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GENERALIZED SEMIAUTOMORPHISM GROUPS OF MODULES

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DEFINITION. Let M and N be right R-modules. Let $\alpha : R \to R$ and $f : M \to N$ be maps. (1) f is a GENERALIZED R-MODULE HOMOMORPHISM with respect to α (or briefly αR-HOMOMORPHISM) if $f(m+q) = f(m) + f(q)$ and $f(ma) = f(m)\alpha(a)$ for all $m, q \in M$ and $a \in R$. f is an R-HOMOMORPHISM if $\alpha = I_R$(identity). f is a SEMIR-HOMOMORPHISM if $f(ma) = f(m)a$ for all $m \in M$ and $a \in R$. (2) f is an αR-MONOMORPHISM [resp. EPIMORPHISM, ISOMORPHISM] if f is an αR-homomorphism and both f and α are injective [resp. surjective, bijective]. (3) f is an αR-ENDOMORPHISM if $M = N$ and it is an αR-homomorphism. (4) f is an αR-AUTOMORPHISM if is an αR-isomorphism and $M = N$. (5) Let f and g be an αR-homomorphism and an βR-homomorphism respectively. Then we define $f = g$ if $f = g$ with $\alpha = \beta$. (6) f is a SEMIαR-HOMOMORPHISM if $f(ma) = f(m)\alpha(a)$ for all $m \in M$ and $a \in R$. (7) f is a SEMIαR-ENDOMORPHISM [resp. SEMIαR-AUTOMORPHISM] if $M = N$ and f is a semiαR-homomorphism [resp. semiαR-endomorphism and f, α are bijective].

NOTATION.

$\mathrm{END}_R(M) = \{f \mid f$ is an $\alpha R-$ endomorphism with a map $\alpha : R \to R\}$

$\mathrm{SEND}_R(M) = \{f \mid f$ is a semi$\alpha R-$ endomorphism with a map $\alpha : R \to R\}$

$\mathrm{AUT}_R(M) = \{f \mid f$ is an $\alpha R-$ automorphism with a map $\alpha : R \to R\}$

$\mathrm{SAUT}_R(M) = \{f \mid f$ is a semi$\alpha R-$ automorphism with a map $\alpha : R \to R\}$

$\mathrm{End}_R(M) = \{f \mid f : M \to M$ is an $R-$ endomorphism$\}$

$\mathrm{SEnd}_R(M) = \{f \mid f : M \to M$ is a semi$R-$ endomorphism$\}$

$\mathrm{Aut}_R(M) = \{f \mid f : M \to M$ is an $R-$ automorphism$\}$

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$\mathrm{SAut}_R(M) = \{f \mid f : M \to M \text{ is a semi}R - \text{automorphism}\}$
$\mathrm{Hom}_R(M,N) = \{f \mid f : M \to N \text{ is an } R - \text{homomorphism}\}$
$\mathrm{SHom}_R(M,N) = \{f \mid f : M \to N \text{ is a semi}R - \text{homomorphism}\}$
$\mathrm{SHOM}_R(M,N) = \{f \mid f : M \to N \text{ is a semi}\alpha R - \text{homomorphism with a map } \alpha : R \to R\}.$

We note that for any $f, g \in \mathrm{END}_R(M)$, if f is an αR-homomorphism and g is an βR-homomorphism, then fg is an $\alpha\beta R$-homomorphism. From now on, unless specified otherwise, "R – module" means "right R-module".

PROPOSITION 1. *Let M be an R-module. Then*
(1) $AUT_R(M), SAUT_R(M), Aut_R(M)$ and $SAut_R(M)$ are groups;
(2) $END_R(M), SEND_R(M), End_R(M)$, and $SEnd_R(M)$ are monoids.

