases}, \tag{4.11}$$

where $\mu_0, \mu_1, \mu_2, \varphi_0$ and φ_1 are to be determined.

Substituting (4.11) into (4.10),

$$2\pi i(\mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3) - \frac{\sqrt{3}}{4} i a_3 r^2 + \frac{a_4}{8\sqrt{3}} r^3 e^{6\pi i\varphi} - \frac{a_4}{2\sqrt{3}} r^3 e^{-6\pi i\varphi} + \mathcal{O}(r^4) = 0.$$

First, choose μ_0 so that $2\pi i\mu_0 - \frac{\sqrt{3}}{4} i a_3 = 0$, then

$$\mu_0 = \frac{\sqrt{3}}{8\pi} a_3. \tag{4.12}$$

With this choice of μ_0, we have

$$[2\pi i\mu_1 + \frac{a_4}{8\sqrt{3}}(e^{6\pi i\varphi} - 4e^{-6\pi i\varphi})] r^3 + 2\pi i\mu_2 r^3 + \mathcal{O}(r^4) = 0. \tag{4.13}$$

Next, we choose μ_1 and φ_0 so that

$$2\pi i\mu_1 + \frac{a_4}{8\sqrt{3}}(e^{6\pi i\varphi_0} - 4e^{-6\pi i\varphi_0}) = 0.$$

Note that since μ_1 and a_4 are real, $e^{6\pi i\varphi_0} - 4e^{-6\pi i\varphi_0}$ must be pure imaginary and this happens only when $e^{6\pi i\varphi_0}$ is pure imaginary, that is,

$$6\pi\varphi_0 = \pm\frac{\pi}{2} \qquad (\mathrm{mod}\ 2\pi)$$

or

$$\varphi_0^{(1),(2)} = \pm\frac{1}{12} \qquad (\mathrm{mod}\ \frac{1}{3}). \tag{4.14}$$

If $\varphi_0 = \varphi_0^{(1)} = \frac{1}{12}$, then

$$\mu_1 = \mu_1^{(1)} = -\frac{5a_4}{16\pi\sqrt{3}}. \tag{4.15}$$

If $\varphi_0 = \varphi_0^{(2)} = -\frac{1}{12}$, then

$$\mu_1 = \mu_1^{(2)} = \frac{5a_4}{16\pi\sqrt{3}}. \tag{4.16}$$

Now, from (4.13), we let

$$h(\mu_2, \varphi, r) = 2\pi i\mu_1 + \frac{a_4}{8\sqrt{3}}(e^{6\pi i\varphi} - 4e^{-6\pi i\varphi}) + 2\pi i\mu_2 + \mathcal{O}(r).$$

Then,

$$\begin{aligned}
&h(0, \varphi_0, 0) = 0\\
&\frac{\partial h}{\partial \mu_2}(0, \varphi_0, 0) = 2\pi i\\
&\frac{\partial h}{\partial \varphi}(0, \varphi_0, 0) = \frac{a_4}{4\sqrt{3}} \cdot 3\pi i(e^{6\pi i\varphi_0} + 4e^{-6\pi i\varphi_0}) = \pm\frac{a_4}{4\sqrt{3}}9\pi,
\end{aligned}$$

and by the implicit function theorem, we know that $\mu_2 = \mu_2(r)$, $\varphi = \varphi(r)$ and $\mu_2(0) = 0$, $\varphi_1(0) = 0$. Thus, we have *a pair of 3-cycles* $z = re^{2\pi i\varphi(r)}$, where r is regarded as a parameter which is related to μ as

$$\begin{cases} \mu^{(1)} = \frac{\sqrt{3}}{8\pi}a_3r^2 - \frac{5}{16\pi\sqrt{3}}a_4r^3 + \mathcal{O}(r^4) \\ \varphi^{(1)} = \frac{1}{12} + \mathcal{O}(r) \quad (\mathrm{mod}\ \frac{1}{3}) \\ \mu^{(2)} = \frac{\sqrt{3}}{8\pi}a_3r^2 + \frac{5}{16\pi\sqrt{3}}a_4r^3 + \mathcal{O}(r^4) \\ \varphi^{(2)} = -\frac{1}{12} + \mathcal{O}(r) \quad (\mathrm{mod}\ \frac{1}{3}) \end{cases}. \tag{4.17}$$

