

ON CERTAIN REAL HYPERSURFACES OF A COMPLEX SPACE FORM*

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Introduction

A complex n -dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC , a complex Euclidean space C^n or a complex hyperbolic space H_nC , according as $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehler metric and the almost complex structure J of $M_n(c)$. We denote by A and S the shape operator and the Ricci tensor of type (1,1) on M , respectively.

On his study of real hypersurfaces of a complex projective space P_nC , Takagi [7] classified all homogeneous real hypersurfaces and showed that they are realized as the tubes of constant radius over Kaehler submanifolds if ξ is principal. Namely, he proved the following

THEOREM A. *Let M be a homogeneous real hypersurface of P_nC . Then M is a tube of radius r over one of the following Kaehler submanifolds;*

- (A₁) a hyperplane $P_{n-1}C$, where $0 < r < \pi/2$,
- (A₂) a totally geodesic P_kC ($1 \leq k \leq n-2$), where $0 < r < \pi/2$,
- (B) a complex quadric Q_{n-1} , where $0 < r < \pi/4$,
- (C) $P_1C \times P_{(n-1)/2}C$, where $0 < r < \pi/4$ and $n(\geq 5)$ is odd,
- (D) a complex Grassmann $G_{2,5}C$, where $0 < r < \pi/4$ and $n = 9$,
- (E) a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.

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According to Takagi's classification, the principal curvatures and their multiplicities of the above homogeneous real hypersurfaces are given.

Real hypersurfaces of $P_n C$ have been studied by many differential geometers ([2], [6], [8] and [18] etc.) and as one of them, Kimura [6] asserts that M has constant principal curvatures and ξ is principal if and only if M is locally congruent to a homogeneous real hypersurface.

On the other hand, real hypersurfaces of a complex hyperbolic space $H_n C$ have also been investigated by Berndt [1], Montiel [13], Montiel and Romero [14] and so on. Berndt [1] classified all homogeneous real hypersurfaces of $H_n C$ and showed that they are realized as the tubes of constant radius over Kaehler submanifolds if ξ is principal. Namely, he proved the following

THEOREM B. *Let M be a homogeneous real hypersurface of $H_n C$. Then M is congruent to one of the following hypersurfaces :*

- (A₀) a horosphere in $H_n C$,
- (A₁) a tube of a complex hyperbolic hyperplane $H_{n-1} C$ of arbitrary radius,
- (A₂) a tube of a totally geodesic $H_k C$ ($1 \leq k \leq n - 2$) of arbitrary radius,
- (B) a tube of a totally real hyperbolic space $H_n R$.

For principal curvatures and their multiplicities of the above hypersurfaces are also given by Berndt [1].

Now it is seen that there exist no real hypersurfaces on $M_n(c)$, $c \neq 0$ with parallel second fundamental form. Kimura and Maeda [8] pointed out the importance of the distribution ξ^\perp of the tangent bundle TM orthogonal to ξ . Using the covariant derivative $\nabla_\xi A$ of the shape operator A in the direction ξ , Maeda and Udagawa [11] proved the following

THEOREM C. *Let M be a real hypersurface of $P_n C$. If ξ is principal with nonzero principal curvature and if it satisfies $\nabla_\xi A = 0$, then M is locally congruent to one of the following homogeneous real hypersurfaces which lies on a tube of radius r .*

- (A₁) a hyperplane $P_{n-1} C$, where $0 < r < \pi/2$ and $r \neq \pi/4$,
- (A₂) a totally geodesic $P_k C$ ($1 \leq k \leq n - 2$), where $0 < r < \pi/2$ and $r \neq \pi/4$.