LEMMA 2. *Let M be an R-module. For any $f, g \in SEND_R(M)$ let f and g be semiαR-endomorphism and semiβR-endomorphism respectively. We define two relations on $SEND_R(M)$ as follows :*

$$(f,g) \in \sigma_E \iff f = g \text{ on } SEND_R(M).$$
$$(f,g) \in \tau_E \iff \alpha = \beta \text{ on } SEND_R(M).$$

Then σ_E and τ_E are congruences relations on $SEND_R(M)$.

Proof. We will show that τ_E is a congruence relation on $\mathrm{SEND}_R(M)$. It is easy to show that τ_E is an equivalence relation. To show τ_E is a congruence relation, let $(f,g) \in \tau_E$ where f and g are semiαR-endomorphisms. For any $h \in \mathrm{SEND}_R(M)$, let h be a semiβR-endomorphism. Then $(f,g)h = (fh, gh) \in \tau_E$ and also, $h(f,g) = (hf, hg) \in \tau_E$. Similarly, it is easy to show that σ_E is a congruence relation.

NOTE. Similarly, for any $f, g \in \mathrm{SAUT}_R(M)$, let f and g be semiαR-automorphism and semiβR-automorphism respectively. We can define two congruence relations on $\mathrm{SAUT}_R(M)$ as follows :

$$(f,g) \in \sigma_A \iff f = g \text{ on } \mathrm{SAUT}_R(M)$$
$$(f,g) \in \tau_A \iff \alpha = \beta \text{ on } \mathrm{SAUT}_R(M).$$

Then
(1) σ_A and τ_A are congruence relations on $\mathrm{SAUT}_R(M)$.
(2) $\mathrm{SAUT}_R(M)/\tau_A = \mathrm{SAUT}_R(M)/\mathrm{SAut}_R(M)$.

DEFINITION. Let M be an R-module and let $a \in R$.

(1) $T_a : M \to M$ is called a RIGHT TRANSLATION if $T_a(m) = ma$ for all $m \in M$.

(2) We define a congruence $\mu_M \subset R \times R$ on R through $(a, b) \in \mu_M \iff T_a = T_b$ for $a, b \in R$.

(3) M is CYCLIC iff $M = mR$ for some $m \in M$. Also, m is called a GENERATOR.

(4) M is STRONGLY CONNECTED iff every element of M is a generator (or for any $m,\ q \in M$, $ma = q$ for some $a \in R$).

(5) M is PERFECT iff M is strongly connected and R is a commutative ring.

LEMMA 3. *Let M and N be R-modules. For any $f, g \in SHOM_R(M, N)$ let f and g be semiαR-homomorphism and semiβR-homomorphism respectively. If M is strongly connected, then*

$$f = g \iff \alpha = \beta \text{ and } f(p) = g(p) \text{ for some } p \in M.$$

Proof. We will show that $f(m) = g(m)$ for all $m \in M$. Since M is strongly connected, $M = qR$ for every $q \in M$. So, we have $M = pR$. This implies that for any $m \in M$, $m = pa$ for some $a \in R$. Hence $f(m) = f(pa) = f(p)\alpha(a) = g(p)\beta(a) = g(pa) = g(m)$. i.e., $f = g$. The converse is trivial.

COROLLARY 3.1. *Let M and N be R-modules. Let M be strongly connected. Then for every $f, g \in SHom_R(M, N)$,*

$$f = g \iff f(m) = g(m) \text{ for some } m \in M.$$

COROLLARY 3.2. *Let M be a strongly connected R-module. Then for any $f, g \in SEnd_R(M)$, $f = g \iff f(m) = g(m)$ for some $m \in M$.*

LEMMA 4. *Let M be an R-module and let $T_R = \{T_a : a \in R\}$. Then*

(1) $T_R \subset End_R(M)$ if R is commutative.

(2) $T_R = SEnd_R(M)$ if M is perfect.

(3) $T_R \subset End_R(M) \iff R$ is commutative if M is free of rank 1.