Note that if $a_3 > 0$, μ must be greater than 0 and we have a supercritical bifurcation and if $a_3 < 0$, μ must be less than 0 and have a subcritical bifurcation (Fig. 2).

To study the stability of the pair of 3-cycles for the map

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5),$$

we examine the eigenvalues of the map

$$\begin{aligned} z' = F_\mu^3(z) =&(1 + 3\mu\lambda_1)z + 3\bar{\lambda}_0\alpha_0z^2\bar{z} \\ &+ 3\bar{\lambda}_0\beta_0z^4 + 3\bar{\lambda}_0\gamma_0z\bar{z}^3 \\ &+ \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0. \end{aligned} \tag{4.18}$$

Let σ_1, σ_2 are the eigenvalues of the Jacobian $A = \partial(z', \bar{z}')/\partial(z, \bar{z})$ at one of the 3 fixed points x of one family for $F_\mu^3(z)$. If we assume that we used an area-preserving transformation of the form (2.11), then we can easily check that if $a_3a_4 > 0$, σ_1 and σ_2 are real and reciprocal for $\mu = \mu^{(2)}$, and σ_1 and σ_2 are complex conjugate on the unit circle for $\mu = \mu^{(1)}$. If $a_3a_4 < 0$, the situation is reversed.

Therefore, we can state the following theorem.

THEOREM 2. *Let $F_\mu : \mathbf{C} \to \mathbf{C}$ be the map given in (2.7) and assume that $\lambda_0^3 = 1 (\lambda_0 \neq 1)$ and $a_2 = 0$. Then a pair of two one-parameter families of 3-periodic fixed points*

$$\{(x_1^{(j)}(r), x_2^{(j)}(r), x_3^{(j)}(r) | r \in \mathbf{R}^+\} \qquad (j = 1, 2)$$

bifurcate from the origin on the same side of $\mu = 0$. If $a_3 > 0$, or < 0 we have a supercritical or subcritical bifurcation respectively. Those 3-periodic points are given by

$$
\begin{aligned}
x_1^{(j)} &= x_1^{(j)}(r) = re^{2\pi i\varphi^{(j)}(r)} + \mathcal{O}(r^2) \\
x_2^{(j)} &= x_2^{(j)}(r) = re^{2\pi i(\varphi^{(j)}(r)+\frac{1}{3})} + \mathcal{O}(r^2) \\
x_3^{(j)} &= x_3^{(j)}(r) = re^{2\pi i(\varphi^{(j)}(r)+\frac{2}{3})} + \mathcal{O}(r^2) \qquad \text{for } j = 1, 2.
\end{aligned}
$$

where r is related to μ as in (4.17).

Moreover, those 3-periodic points with smaller r is hyperbolic (saddle) and those with larger r is elliptic.

(iii) The case $n = 4$

Let $\lambda_0 = e^{2\pi i/4} = i$.

Then the normal form of $F_\mu(z)$ is

$$
F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + \mathcal{O}(|z|^5), \tag{4.19}
$$

where $\alpha(0) \equiv \alpha_0$ and $\beta(0) \equiv \beta_0$ are related to the coefficients of the original equation as follows

$$
\begin{aligned}
\alpha_0 &= \frac{1}{8}(3a_3 + a_2^2) \\
\beta_0 &= \frac{1}{8}(a_3 - a_2^2).
\end{aligned}
$$