On the other hand, Okumura [15] proved that in order for a real hypersurface of $P_n C$ to be of type A_1 or of type A_2 , it is necessary and sufficient that it satisfies $L_\xi g = 0$, where L_ξ denotes the Lie derivative in the direction of the structure vector field ξ . For simplicity, we shall say that a real hypersurface M of $M_n(c)$, $c \neq 0$ is of type A if it is of type A_1 or of type A_2 in $P_n C$, or it is of type A_0 , of A_1 or type A_2 in $H_n C$. Giving attention to this fact and Theorem C, Ki, Kim and Lee [3] proved the following

THEOREM D. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If it satisfies $L_\xi A = 0$, then M is of type A .*

Under certain conditions for the Ricci tensor of M , real hypersurfaces of a complex space form were studied by many geometers [4], [7], [10], [16], etc. Taking account of such a situation, we shall investigate a real hypersurface M of $M_n(c)$, $c \neq 0$ with certain conditions about the Ricci tensor S of M . The main purpose of the present paper is to classify real hypersurfaces of $M_n(c)$, $c \neq 0$, which satisfy $L_\xi S = 0$.

1. Preliminaries

Let M be a real hypersurface of a complex n -dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c , and let C be a unit normal vector field on a neighborhood of a point x in M . We denote by $\bar{\nabla}$ and ∇ the Riemannian connection in $M_n(c)$ and in M , respectively. Then by the Gauss formula, we have the relationship between $\bar{\nabla}$ and ∇ : For any vector fields X and Y on M

$$(1.1) \quad \bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)C,$$

where g is the Riemannian metric tensor of M induced from that of $M_n(c)$ and A denotes the shape operator with respect to C of M in $M_n(c)$. Furthermore we have another equation which is called the Weingarten formula:

$$(1.2) \quad \bar{\nabla}_X C = -AX.$$

For any local vector field X on a neighborhood of x in M , the transformations of X and C under the complex structure J in $M_n(c)$ can be given by

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M , where η and ξ denote a 1-form and a vector field on a neighborhood of x in M respectively. Then it is seen that $g(\xi, X) = \eta(X)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M . They satisfy the following

$$(1.3) \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation and \otimes denotes the tensor product. Furthermore the covariant derivatives of the structure tensors are given by

$$(1.4) \quad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.$$

Since the ambient space is of constant holomorphic sectional curvature c , equations of the Gauss and Codazzi are respectively given as follows;

$$(1.5) \quad \begin{aligned} R(X, Y)Z &= c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &\quad - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}/4 \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(1.6) \quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X .

The Ricci tensor S' of M is a tensor of type $(0, 2)$ given by $S'(X, Y) = \text{tr}\{Z \rightarrow R(Z, X)Y\}$. Also it may be regarded as the tensor of type $(1, 1)$ and denoted by $S : TM \rightarrow TM$; it satisfies $S'(X, Y) = g(SX, Y)$. From (1.5) we see that the Ricci tensor S of M is given by

$$(1.7) \quad S = c\{(2n + 1)I - 3\eta \otimes \xi\}/4 + hA - A^2,$$

where we have put $h = \text{tr } A$. A real hypersurface M of $M_n(c)$ is said to be *pseudo-Einstein* if the Ricci tensor S satisfies

$$(1.8) \quad S = aI + b\eta \otimes \xi$$

for some functions a and b on M . Moreover, using (1.4) we get

$$(1.9) \quad \begin{aligned} \nabla_X S(Y) = & -3c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}/4 \\ & + dh(X)AY + (hI - A)\nabla_X A(Y) \\ & - \nabla_X A(AY), \end{aligned}$$

where d denotes the exterior differential

Now we assume that the structure vector ξ is principal with corresponding principal curvature α . Then it is seen in Maeda [12] and Ki and Suh [5] that α is constant on M and it satisfies

$$(1.10) \quad 2A\phi A = \frac{c}{2}\phi + \alpha(A\phi + \phi A).$$

Then the second formula of (1.4) gives $(\nabla_X A)\xi = \alpha\phi AX - A\phi AX$, which together with (1.10) implies that

$$(1.11) \quad \nabla_X A(\xi) = -\frac{c}{4}\phi X - \frac{\alpha}{2}(A\phi - \phi A)X.$$