(4) $T_{ab} = T_a T_b$ *for any* $a, b \in R$ *if* R *is commutative.*

Proof. For (1), for any $T_a \in T_R$ we will show that T_a is an R-homomorphism. For all $m \in M$ and $b \in R$, $T_a(mb) = (mb)a = m(ba) = m(ab) = (ma)b = T_a(m)b$. Also, for all $m, q \in M$, $T_a(m+q) = (m+q)a = ma + qa = T_a(m) + T_a(q)$. Hence $T_a \in \mathrm{End}_R(M)$. For (2), it is enough to show $T_R \supset \mathrm{SEnd}_R(M)$. Choose any $f \in \mathrm{SEnd}_R(M)$. Claim: $f = T_a$ for some $a \in R$. To prove this, let $m \in M$ and $f(m) = q$ for some $q \in M$. Since M is strongly connected, we have $ma = q$ for some $a \in R$. So, $f(m) = q = ma = T_a(m)$. Hence $f = T_a$ by Corollary 3.2. For (3), to show $ab = ba$ for all $a, b \in R$, let $\{m\}$ be a basis for M. Now, $m(ab) = (ma)b = T_a(m)b = T_a(mb) = (mb)a = m(ba)$. So, we have $m(ab - ba) = 0$. Hence $ab = ba$. (4) is trivial.

PROPOSITION 5. *Let* M *be an* R*-module. Then the following conditions are equivalent :*

(1) $\mu_M = 0$ *on* R *where 0 is the identity relation.*
(2) *For all* $a, b \in R$, $T_a = T_b \Longrightarrow a = b$.
(3) $\sigma_A = 0$ *on* $SAUT_R(M)$.
(4) $\sigma_E = 0$ *on* $SEND_R(M)$ *if* M *is perfect.*

Proof. (1) $\Longleftrightarrow$ (2) : Trivial. (2) $\Longrightarrow$ (3) : Let $(f, g) \in \sigma_A$ where f and g are semiαR-automorphism and semiβR-automorphism respectively. Since $f, g \in \mathrm{SAUT}_R(M)$ and $f = g$, $f(ma) = f(m)\alpha(a) = f(m)\beta(a)$ for all $m \in M$ and $a \in R$. This means $T_{\alpha(a)}(f(m)) = T_{\beta(a)}(f(m))$. Since f is bijective, $T_{\alpha(a)}(m) = T_{\beta(a)}(m)$ for all $m \in M$. So, we have $T_{\alpha(a)} = T_{\beta(a)}$. By assumption, $\alpha(a) = \beta(a)$ for all $a \in R$. Hence $\alpha = \beta$. i.e., $\sigma_A = 0$. (3) $\Longrightarrow$ (2) : We define a map $\alpha : R \to R$ given by $\alpha(a) = b$, $\alpha(b) = a$ and $\alpha(t) = t$ for all $t \in R - \{a, b\}$. Then α is bijective with $\alpha(\alpha(a)) = a$ and $\alpha(\alpha(b)) = b$. Let $I : M \to M$ be the identity map. Then I is a semiαR-automorphism (it is easy to show this, using $T_a = T_b$). So, $I \in \mathrm{SAUT}_R(M)$. Also, I is a semi$I_R R$-automorphism where $I_R : R \to R$ is the identity map. Hence $I \in \mathrm{SAUT}_R(M)$. So, $(I, I) \in \sigma_A = 0$. This means $\alpha = I_R$. Hence $a = b$. (2) $\Longrightarrow$ (4) : For any $(f, g) \in \sigma_E$ let f and g be semiαR-endomorphism and semiβR-endomorphism respectively. Then for all $m \in M$ and $a \in R$, $f(ma) = f(m)\alpha(a)$ and $g(ma) = g(m)\beta(a)$. From $f = g$, we have $f(m)\alpha(a) = f(m)\beta(a)$ for all $m \in M$ and $a \in R$. This implies $T_{\alpha(a)}(f(m)) = T_{\beta(a)}(f(m))$. Since M is perfect, from Lemma 4 and Corollary 3.2 we have $T_{\alpha(a)} = T_{\beta(a)}$. By assumption, $\alpha(a) = \beta(a)$ for all $a \in R$. Hence $\alpha = \beta$. i.e., $\sigma_E = 0$. (4) $\Longrightarrow$ (2) : Clear from $\sigma_A \leq \sigma_E = 0$.