Then eq. (3.15) becomes

$$
2\pi i\mu z + c_1 z^2\bar{z} + c_2\bar{z}^3 + g_1(\mu, z, \bar{z}) = 0, \tag{4.20}
$$

where

$$
\begin{aligned}
c_1 &= \bar{\lambda}_0\alpha_0 = -\frac{i}{8}(3a^3 + a_2^2) \\
c_2 &= \bar{\lambda}_0\beta_0 = \frac{i}{8}(a_2^2 - a_3) \\
g_1(\mu, z, \bar{z}) &= \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |z|^5).
\end{aligned}
$$

Setting $z = re^{2\pi i\varphi}$ and separating the trivial solution $r = 0$, we have

$$2\pi i\mu + c_1 r^2 + c_2 r^2 e^{-8\pi i\varphi} + g(\mu, r, \varphi) = 0, \tag{4.21}$$

where $g(\mu, r, \varphi) = \mathcal{O}(|\mu|^2 + |\mu|r^2 + r^4)$.

To look for the principal part, put

$$\begin{cases} \mu = \mu_0 r^2 + \mu_1 r^2 \\ \varphi = \varphi_0 + \varphi_1 \end{cases}, \tag{4.22}$$

where μ_0, μ_1, φ_0 and φ_1 are to be determined.

Substituting (4.22) in (4.21) and dividing by r^2, we have

$$(2\pi i\mu_0 + c_1 + c_2 e^{-8\pi i\varphi}) + 2\pi i\mu_1 + f_1(\mu_1, r, \varphi) = 0, \tag{4.23}$$

where $f_1(\mu_1, r, \varphi) = \mathcal{O}(r^2)$. We choose μ_0 and φ_0 so that

$$2\pi i\mu_0 + c_1 + c_2 e^{-8\pi i\varphi_0} = 0.$$

If $c_2 \neq 0$, i.e. $a_3 \neq a_2^2$, we have

$$e^{-8\pi i\varphi_0} = -\frac{2\pi i\mu_0 + c_1}{c_2} = \frac{a_2^2 + 3a_3 - 16\pi\mu_0}{a_2^2 - a_3}.$$

Since $e^{-8\pi i\varphi_0}$ must be real, we must have

$$\varphi_0 = \varphi_0^{(1)} = 0 \quad \text{or} \quad \varphi_0 = \varphi_0^{(2)} = \frac{1}{8} \quad (\text{mod } \frac{1}{4}), \tag{4.24}$$

and for each value of φ_0, μ_0 can be determined as

$$\mu_0 = \mu_0^{(1)} = \frac{a_3}{4\pi} \quad \text{for } \varphi_0^{(1)} \quad \text{and} \tag{4.25}$$

$$\mu_0 = \mu_0^{(2)} = \frac{a_3 + a_2^2}{8\pi} \quad \text{for } \varphi_0^{(2)}.$$

If $c_2 = 0$, i.e., $a_3 = a_2^2$, we have one solution for μ_0

$$\mu_0 = -\frac{c_1}{2\pi i} = \frac{3a_3 + a_2^2}{16\pi}. \tag{4.26}$$

However, this is not the generic case. Furthermore, if we define $h(\mu_1, r, \varphi)$ as following

$$h(\mu_1,r,\varphi) = (2\pi i\mu_0 + c_1 + c_2 e^{-8\pi i\varphi}) + 2\pi i\mu_1 + f_1(\mu_1,r_1,\varphi),$$

we have

$$\begin{aligned} &h(0,0,\varphi_0) = 0 \\ &\frac{\partial h}{\partial \mu_1}(0,0,\varphi_0) = 2\pi i \\ &\frac{\partial h}{\partial \varphi}(0,0,\varphi_0) = -8\pi i c_2 e^{-8\pi i\varphi_0} \\ &\qquad\qquad = \pm\pi(a_2^2 - a_3) \quad (\pm \text{ according as } \varphi_0 = \varphi_0^{(1)} \text{ or } \varphi_0^{(2)}), \end{aligned}$$

and hence the implicit function theorem is applicable only if $a_3 \neq a_2$. Thus, in this generic case, from the evenness of $f_1(\mu_1, r, \varphi)$ in r, we have

$$\mu_1 = \mu_1(r) = \mathcal{O}(r^2), \qquad \varphi_1 = \varphi_1(r) = \mathcal{O}(r^2).$$

