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ON THE ESTIMATE OF THE FIRST EIGENVALUE OF THE LAPLACIAN ON A COMPACT RIEMANNIAN MANIFOLD

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§0. Introduction

Throughout this paper, let M be an $n-$dimensional compact Riemannian manifold with or without boundary ∂M and be assumed that K and H are non-negative constants and d denotes the diameter of M.

We shall consider the solution of the equation

$$\Delta u = -\lambda u \tag{0.1}$$

defined on M. In case M is a manifold with boundary ∂M, we impose the following boundary condition:

$$\frac{\partial u}{\partial \nu}|_{\partial M} = 0, \tag{0.2}$$

where ν is the unit normal vector to ∂M.

Our purpose is largely to show that results of a lower bound of the first non-zero eigenvalue obtained for the Laplacian on a compact Riemannian manifold by P.Li, S.T.Yau, and R.Chen can be improved by a simple but sharper inequality. More precisely, this article has triple purposes: Firstly, we generalize the first non-zero eigenvalue estimate of (0.1) obtained in P.Li-S.T.Yau [4,5] with $Ricc(M) \geq 0$ to the case $Ricc(M) \geq -(n-1)K$ ([Theorem 0.1]). Secondly, we improve the first non-zero eigenvalue estimate of (0.1) given by P.Li-S.T.Yau [4,5] via the inequality $(a-b)^2 \geq \xi a^2 - \frac{\xi}{(1-\xi)} b^2 (0 < \xi < 1)$ without using a less sharp one $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ ([Theorem 0.2]). And finally, we introduce (probably improve) a lower bound of the non-zero Neumann

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eigenvalue of (0.1) with (0.2) [2] which is a generalization of [4, [Theorem 9]] to a compact Riemannian manifold with possibly non-convex ∂M ([Theorem 0.3]).

Using a method similar to those of P.Li–S.T.Yau and R.Chen, we have the followings:

THEOREM 0.1. *Let M be a compact Riemannian manifold without boundary. Let λ_1 be the first non-zero eigenvalue of (0.1). Suppose $Ricc(M) \geq -(n-1)K$. Then*

$$\lambda_1 \geq \Big[\frac{2\xi(n)}{(n-1)d^2}B^2 - (n-1)K\Big]\exp(-B),$$

where

$$\xi(n) = (n-1)\{\sqrt{(n-1)^2+2} - (n-1)\}$$

and

$$B = 1 + \sqrt{1 + \frac{2(n-1)^2}{\xi(n)}d^2K}$$

REMARK. Note that $\frac{1}{2} < \xi(n) < 1$. Hence this eigenvalue estimate is better than

$$\frac{1}{(n-1)d^2}\exp[-(1+\sqrt{1+4(n-1)^2d^2K})]$$

obtained by Li-Yau [5, p.129].

THEOREM 0.2. *Let M be a compact Riemannian manifold with ∂M. Let η_1 be the first nonzero eigenvalue of (0.1) and (0.2). Let ∂M satisfy the "interior rolling ϵ-ball" condition. Suppose $Ricc(M) \geq -(n-1)K$ and the second fundamental form elements of $\partial M \geq -H$. By choosing ϵ small, we have*

$$\eta_1 \geq \frac{1}{(1+H)^2}\Big[\frac{(1-\alpha^2)\xi(n,\alpha)}{2(n-1)d^2}B_1^2 - C\Big]\exp(-B_1)$$

where α and $\epsilon < 1$,

$$B_1 = 1 + \Big(1 + \frac{2(n-1)d^2C}{(1-\alpha^2)\xi(n,\alpha)}\Big)^{1/2},$$

$$C = (1+H)C_1 + (1+H)^2(n-1)K + \frac{H^2[(2n-2-2\xi(n,\alpha))^2 + (8n-8-4\xi(n,\alpha))\xi(n,\alpha)\alpha^2]}{\xi(n,\alpha)\alpha^2\epsilon^2(n-1)},$$

$$C_1 = \frac{2(n-1)H(3H+1)(H+1)}{\epsilon} + \frac{H+H^2}{\epsilon^2},$$

and

$$\xi(n,\alpha) = \frac{(n-1)}{1-\alpha^2}\{\sqrt{(n-1)^2 + 2(1-\alpha^2)} - (n-1)\}.$$

REMARK. When the boundary is convex, our estimate implies the eigenvalue estimate obtained in [Theorem 0.2] (See **Remark** after the [Lemma 2.2] below). In contrast to the result of R.Chen [2], $\xi(n,\alpha)$ has been inserted ($\frac{1}{2} < \xi(n,\alpha) < 1$). But we still don't know whether or not this result is sharper than that of R.Chen [2].

DEFINITION 0.3. Let ∂M be the boundary of a compact Riemannian manifold M. Then ∂M satisfies the interior rolling ϵ- ball condition if for each point $p \in \partial M$, there exists a geodesic ball $B_q(\epsilon/2)$ s.t.

$$p = \overline{B_q(\epsilon/2)} \cap \partial M$$

and

$$B_q(\epsilon/2) \subset M.$$

In §1., we shall give gradient estimates which are essential in proofs of the main results. In §2., we shall give brief proofs of main theorems.

§1. Gradient Estimates

We have the followings:

LEMMA 1.1. *Let M be as above with $Ricc(M) \geq -(n-1)K$ and let u be a non-constant first eigenfunction of (0.1) with λ_1 s.t.*

$$1 = \sup u > \inf u = -k \geq -1 \quad (0 < k \leq 1).$$

Then we get

$$|\nabla u|^2 \leq [\lambda_1 \frac{2}{1+k} + (n-1)K](1-u)(k+u).$$

Proof. This result follows from the proof of P.Li-S.T.Yau [5, p.121] with handling the condition $Ricc(M) \geq -(n-1)K$ carefully.