DEFINITION. An R-module M is called a MODULE with REDUCED R if one of the equivalent statements of Proposition 5 is satisfied.

PROPOSITION 6. *Let M be an R-module. Let $T_R = \{T_a : a \in R\}$ and let $\langle T_R \rangle$ be the semigroup generated by T_R. Then $R/\mu_M \cong \langle T_R \rangle$ where $\cong$ means semigroup isomorphic.*

DEFINITION. R/μ_M is called the characteristic semigroup of a module M.

LEMMA 7. *Let M be an R-module and let $a, b \in R$.*
(1) *If $f \in SAUT_R(M)$ and f is a semiαR-automorphism, then*

$$(a, b) \in \mu_M \Longleftrightarrow (\alpha(a),\ \alpha(b)) \in \mu_M.$$

(2) *Assume M is perfect. If $f \in SEND_R(M)$ and f is a smiαR-endomorphism, then $(a, b) \in \mu_E \Longrightarrow (\alpha(a), \alpha(b)) \in \mu_M$.*

Proof. For (1),

$$\begin{aligned}(a, b) \in \mu_M &\Longleftrightarrow T_a = T_b \Longleftrightarrow T_a(m) = T_b(m) \text{ for all } m \in M\\ &\Longleftrightarrow ma = mb \Longleftrightarrow f(ma) = f(mb)\\ &\Longleftrightarrow f(m)\alpha(a) = f(m)\alpha(b)\\ &\Longleftrightarrow T_{\alpha(a)}(f(m)) = T_{\alpha(b)}(f(m))\\ &\Longleftrightarrow T_{\alpha(a)} = T_{\alpha(b)} \Longleftrightarrow (\alpha(a), \alpha(b)) \in \mu_M.\end{aligned}$$

For (2),

$$\begin{aligned}(a, b) \in \mu_M &\Longleftrightarrow T_a = T_b \Longleftrightarrow T_a(m) = T_b(m) \text{ for all } m \in M\\ &\Longleftrightarrow ma = mb \Longrightarrow f(ma) = f(mb)\\ &\Longleftrightarrow f(m)\alpha(a) = f(m)\alpha(b)\\ &\Longleftrightarrow T_{\alpha(a)}(f(m)) = T_{\alpha(b)}(f(m))\\ &\Longleftrightarrow T_{\alpha(a)} = T_{\alpha(b)} \Longleftrightarrow (\alpha(a), \alpha(b)) \in \mu_M.\end{aligned}$$

LEMMA 8. *Let M be a perfect module and let $\alpha, \beta : R \to R$ be maps. Let Π_α and Π_β be maps defined by $\Pi_\alpha([a]) = [\alpha(a)]$ and $\Pi_\beta([a]) = [\beta(a)]$ for $a \in R$ respectively where $[\ ] = [\ ]_{\mu_M}$. For any $f, g \in SEND_R(M)$ let f and g be a semiαR-endomorphism and a semiβR-endomorphism respectively. Then we have the following statements:*

(1) Π_α and Π_β are endomorphisms if α and β are ring-homomorphisms.

(2) $\Pi_{\beta\alpha} = \Pi_\beta \Pi_\alpha$.

(3) $\Pi_\alpha = \Pi_\beta \iff \alpha = \beta$ if R is reduced

where the product of maps means the composition of maps.