Therefore, generically we have two one-parameter families of 4-cycles, $z = z^{(j)}(r) = re^{2\pi i\varphi^{(j)}(r)}$ $(j = 1, 2)$, bifurcating from the origin, and the parameters μ and r are given as

$$\text{(4.27)} \qquad \begin{cases} \mu^{(j)} = \mu_0^{(j)} r^2 + \mathcal{O}(r^4) \\ \varphi^{(j)} = \varphi_0^{(j)} + \mathcal{O}(r^2) \end{cases},$$

where $\mu_0^{(j)}$ and $\varphi_0^{(j)}$ $(j = 1, 2)$ are given in (4.24) and (4.25). Notice from (4.25) that if $a_3 > 0$ or $a_3 < -a_2^2$, then $\mu_0^{(1)}\mu_0^{(2)} > 0$, so the two families bifurcate on the same side of $\mu = 0$ (supercritical if $a_3 > 0$ and subcritical if $a_3 < -a_2^2$). If $-a_2^2 < a_3 < 0$, then $\mu_0^{(1)} < 0$ and $\mu_0^{(2)} > 0$, so the two families bifurcate on the opposite side of $\mu = 0$ (Fig. 3).

To study the stability of the 4-cycles for the map (4.19), we consider the map

$$\text{(4.28)} \qquad z' = F_\mu^4(z) = \lambda(\mu)^4 z + 4c_1 z^2\bar{z} + 4c_2\bar{z}^3 + \mathcal{O}(|\mu||z|^3 + |z|^5).$$

If σ_1 and σ_2 are the eigenvalues of the Jacobian $A = \partial(z', \bar{z}')/\partial(z, \bar{z})$ at one of the 4 fixed points x of one family for $F_\mu^4(z)$ and also if we assume that we used an area-preserving transformation of the form (2.11) then we can easily see that i) σ_1 and σ_2 are complex conjugate on the unit circle for $\mu^{(2)}$ if $a_3 > a_2^2$ or $a_3 < -a_2^2$, and also for $\mu^{(1)}$ if $0 < a_3 < a_2^2$; ii) σ_1 and σ_2 are real reciprocal each other for $\mu^{(1)}$ if $a_3 > a_2^2$ or $a_3 < -a_2^2$, also for $\mu^{(2)}$ if $0 < a_3 < a_2^2$, and for both $\mu^{(1)}$ and $\mu^{(2)}$ if $-a_2^2 < a_3 < 0$.

From the above results, we can state the following theorem.

THEOREM 3. *Let $F_\mu : \mathbf{C} \to \mathbf{C}$ be the map given in (2.7) and assume that $\lambda_0^4 = 1 (\lambda_0 \neq \pm 1)$ and $a_3 \neq 0$. Then, generically we have two one-parameter families of 4-periodic fixed points $\{x_1^{(j)}(r), x_2^{(j)}(r), x_3^{(j)}(r), x_4^{(j)}(r) | r \in \mathbf{R}^+, \quad j = 1,2\}$ bifurcating from the origin and those 4-periodic points are given by*

$$x_k^{(j)} = x_k^{(j)}(r) = re^{2\pi i[\varphi_0^{(j)} + \frac{k-1}{4}]} + \mathcal{O}(r^3) \qquad (j = 1,2, k = 1,2,3,4).$$

where the parameter r is related to μ as in (4.27).