By the Codazzi equation (1.6) and the above equation we get

$$(1.12) \quad \nabla_\xi A(X) = -\frac{\alpha}{2}(A\phi - \phi A)X,$$

which implies

$$(1.13) \quad dh(\xi) = 0.$$

If X is a principal vector with corresponding principal curvature λ , then (1.10) gives us to

$$(1.14) \quad (2\lambda - \alpha)A\phi X = \left(\frac{c}{2} + \alpha\lambda\right)\phi X.$$

2. Lie derivatives

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$ and let ξ be a principal vector field with corresponding principal curvature α . In this section we assume that the Ricci tensor S satisfies the following

$$(2.1) \quad L_\xi S = 0,$$

where L_ξ denotes the Lie derivative with respect to ξ . By definition we have

$$L_\xi S(X) = L_\xi(SX) - SL_\xi X$$

for any vector field X and hence using (1.7), (1.9), (1.12) and (1.13) we obtain

$$(2.2) \quad \begin{aligned} L_\xi S(X) = & -\frac{\alpha}{2} \{h(A\phi - \phi A) - (A\phi - \phi A)A - A(A\phi - \phi A)\}X \\ & - (\phi AS - S\phi A)X \end{aligned}$$

for any vector field X because the structure vector ξ is principal.

Accordingly, it follows from (1.7), (2.1) and (2.2) that we have

$$\frac{\alpha}{2} \{h(A\phi - \phi A) + \phi A^2 - A^2 \phi\} + \phi A(hA - A^2) - (hA - A^2)\phi A = 0.$$

By using (1.10) repeatedly, the above equation is rewritten as

$$(2.3) \quad c\left(\frac{\alpha}{2} - h\right)\phi + (\alpha^2 + c)A\phi + \alpha(\alpha - 4h)\phi A + 2(\alpha + 2h)\phi A^2 - 4\phi A^3 = 0.$$

First of all, we suppose that there exists a principal vector X orthogonal to ξ and with corresponding principal curvature λ ($\neq \alpha/2$). By (1.14) we get ϕX is also a principal vector and its corresponding principal curvature is given by $\lambda' = (2\alpha\lambda + c)/2(2\lambda - \alpha)$. Then (2.3) is reduced to

$$(2.4) \quad (2\lambda - \alpha)(\lambda' - \lambda)(h - \lambda - \lambda') = 0,$$

which means $\lambda' - \lambda = 0$ or $\lambda' + \lambda = h$. In the first case, λ is the root of the quadratic equation

$$(2.5) \quad 4y^2 - 4\alpha y - c = 0$$

because of the definition of λ' , and hence we get $\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + c}}{2}$. We put

$$(2.6) \quad \lambda_1 = \frac{\alpha - \sqrt{\alpha^2 + c}}{2}, \quad \lambda_2 = \frac{\alpha + \sqrt{\alpha^2 + c}}{2}.$$

Furthermore it is seen that the multiplicity of λ_1 or λ_2 is even. So we denote by $2p$ and $2q$ the multiplicity of λ_1 and λ_2 , respectively. In the other case where $\lambda' + \lambda = h$, λ is also the root of the quadratic equation

$$(2.7) \quad 4y^2 - 4hy + 2\alpha h + c = 0,$$

and hence λ is given by $\lambda = \frac{h \pm \sqrt{h^2 - 2\alpha h - c}}{2}$. Here we put

$$(2.8) \quad \lambda_3 = \frac{h - \sqrt{h^2 - 2\alpha h - c}}{2}, \quad \lambda_4 = \frac{h + \sqrt{h^2 - 2\alpha h - c}}{2},$$

and it is seen that their multiplicities are equal to each other.