Proof. We note that Π_α and Π_β are well-defined from Lemma 7(2). For (1) and (2), it is easy to check them. For (3), for every $t \in R$, $\Pi_\alpha([t]) = \Pi_\beta([t])$. This implies $[\alpha(t)] = [\beta(t)]$. Hence $(\alpha(t), \beta(t)) \in \mu_M$. Moreover, $(\alpha(t), \beta(t)) \in \mu_M \iff T_{\alpha(t)} = T_{\beta(t)}$. From the fact that M is a module with reduced R we can conclude that $T_{\alpha(t)} = T_{\beta(t)} \Longrightarrow \alpha(t) = \beta(t)$. i.e., $\alpha = \beta$. The converse is trivial.

COROLLARY 8.1. *Let M be an R-module. For any $f, g \in SAUT_R(M)$ let f and g be a semiαR-automorphism and a semiβR-automorphism respectively. Then the following statements hold:*

1. *Π_α and Π_β are semigroup-automorphisms if α and β are ring-homomorphisms.*
2. *$\Pi_{\beta\alpha} = \Pi_\beta \Pi_a$.*
3. *$\Pi_\alpha = \Pi_\beta \iff \alpha = \beta$ if R is reduced.*

RECALL. Let S and T be semigroups. Let $f : S \to T$ be a homomorphism. The Kernel of f is the set $\operatorname{Ker} f$ of all the elements of $S \times S$ which are carried by f onto the same element of T. That is, $\operatorname{Ker} f = \{(a, b) \in S \times S : f(a) = f(b)\}$.

LEMMA 9. *Let M be a perfect R-module and let $End(R/\mu_M)$ be the set of all endomorphisms (not R-endomorphisms) on R/μ_M. Let $h : SEND_R(M) \to End(R/\mu_M)$ be a map defined by $h(f) = \Pi_\alpha$ with semiαR-endomorphism f where $\alpha : R \to R$ is a ring-homomorphism. Then*

(1) h is a homomorphism.

(2) *Ker* $h = \tau_E$ *if* M *is with reduced* R.

Proof. (1) is trivial. For (2),

$$\begin{aligned}\operatorname{Ker} h &= \{(f,g) : h(f) = h(g)\} = \{(f,g) : \Pi_\alpha = \Pi_\beta \text{ for semi}\alpha R-\\ &\qquad \text{endomorphism } f \text{ and semi}\beta R - \text{endomorphism } g\}\\ &= \{(f,g) : \alpha = \beta \text{ for semi}\alpha R - \text{endomorphism } f \text{ and semi}\beta R-\\ &\qquad \text{endomorphism } g\} = \tau_E.\end{aligned}$$

From Lemma 9 we have the following proposition.

PROPOSITION 10. *Let* M *be a perfect* R*-module with reduced* R. *Then* $SEND_R(M)/\tau_E$ *is isomorphic to a submonoid of* $End(R/\mu_M)$.

LEMMA 11. *Let* M *be an* R*-module and let* $Aut(R/\mu_M)$ *be the set of all automorphisms (not* R*-automorphisms) on* R/μ_M. *Let* $h : SAUT_R(M) \to Aut(R/\mu_M)$ *be a map defined by* $h(f) = \Pi_\alpha$ *with semi*αR*-automorphism* f *where* $\alpha : R \to R$ *is a ring-homomorphism. Then*

(1) h *is a group-homomorphism.*

(2) *Ker* $h = SAut_R(M)$ *if* R *is reduced.*

Proof. (1) is trivial. For (2),

$$\begin{aligned}\operatorname{Ker} h &= \{f \in \mathrm{AUT}_R(M) : h(f) = I \text{ (identity map)}\}\\ &= \{f \in \mathrm{AUT}_R(M) : \Pi_\alpha = I \text{ for semi}\alpha R - \text{automorphism } f\}\\ &= \{f \in \mathrm{AUT}_R(M) : \alpha = I_R \text{ for semi}\alpha R - \text{automorphism } f\}\\ &= \mathrm{SAut}_R(M).\end{aligned}$$

From Lemma 11 we can obtain the following proposition.