Moreover, if $a_3 > 0$ or $a_3 < -a_2^2$, then the two families bifurcate on the same side of $\mu = 0$ and one family with smaller r is hyperbolic (saddle) and the other with larger r is elliptic. If $-a_2^2 < a_3 < 0$, then the two families bifurcate on the opposite side of $\mu = 0$ and both are hyperbolic (saddle).

(iv) The case $n \geq 5$

when $\lambda_0 = e^{2\pi i/n} (n \geq 5)$, the normal form of $F_\mu(z)$ is

$$(4.29) \quad F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^{n-1} + \gamma(\mu)z^3\bar{z}^2 + \mathcal{O}(|z|^7 + |z|^n)$$

and the coefficient $\alpha(0) \equiv \alpha_0$ can be computed from (2.12) as

$$\alpha_0 = -\frac{\lambda_0}{8(\operatorname{Im}\lambda_0)^2}\left[3ia_3 \operatorname{Im}\lambda_0 - a_2^2 \cdot \frac{(\lambda_0+1)(2\lambda_0^2+\lambda_0+2)}{\lambda_0^3 - 1}\right].$$

Since

$$\begin{aligned} \frac{(\lambda_0+1)(2\lambda_0^2+\lambda_0+2)}{\lambda_0^3-1} &= \frac{\lambda_0+1}{\lambda_0-1}\left(1 + \frac{\lambda_0^2+1}{\lambda_0^2+\lambda_0+1}\right) \\ &= \frac{\lambda_0+1}{\lambda_0-1}\left(1 + \frac{1}{1+\frac{1}{\lambda_0+\lambda_0}}\right) \\ &= -i\cot\frac{\pi}{n} \cdot \frac{1+4\cos\frac{2\pi}{n}}{1+2\cos\frac{2\pi}{n}} \qquad (n \geq 5), \end{aligned}$$

α_0 can be rewritten as

$$\alpha_0 = \lambda_0 i \xi_n \qquad (n \geq 5), \tag{4.30}$$

where

$$\xi_n = -\frac{1}{16}\csc\frac{2\pi}{n}\left(6a_3 + a_2^2 \cdot \csc^2\frac{\pi}{n} \cdot \frac{1+4\cos\frac{2\pi}{n}}{1+2\cos\frac{2\pi}{n}}\right) \quad (n \geq 5). \tag{4.31}$$

Notice that (4.30) also covers the case $n = 4$.

The bifurcation equation (3.15) becomes

$$2\pi i\mu z + c_1 z^2 \bar{z} + c_2 \bar{z}^{n-1} + \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^{n-1} + |z|^5) = 0, \tag{4.32}$$

where $c_1 = \bar{\lambda}_0 \alpha_0 = i\xi_n$, $\quad c_2 = \bar{\lambda}_0 \beta_0$.

Setting $z = re^{2\pi i\varphi}$ and separating the trivial solution $r = 0$, we have

$$2\pi i\mu + i\xi_n r^2 + c_2 r^{n-2} e^{-2n\pi i\varphi} + \mathcal{O}(|\mu|^2 + |\mu|r^2 + |\mu|r^{n-2} + r^4) = 0. \tag{4.33}$$

For $n = 5$, we set

$$\begin{cases} \mu = \mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3 \\ \varphi = \varphi_0 + \varphi_1 \end{cases}, \tag{4.34}$$

and take μ_0 as

$$\mu_0 = -\frac{\xi_5}{2\pi}. \tag{4.35}$$

Then (4.33) becomes

$$2\pi i\mu_1 + c_2 e^{-10\pi i\varphi} + 2\pi i\mu_2 + \mathcal{O}(r) = 0. \tag{4.36}$$

Now assume that $c_2 \neq 0$. Then we can take μ_1 and φ_0 such that

$$2\pi i\mu_1 + c_2 e^{-10\pi i\varphi_0} = 0,$$

that is,

$$\begin{cases} \mu_1 = -\frac{|c_2|}{2\pi}, \\ \varphi_0 = \frac{1}{10\pi} arg(\frac{c_2}{2\pi i}) \qquad (\text{mod } \frac{1}{5}) \end{cases}. \tag{4.37}$$