Secondly, we suppose that there exists a principal vector Y orthogonal to ξ and with corresponding principal curvature $\alpha/2$. Then, by (1.14), it turns out that $\alpha^2 + c = 0$, which means that c is negative. Moreover, suppose that there exists a principal vector X orthogonal to ξ and with corresponding principal curvature $\lambda (\neq \alpha/2)$. Then, by using the above discussion, another principal curvature λ' is determined. If $\lambda' = \lambda$, then (2.6) and the fact that $\alpha^2 + c = 0$ imply that $\lambda_1 = \lambda_2 = \alpha/2$, a contradiction. So we have $\lambda' + \lambda = h$ and

$$\lambda_3 \quad \text{or} \quad \lambda_4 = \frac{h \pm |h - \alpha|}{2},$$

which means that one of λ_3 and λ_4 is equal to $\alpha/2$. Taking account of the fact that the multiplicity of λ_3 is equal to that of λ_4 , we verify that their multiplicities are equal to zero. Thus it is seen that there are no principal curvatures different from $\alpha/2$. Namely, we have distinct principal curvatures α and $\alpha/2$, whose multiplicities are 1 and $2n - 2$ respectively.

An eigenvalue of the linear transformation on the tangent bundle TM is said to be *simple* if the multiplicity is equal to one, and the transformation is said to be of no simple roots if each eigenvalues is not simple. Under these preparations we will prove the following

THEOREM 2.1. *Let M be a real hypersurface of $P_n C$, $n \geq 3$, with no simple roots. If the structure vector ξ is principal and if it satisfies $L_\xi S = 0$, then M is a tube of radius r over one of the following Kaehler submanifolds;*

- (A₁) a hyperplane $P_{n-1}C$, where $0 < r < \pi/2$ and $r \neq \pi/4$,
- (A₂) a totally geodesic $P_k C$, $1 \leq k \leq n-2$, where $0 < r < \pi/2$ and $r \neq \pi/4$,
- (B) a complex quadric Q_{n-1} , where $0 < r < \pi/4$ and $\cot^2 2r = n-2$,
- (C) $P_1 C \times P_{(n-1)/2} C$, where $0 < r < \pi/4$, $\cot^2 2r = 1/(n-2)$ and $n(\geq 5)$ is odd,
- (D) a complex Grassmann $G_{2,5}C$, where $0 < r < \pi/4$, $\cot^2 2r = 3/5$ and $n = 9$,
- (E) a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$, $\cot^2 2r = 5/9$ and $n = 15$.

Proof. Let M_0, M_1 and M_2 be subsets of M consisting of points at which there exists a principal curvature $\alpha/2, \lambda_1$ or λ_2 , and λ_3 or λ_4 respectively. Then each set is closed and $\{M_0, M_1, M_2\}$ is the covering of M because of (2.4). The above consideration in the latter half implies that the intersection of M_1 with M_2 is empty.

The proof is divided into the following four cases;

- (1) $M = M_0$,
- (2) $M_1 - M_2 \neq \phi$,
- (3) $M_2 - M_1 \neq \phi$,
- (4) $M_1 \cap M_2 \neq \phi$,

The case (1). As is already discussed, for any point in $M = M_0$, distinct principal curvatures are α and $\alpha/2$, where multiplicities are 1 and $2n-2$, respectively. Accordingly we have $A = \frac{\alpha}{2}I + \frac{\alpha}{2}\eta \otimes \xi$. Since all principal curvatures are constant on M , we can apply the structure theorem due to Berndt [1] and Kimura [6] to our situation and it is seen that M is locally congruent to one of homogeneous real hypersurfaces. Then the classification theorem due to Berndt [1] and Takagi [17] means that $c < 0$ and M is of type A_0 .

The case (2). The set $M_1 - M_2 = M - (M_0 \cup M_2)$ is open. By means of (2.6), distinct principal curvatures on $M_1 - M_2$ are at most α, λ_1 and λ_2 and they are constant on the set, which implies that $M_1 - M_2$ is closed and hence it is identical with the whole M by the assumption of that $M_1 - M_2$ is not empty. Namely, we get $M_1 = M, M_0 = \phi, M_2 = \phi$. Consequently distinct principal curvatures are at most α, λ_1 and λ_2 on the whole space M , all of which multiplicities are constant. Similar to the case (1), the structure theorem and the classification theorem due to Berndt [1], Kimura [6] and Takagi [17] show that M is of type A .