PROPOSITION 12. *Let* M *be an* R*-module with reduced* R. *Then the factor group* $SAUT_R(M)/SAut_R(M)$ *is isomorphic to a subgroup of* $Aut(R/\mu_M)$.

DEFINITION. Let M be an R-module. Let $\Omega_M = \{f : M \to M$ is a transformation map$\}$. i.e., the semigroup of all transformation maps of M into M.

(1) We define the CENTRALIZER $C(T_R)$ and the NORMALIZER $N(T_R)$ of T_R in Ω_M as follows:

$$C(T_R) = \{f \in \Omega_M : T_a f = f T_a \text{ for all } T_a \in T_R\}$$
$$N(T_R) = \{f \in \Omega_M : T_R f = f T_R\}.$$

(2) We define the PERMUTATION CENTRALIZER (briefly p-CENTRALIZER) $C_p(T_R)$ and the PERMUTATION NORMALIZER (briefly p-NORMALIZER) $N_p(T_R)$ of T_R as follows:

$$C_p(T_R) = C(T_R) \cap S_M \text{ and } N_p(T_R) = N(T_R) \cap S_M$$

where S_M is the symmetric group over M.

NOTE. $N(T_R)$ is a monoid and $C(T_R) \leq N(T_R)$ (a submonoid of $N(T_R)$).

LEMMA 13. *Let M be an R-module with reduced R. Let $f \in N_p(T_R)$. Then for any $T_a \in T_R$ $\exists! T_b \in T_R$ such that $fT_b = T_a f$ (or $fT_a = T_b f$).*

Proof. Suppose there is another $T_c \in T_R$ such that $T_a f = fT_c$. Then $fT_b = fT_c$ and $fT_b(m) = fT_c(m)$ for all $m \in M$. This implies that $f(mb) = f(mc)$. Since f is $1-1$, $mb = mc$. This means that $T_b(m) = T_c(m)$ for all $m \in M$. i.e., $T_b = T_c$. Hence $b = c$.

PROPOSITION 14. *Let M be an R-module. Then*
(1) $SEnd_R(M) = C(T_R)$ and $SAut_R(M) = C_p(T_R)$.
(2) $C_p(T_R)$ is a normal subgroup of $N_p(T_R)$.
(3) $SAUT_R(M) = N_p(T_R)$ if R is reduced.

Proof. For the first part of (1), $\mathrm{SEnd}_R(M) \subset C(T_R)$: For any $f \in \mathrm{SEnd}_R(M)$, it is enough to show $fT_a = T_a f$ for all $T_a \in T_R$. To do this, choose any $m \in M$. Then $fT_a(m) = f(ma) = f(m)a = T_a f(m)$. Hence it holds. Similarly, the converse can be shown easily.

The second part of (1) follows from the first part of (1). For (2), for any $f \in N_p(T_R)$, $g \in C_p(T_R)$ and $T_a \in T_R$,

$$\begin{aligned} T_a f g f^{-1} &= f T_b g f^{-1} \quad \text{for some} \quad T_b \in T_R \\ &= f g T_b f^{-1} \\ &= f g f^{-1} T_a. \end{aligned}$$

Hence it holds. For (3), $\mathrm{SAUT}_R(A) \subset N_p(T_R)$: To prove this, choose any $f \in \mathrm{SAUT}_R(M)$ and let f be a semiαR-automorphism. Then we have $f(ma) = f(m)\alpha(a)$ for all $m \in M$ and $a \in R$. This means that $f[T_a(m)] = T_{\alpha(a)}[f(m)]$. Also, this implies that $fT_a = T_{\alpha(a)}f$. Hence since α is bijective, $fT_R = T_R f$. i.e., $f \in N_p(T_R)$. $\mathrm{SAUT}_R(A) \supset N_p(T_R)$: By Lemma 13, for any $f \in N_p(T_R)$ and $T_a \in T_R$ $\exists! T_b \in T_R$ such that $fT_a = T_b f$. Let $\alpha : R \to R$ be a map defined by $\alpha(a) = b$ with $fT_a = T_b f$.