From (4.36), we define

$$h(\mu_2, \varphi, r) = 2\pi i \mu_1 + c_2 e^{-10\pi i \varphi} + 2\pi i \mu_2 + \mathcal{O}(r).$$

Then,

$$h(0, \varphi_0, 0) = 0, \qquad \frac{\partial h}{\partial \mu_2}(0, \varphi_0, 0) = 2\pi i,$$

$$\frac{\partial h}{\partial \varphi}(0, \varphi_0, 0) = -10\pi i c_2 e^{-10\pi i \varphi_0} = 10\pi |c_2| \neq 0.$$

Hence, by the implicit function theorem, we have

$$\mu_2 = \overline{\mu}_2(r) = \mathcal{O}(r), \qquad \varphi_1 = \varphi_1(r) = \mathcal{O}(r).$$

Therefore we have a one-parameter family of 5-cycles bifurcating from the origin, given by

$$\text{(4.38)} \qquad \begin{cases} \mu = -\frac{\xi_5}{2\pi} r^2 - \frac{|c_2|}{2\pi} r^3 + \mathcal{O}(r^4) \\ \varphi = \frac{1}{10\pi} arg(\frac{c_2}{2\pi i}) + \mathcal{O}(r) \qquad (\text{mod } \frac{1}{5}) \end{cases}.$$

For $n \geq 6$, we set

$$\mu = -\frac{\xi_5}{2\pi} r^2 + \mu_1 r^4,$$

and can proceed as before by imposing more conditions on the coefficients of the higher order terms.

Thus, we have the following theorem.

THEOREM 4. *Let $F_\mu : \mathbf{C} \to \mathbf{C}$ be the map given in (2.7) and assume that*

$$\lambda_0^n = 1 \qquad (\lambda_0 \neq \pm 1) \quad (n \geq 5).$$

Then, generically, we have a one-parameter family of n-periodic fixed points bifurcating from the origin.

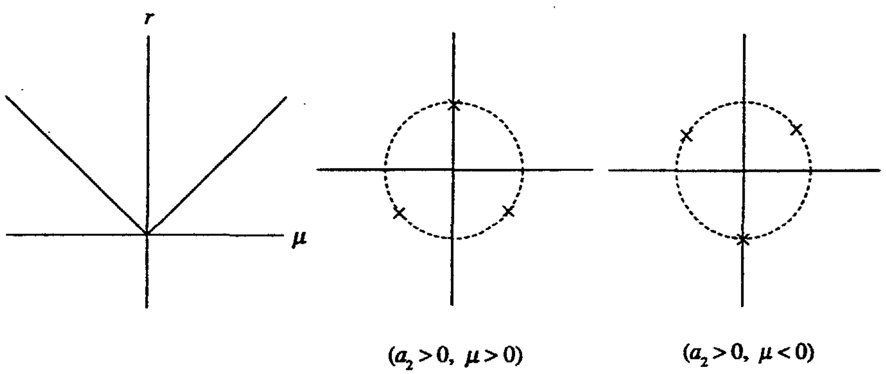


Fig. 1. The bifurcation diagrams and the positions of the 3 - periodic fixed points for $\theta_0=\frac{1}{3}$ and $a_2\neq 0$