The case (3). The set M_0 is also empty and $M_2 - M_1 = M - (M_0 \cup M_1)$ is open. Taking account of (2.8), the distinct principal curvatures on $M_2 - M_1$ are α, λ_3 and λ_4 whose multiplicity is equal to 1, $n - 1$ and $n - 1$, respectively. Because the trace h of A is given by $h = \alpha + (n - 1)(\lambda_3 + \lambda_4)$ we then obtain $(n - 2)h + \alpha = 0$. It yields that h is constant on $M_2 - M_1$ and hence λ_3 and λ_4 so are. Thus, the set $M_2 - M_1$ becomes closed and consequently $M_2 = M, M_0 = \phi, M_1 = \phi$. By developing the similar argument as that of the case (2), this shows that M is of type B .

The case (4). By the cases (1) and (2) we see that the subsets $M_1 - M_2$ and $M_2 - M_1$ are not empty and $M = M_0 \cup (M_1 \cap M_2)$. The set $M_1 \cap M_2$ is of course closed and M_1, M_2 contains a non void open set. In fact, if not so, then the continuity of principal curvatures tells us that M must coincides with the subset M_0 . In the case (4) this fact is rejected. Accordingly it may be supposed that there exists an interior of $M_1 \cap M_2$. On the interior of the intersection, distinct principal curvatures are at most $\alpha, \lambda_1, \lambda_2, \lambda_3$ and λ_4 which satisfies $\alpha^2 + c \neq 0$. We may suppose that the multiplicity of λ_1 or λ_2 is equal to $2p$ or $2q$ and that of λ_3 and λ_4 is equal to s . λ_1 and λ_2 being constant on $M_1 \cap M_2$, p, q and s are also constant on $M_1 \cap M_2$. Since λ_3 and λ_4 satisfy $h = \lambda_3 + \lambda_4$, we obtain $h = \alpha + 2p\lambda_1 + 2q\lambda_2 + s(\lambda_3 + \lambda_4)$ and thus

$$(2.9) \quad (1 - s)h = (1 + p + q)\alpha + (q - p)\sqrt{\alpha^2 + c}$$

on $M_1 \cap M_2$ because of $p + q + s = n - 1$. Since there are no simple roots for the shape operator A , this is equivalent to $s > 1$ and hence h

is constant on $M_1 - M_2$. This means that all principal curvatures are constant on $M_1 - M_2$ and therefore it is closed. Accordingly we have $M_1 \cap M_2 = M$, which shows that M is of type C , type D or type E .

By the above discussion, the classification theorem due to Takagi [18] tells us that the case where $M = M_0$ cannot occur and in the case where $M = M_1$ it is of type A_1 or of type A_2 . Of course, a homogeneous real hypersurface of type A_1 or type A_2 , which is a tube of any radius r ($r \neq \pi/4$) satisfies the assumptions of Theorem 2.1. Suppose that $M = M_2$, then M is of type B and it satisfies $\lambda_3 + \lambda_4 = h$. Let M be a homogeneous real hypersurface of $P_n C$ with principal curvature $\alpha = \sqrt{c} \cot 2r$ ($0 < r < \pi/4$), which is a tube of radius r . Then other distinct principal curvatures λ_3 and λ_4 satisfy $\lambda_3 = \frac{\sqrt{c}}{2} \cot(r - \pi/4)$ and $\lambda_4 = -\frac{\sqrt{c}}{2} \tan(r - \pi/4)$ and their multiplicities are equal to $n - 1$. Hence we have $h = \alpha + (n - 1)(\lambda_3 + \lambda_4) = \alpha - (n - 1)c/\alpha$, where we have used $\lambda_3 + \lambda_4 = -c/\alpha$. Since we have $\alpha^2 = (n - 2)c$, and it turns out that $\cot^2 2r = n - 2$.

Similarly we consider a homogeneous real hypersurface of type C, D or E of $P_n C$ which satisfies $\lambda_3 + \lambda_4 = h$. By the classification theorem of Takagi [17], we get

$$h = \alpha + (n - 3)\alpha + 2(-c/\alpha),$$

$$h = \alpha + 4\alpha + 4(-c/\alpha),$$

$$h = \alpha + 8\alpha + 6(-c/\alpha),$$

respectively, where we have used $\lambda_1 + \lambda_2 = \alpha$, which means

$$\cot^2 2r = \frac{1}{n - 2}, \quad \frac{3}{5} \quad \text{or} \quad \frac{5}{9}$$

respectively. This completes the proof of Theorem 2.1.