Claim: α is bijective. (i) α is well-defined : To prove this, let $t = u$ for $t, u \in R$. By Lemma 13, for T_t and T_u $\exists! T_c, T_d \in T_R$ such that $fT_t = T_c f$ and $fT_u = T_d f$. This implies $T_c f = T_d f$. Hence $T_c = T_d$. So, we have $c = d$ since R is reduced. Thus, $\alpha(t) = c = d = \alpha(u)$. (ii) $\alpha = 1-1$: Suppose $\alpha(t) = \alpha(u)$. Let $\alpha(t) = c$ with $fT_t = T_c f$ and let $\alpha(u) = d$ with $fT_u = T_d f$. Then from $c = d$ $fT_t = fT_u$. Hence $T_t = T_u$. Thus, we have $t = u$. (iii) α is onto : For any $b \in R$, consider $T_b \in T_R$. By Lemma 13 $\exists! T_a \in T_R$ such that $T_b f = fT_a$. Hence $\exists a \in R$ such that $\alpha(a) = b$ with $fT_a = T_b f$.

Now, we will show that f is a semiαR-homomorphism. For any $m \in M$ and $a \in R$,

$$\begin{aligned} f(m)\alpha(a) &= f(m)b \text{ with } fT_a = T_b f \\ &= T_b f(m) = fT_a(m) = f(ma). \end{aligned}$$

Hence $f \in \mathrm{SAUT}_R(A)$.

COROLLARY 14.1. *Let M be an R-module with reduced R. Then the following statements hold:*

(1) *$N_p(T_R)/C_p(T_R) \cong$ a subgroup of $Aut(S/\mu_M)$.*

(2) *$SAut_R(M)$ is a normal subgroup of $SAUT_R(M)$.*

NOTATION. Let M be an R-module and $\alpha : R \to R$ be a map. For $m, q \in M$, $H_{m\alpha q} = \{a \in R : m\alpha(a) = q\}$ and $H_{mq} = \{a \in R : ma = q\}$.

LEMMA 15. *Let M and N be R-modules. Let $m \in M$ be a fixed element and let $\alpha : R \to R$ be a map. If $f : M \to N$ is any map, then the following statements hold:*

(1) *If $f(mt) = f(m)\alpha(t)$ for all $t \in R$, then $H_{mq} \subset H_{f(m)\alpha f(q)}$ for all $q \in M$.*
(2) *If $H_{mq} \subset H_{f(m)\alpha f(q)}$ for some $q \in M$, then $f(mt) = f(m)\alpha(t)$ for all $t \in H_{mq}$.*
(3) *$f(mt) = f(m)\alpha(t)$ for all $t \in H_{mq} \Longleftrightarrow H_{mq} \subset H_{f(m)\alpha f(q)}$ for all $q \in M$.*
(4) *Assume M is strongly connected. Then $f(mt) = f(m)\alpha(t)$ for all $t \in R \Longleftrightarrow H_{mq} \subset H_{f(m)\alpha f(q)}$ for all $q \in M$.*

Proof. For (1), for every $a \in H_{mq}$ we have $ma = q$. This implies $f(q) = f(ma) = f(m)\alpha(a)$. Hence $a \in H_{f(m)\alpha f(q)}$.

For (2), for every $t \in H_{mq}$ we have $mt = q$ and also, since $t \in H_{f(m)\alpha f(q)}$, we have $f(m)\alpha(t) = f(q)$. This implies $f(m)\alpha(t) = f(mt)$.

(3) is clear from (1) and (2). For (4), suppose M is strongly connected. Then we have $M = mR$. So, for every $t \in R$, we have $k = mt$ for some $k \in M$. This implies $t \in H_{mk} \subset H_{f(m)\alpha f(k)}$. Thus, $f(m)\alpha(t) = f(k)$. Hence $f(mt) = f(k) = f(m)\alpha(t)$. The converse is clear from (1).