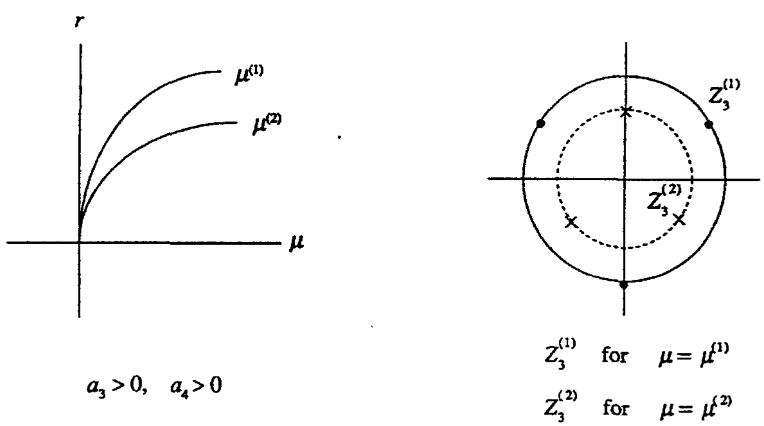
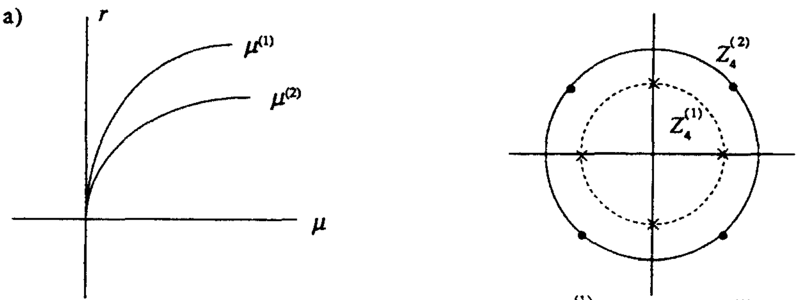


Fig. 2. The bifurcation diagrams and the positions of the 3 - periodic fixed points for $\theta_0=\frac{1}{3}$ and $a_2=0$

a)



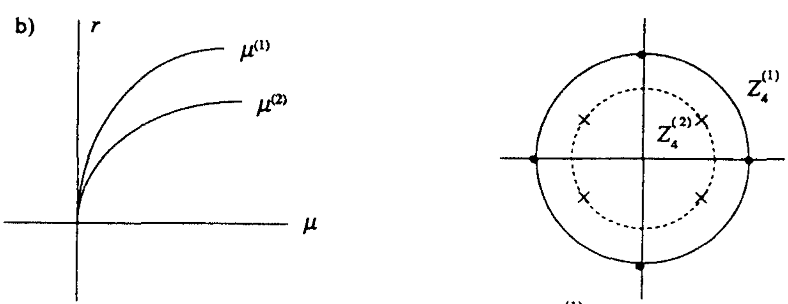
$a_3 > a_2^2$

$Z_4^{(1)}$ for $\varphi = \varphi^{(1)}$

$Z_4^{(2)}$ for $\varphi = \varphi^{(2)}$

$\mu > 0$

b)