REMARK 1. A homogeneous real hypersurface of type B of $P_n C$ satisfying $\cot^2 2r = n - 2$ is identified with a real hypersurface $M(2n - 1, \frac{1}{n-1})$ which is constructed by the Hope fibration and it is η -Einstein (cf. Yano and Kon [19]).

In the previous case (4), by using $h = \lambda_3 + \lambda_4$ we see that the trace of A^2 is given by

$$\text{tr } A^2 = (1 + p + q)\alpha^2 + \frac{c}{2}(p + q) + \alpha\sqrt{\alpha^2 + c}(q - p) + sh^2 - 2s\lambda_3\lambda_4,$$

which together with (2.8) and (2.9) yields $\text{tr } A^2 = \frac{c}{2}(n - 1 - 2s) + (1 - 2s)\alpha h + sh^2$. Because of (1.7) the scalar curvature k of M is obtained : $k = c(n^2 - 1) + h^2 - \text{tr } A^2$. Therefore we have

$$k = \frac{c}{2}(2n^2 - n + 1 + 2s) + (2s - 1)\alpha h - (s + 1)h^2.$$

Thus, if the scalar curvature of M is constant, then h is constant on $M_1 \cap M_2$. As in the proof of Theorem 2.1, we have

THEOREM 2.2. *Let M be a real hypersurface $P_n C$, ($n \geq 3$) with constant scalar curvature. If the structure vector ξ is principal and if it satisfies $L_\xi S = 0$, then M is the same types as those in Theorem 2.1.*

For a tube of radius r over the submanifold of a complex space form $M_n(c)$, cf. see Cecil and Ryan [2] and Montiel [13]. In particular, a Montiel tube of a complex hyperbolic space is only defined here. Let H_1^{2n+1} be a $(2n + 1)$ -dimensional anti-de Sitter space in C^{n+1} , which is a Lorentz manifold of constant curvature $c/4$ (< 0). Given the real hypersurface M of a complex hyperbolic space $H_n C$, one can construct a Lorentz hypersurface N of H_1^{2n+1} which is a principal S^1 -bundle over M with time-line totally geodesic fibers and projection $\pi : N \rightarrow M$ in such a way that the diagram

$$\begin{array}{ccc} N & \xrightarrow{i'} & H_1^{2n+1} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{i} & H_n C \end{array}$$

is commutative (i, i' being the isometric immersions). In particular, let $N(t)$ be the Lorentz hypersurface of H_1^{2n+1} in C^{n+1} given by

$$-|z_0|^2 + \sum_{j=1}^n |Z_j|^2 = 1, \quad |z_0 - z_1|^2 = t.$$

It is seen in [13] that $M_n^*(t) = \pi(N(t))$ is the pseudo-Einstein real hypersurface of $H_n C$. Then $M_n^* = M_n^*(1)$ is called a *Montiel-tube*.

For the real hypersurface of $H_n C$, we can prove the following

THEOREM 2.3. *Let M be a $(2n - 1)$ -dimensional real hypersurface of $H_n C$, $n \geq 3$. If the structure vector ξ is principal and if it satisfies $L_\xi S = 0$, then M is congruent to one of the following hypersurfaces :*

- (A₀) a Montiel tube,
- (A₁) a tube of a totally geodesic hyperplane $H_k C$ ($k = 0$ or $n - 1$),
- (A₂) a tube of a totally geodesic $H_k C$, where $1 \leq k \leq n - 2$.