PROPOSITION 16. *Let M and N be R-modules. Let $f : M \to N$ and $\alpha : R \to R$ be maps. Then the following statements are equivalent:*

(1) *$f : M \to N$ is a semiαR-homomorphism.*
(2) *$H_{mq} \subset H_{f(m)\alpha f(q)}$ for any $m, q \in M$.*
(3) *$f(qa) = f(q)\alpha(a)$ for some $q \in M$ and all $a \in R$ if M is strongly connected and α is a semigroup-homomorphism.*

Proof. (1) $\Longrightarrow$ (2): For all $m \in M$ and $t \in R$, $f(mt) = f(m)\alpha(t)$. Hence it holds by Lemma 15(1). (2) $\Longrightarrow$ (1): To show $f(mt) = f(m)\alpha(t)$ for all $m \in M$ and $t \in R$, we recall $R = \cup_{q \in M} H_{mq}$. Now, for any $t \in R$, we have $t \in H_{mq}$ for some $q \in M$. By the assumption, $t \in H_{mq} \subset H_{f(m)\alpha f(q)}$. Hence it holds from (2) of Lemma 15. (2) $\Longrightarrow$ (3): Since M is strongly connected, we have $M = qR$ for some $q \in M$. This means that for any $a \in R$, there is an $k \in M$ such that $k = qa$. This implies $a \in H_{qk} \subset H_{f(q)\alpha f(k)}$. So, we have $f(q)\alpha(a) = f(k) = f(qa)$. (3) $\Longrightarrow$ (1): We have $M = qR$ from the strong connectedness. This implies that for any $m \in M$ there is an $b \in R$ such that $m = qb$. So, we have $ma = (qb)a$. Hence for any $m \in M$ and $a \in R$ we have $f(ma) = f((qb)a) = f(q(ba)) = f(q)\alpha(ba) = f(q)\alpha(b)\alpha(a) = [f(q)\alpha(b)]\alpha(a) = f(qb)\alpha(a) = f(m)\alpha(a)$.

COROLLARY 16.1. Let M be an R-module. Then $f : M \to M$ is a semiαR-automorphism $\Longleftrightarrow$ f and α are permutations on M and R respectively and $H_{mq} \subset H_{f(m)\alpha f(q)}$ for any $m, q \in M$.

NOTE. If $f \in \mathrm{SAUT}_R(M)$, then $f^n \in \mathrm{SAUT}_R(M)$ for any nonnegative interger n where $f^n = fff\ldots f$ (n times) and the product means the composition of f's.

DEFINITION. Let M be an R-module. Then we say that a mapping $\alpha : R \to R$ is an M-HOMOMORPHISM if $m\alpha(a) = ma$ for all $m \in M$ and $a \in R$. We recall that f is a REGULAR PERMUTATION on a set M if f is a permutation on M and for every power, say f^n, of f, it is the case that $f^n(p) = p$ for some $p \in M$ implies $f^n = I$ (identity).

PROPOSITION 17. *Let M be a strongly connected R-module. For every $f \in SAUT_R(M)$ let f be a semiαR-automorphism. Then f is a regular permutation on M if $\alpha : R \to R$ is an M-homomorphism.*

Proof. Suppose that for any $n \in \mathbb{N}$, $f^n(x) = x$ for some $x \in M$.
Claim: $f^n = I$ (identity). Proof. Since $f \in \mathrm{SAUT}_R(M)$, f^n is a semi$\alpha^n R$-automorphism and $f^n \in \mathrm{SAUT}_R(M)$. This implies $f^n \in \mathrm{SEND}_R(M)$. Also, for all $m \in M$ and $a \in R$ $I(ma) = ma = m\alpha(a) = I(m)\alpha(a)$. This implies that I is a semiαR-automorphism. Hence I^n is a semi$\alpha^n R$-automorphism and $I^n \in \mathrm{SEND}_R(M)$. From Lemma 3 we have $f^n = I$.

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