$0 < a_3 < a_2^2$

$Z_4^{(1)}$ for $\varphi = \varphi^{(1)}$

$Z_4^{(2)}$ for $\varphi = \varphi^{(2)}$

$\mu > 0$

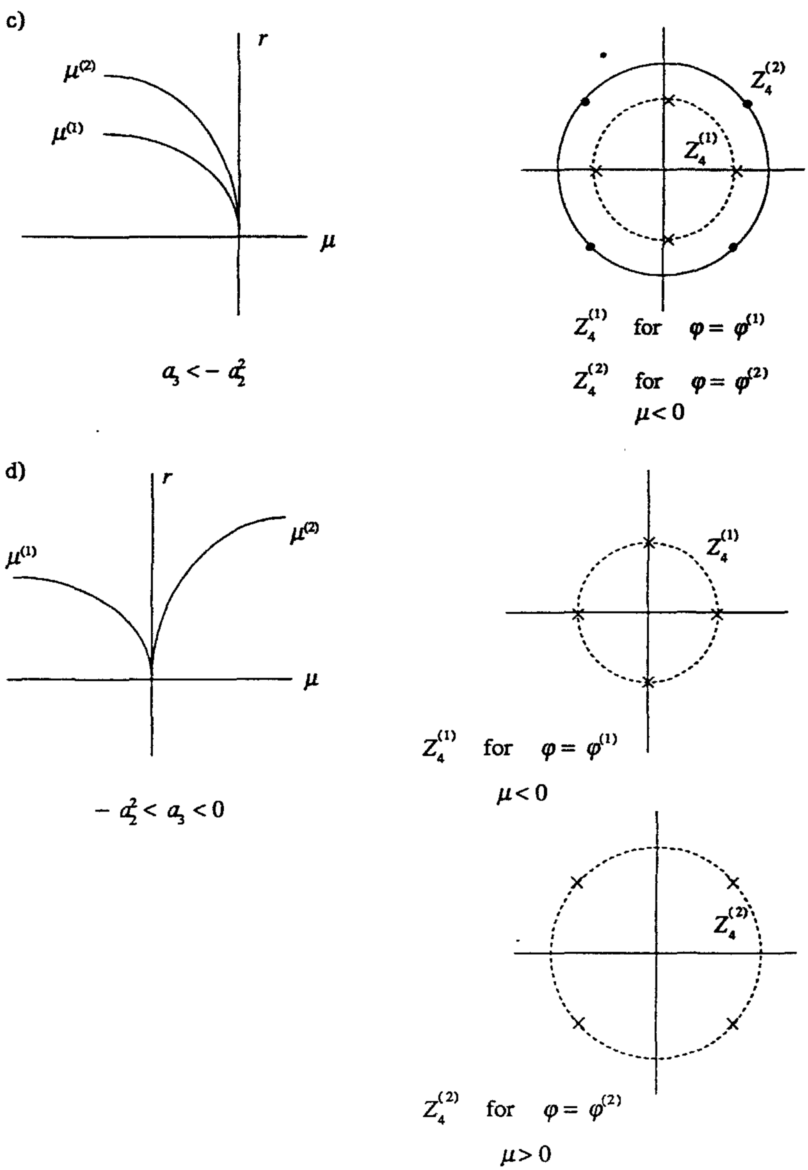


Fig. 3. The bifurcation diagrams and the positions of the 4 - periodic fixed points

References

1. Mackey. R., *Renormalization in area preserving maps,* Ph. D. thesis, Princeton (1982).
2. Zisook. A., *Intermittency in area-preserving mappings,* Phys. Rev. **A25**(1982), 2289–2292.
3. Van Der Weele. J., Capel. H., Valking. T. and Post. T., *The squeeze effect in non-integrable hamiltonian systems,* Phys. A**147**(1988), 499–532.
4. Meyer. K., *Generic bifurcation of periodic points,* Trans. AMS, **149**(1970), 95–107.
5. Arnold. V., *Geometrical Methods in the Theory of Ordinary Differential Equation* Springer, NY. 1983.
6. Siegel. C. and Moser, J., *Lectures on Celestial Mechanics*, Springer, NY.
7. Ioose. G., *Bifurcation of Maps and Applications*, North-Holland. 1979.
8. Guckenheimer. J. and Holmes. P., *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer, NY. 1983.
9. Cushman. A., Deprit. A. and Mosak. R., *Normal forms and representation theory,* J. Math. Phys., **24**(1983), 2102–2117.
10. Chow. S. and Hale. J., *Methods of Bifurcation Theory*, Springer, NY. 1982.
11. Golubitsky. M., Stewart. I., Schaeffer. G., *Singularities and Groups in Bifurcation Theory*, **I** & **II**, Springer, NY. 1988.
12. Schuster. H., *Determinstic Chaos, an Introduction*, VCH, P.17. 1988.
13. Lichitenberg. A. and Lieberman. M., *Regular and Stochastic Motion*, Springer, NY. P.422(1983).

Department of Mathematics
University of Ulsan
Ulsan 680–749, Korea

Department of Chemistry
Korea Advanced Institute of Science and Technology
Taejon 305–701, Korea