Proof. The proof may be divided into the same four cases as those stated in the proof of Theorem 2.1. Accordingly, in the case where (1), (2) and (3) we see that all principal curvatures are constant. In the case (4), we suppose that $s = 1$. Then (2.9) leads to $(1 + 2p)^2(1 + 2q)\alpha^2 = (p - q)^2c$. So, we have $p = q$ and $\alpha = 0$ because of $c < 0$. By the assumption $n \geq 3$ there is always a principal curvature $\lambda_1 = -\frac{\sqrt{c}}{2}$ or $\lambda_2 = \frac{\sqrt{c}}{2}$, a contradiction. It implies that $s > 1$, and hence h is constant on $M_1 \cap M_2$. Thus we have distinct principal curvatures $\alpha, \lambda_1, \lambda_2, \lambda_3$ and λ_4 on $M_1 \cap M_2$ and all principal curvatures are constant, which means that $M_1 \cap M_2$ coincide with the whole space M . Thus, by Berndt [1] M is locally congruent to one of homogeneous real hypersurfaces.

In the case where $M = M_0$, the hypersurface is of type A_0 , and of course a homogeneous real hypersurface of type A_0 satisfies the assumptions of Theorem 2.3. In the case where $M = M_1$, we have the similar situation to that in Theorem 2.1, and M is of type A_1 or type A_2 . Conversely, the hypersurface of type A_1 or type A_2 satisfies the hypothesis of Theorem 2.3. In the last case, M is of type B and it satisfies $h = \lambda_3 + \lambda_4$. By the classification theorem due to Berndt [1], a homogeneous real hypersurface of $H_n C$ has distinct principal curvatures

$$\lambda_3 = \frac{\sqrt{-c}}{2} \coth r, \quad \lambda_4 = \frac{\sqrt{-c}}{2} \tanh r,$$

different from $\alpha = -c \coth 2r$, and their multiplicities are equal to $n - 1$. Consequently we have $h = \alpha - (n - 1)(c/\alpha)$ because of $\lambda_3 + \lambda_4 = -c/\alpha$. This implies $\alpha^2 = (n - 1)c$, a contradiction. So it cannot occur. This completes the proof.

3. Covariant derivatives

This section is concerned with the covariant derivative of the Ricci tensor S in the direction of ξ . Let M be a real hypersurface of $M_n(c)$,

$c \neq 0$, $n \geq 3$ and ξ be a principal vector with corresponding principal curvature α . We suppose that

$$(3.1) \quad \nabla_{\xi} S = 0$$

for the Ricci tensor S . Since α is constant and the Ricci tensor S is given by (1.7), we have

$$(3.2) \quad \nabla_{\xi} S(Y) = (hI - A)\nabla_{\xi} A(Y) - \nabla_{\xi} A(AY) = 0$$

for any vector field Y because of (1.9) and (1.13). By (1.11) this is equivalent to

$$(3.3) \quad \alpha\{h(A\phi - \phi A) - A^2\phi + \phi A^2\} = 0.$$

If X is orthogonal to ξ , and it is principal vector with corresponding principal curvature λ , then ϕX is also principal vector with corresponding principal curvature $\lambda' = \frac{\alpha\lambda + c/2}{2\lambda - \alpha}$ provided $\lambda \neq \alpha/2$. Thus, (3.3) tells us that $(2\lambda - \alpha)(\lambda - \lambda')(\lambda + \lambda' - h) = 0$ because $\alpha \neq 0$ is assumed. Thus we can apply the proofs of Theorem 2.2 and Theorem 2.3 in the previous section to our situation and the same conclusions are obtained. Thus we have

THEOREM 3.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. Suppose that ξ is principal and that the corresponding principal curvature is nonzero and satisfies $\nabla_{\xi} S = 0$. If $c > 0$ and M has no simple roots, or if $c < 0$, then M is the same types as those in Theorem 2.1 and Theorem 2.3.*

REMARK 2. In the case where the ambient space is complex projective space, Kimura and Maeda [9] proved Theorem 3.1 under the condition that the mean curvature is constant in place of the condition that M has no simple roots.

REMARK 3. The condition $\nabla_{\xi} S = 0$ and $\alpha \neq 0$ is equivalent to $S\phi = \phi S$. Kimura [7] classified real hypersurfaces with constant mean curvature in $P_n C$ which satisfies $S\phi = \phi S$.

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