LEMMA 1.2. *If u is a non-constant first eigenfunction of (0.1) with* $\sup u = 1$ *and* λ_1, *then*

$$|\nabla u| \leq \Big[\frac{2(n-1)}{\xi(n)}\big(\frac{\beta}{\beta-1}\lambda_1 + (n-1)K\big)\Big]^{1/2}(\beta - u),$$

where $\beta > 1$.

REMARK. If M is a compact Riemannian manifold with ∂M being convex, then we have the same gradient estimate with the same $G(x)$ defined amid the proof of [Lemma 1.2] under the Newmann condition by the maximum principle. Thus in that case the same first eigenvalue estimate as in [Theorem 0.2] can be obtained.

Proof. The proof is a slight modification of the proof given by P.Li-S.T.Yau [5]. As in [5], consider the function defined by

$$G(x) = \frac{|\nabla u|^2}{(\beta - u)^2}$$

By applying the maximum principle and the Bochner identity in line with the proof in [5], it is easy to get

$$\sum u_{ij}^2 - (\lambda_1 + (n-1)K)|\nabla u|^2 - G[\lambda_1 u(\beta - u) + |\nabla u|^2] \leq 0 \tag{1.1}$$

at x_0, at which G attains its maximum.

If we choose an orthonormal frame at x_0 s.t.$u_1 = |\nabla u|$ and $u_i = 0$ for all $i \neq 1$,then, from $\nabla G(x_0) = 0$,

$$u_{11} = -\frac{|\nabla u|^2}{(\beta - u)} \tag{1.2}$$

and

$$u_{ii} = 0,\ i \neq 1 \tag{1.3}$$

Also, at x_0,

(1.4)

$$\begin{aligned}\sum u_{ij}^2 \geq \sum u_{ii}^2 &= u_{11}^2 + \sum_{i\neq 1} u_{ii}^2 \\ &\geq u_{11}^2 + \frac{(\sum_{i\neq 1} u_{ii})^2}{(n-1)} \\ &= u_{11}^2 + \frac{(\Delta u - u_{11})^2}{(n-1)} \\ &\geq u_{11}^2 + \frac{\xi}{(n-1)}u_{11}^2 - \frac{\xi}{(n-1)(1-\xi)}(\Delta u)^2 \\ &= \frac{(n-1+\xi)}{(n-1)}u_{11}^2 - \frac{\xi}{(n-1)(1-\xi)}(\Delta u)^2\end{aligned}$$

for $0 < \xi < 1$. Substituting (1.2),(1.3) and (1.4) into (1.1), we get

$$\frac{(n-1+\xi)}{(n-1)}\frac{|\nabla u|^4}{(\beta-u)^2} - \frac{\xi\lambda_1^2 u^2}{(n-1)(1-\xi)} - (\lambda_1 + (n-1)K)|\nabla u|^2 - \lambda_1 u\frac{|\nabla u|^2}{(\beta-u)} - \frac{|\nabla u|^4}{(\beta-u)^2} \leq 0$$

Since $\frac{u}{(\beta-u)} \leq \frac{1}{(\beta-1)}$, simple computations show

$$G \leq \max\{\frac{2(n-1)}{\xi}[\lambda_1 + (n-1)K + \lambda_1\frac{1}{\beta-1}],\ (\frac{2}{1-\xi})^{1/2}\lambda_1\frac{1}{\beta-1}\},$$

so that

$$|\nabla u|^2 \leq \frac{2(n-1)}{\xi(n)}[\frac{\beta}{\beta-1}\lambda_1 + (n-1)K])(\beta-u)^2,$$

for $\xi(n)$ defined previously.

LEMMA 1.3. *Let M be the same as* [Theorem 0.3]. *Let u be a non-costant solution of (0.1) and (0.2) with* $\sup u = 1$ *and* η_1. *If $\beta > 1$ and ϵ is "small", then*

$$|\nabla u|^2 \leq \frac{2(n-1)}{(1-\alpha^2)\xi(n,\alpha)}\left(C + (1+H)^2\eta_1\frac{\beta}{\beta-1}\right)(\beta-u)^2,$$

where

$$C,\ \xi(n,\alpha) = \textit{the same as } [\text{Theorem 0.3}].$$

Proof. Let $\psi(r), r, \phi, G(x)$ be the same completely as in the proof of [2]. The same kind of reasoning used in [2] gives

$$\begin{aligned}
(1.5) \qquad 0 \geq & -\frac{C_1}{1+\phi} - \frac{2(\psi')^2}{\epsilon^2(1+\phi)^2} - \frac{2\psi' u_1 r_1}{\epsilon(1+\phi)(\beta-u)} \\
& + \frac{\sum_{i\neq 1} u_{ii}^2}{u_1^2} + Ric_{11} - \frac{\eta_1 \beta}{\beta - u},
\end{aligned}$$

at the interior point x_0, where G attains its maximum and we choose an orthonormal farme $\{e_i\}$ $(1 \leq i < n)$ s.t. $u_1(x_0) = |\nabla u|(x_0)$. Then, at x_0, $\nabla G(x_0) = 0$ gives

$$(1.6) \qquad u_{1j} = -\frac{\psi' u_1 r_j}{\epsilon(1+\phi)} - \frac{u_1 u_j}{\beta - u}$$

It is obvious with (1.6) that

(1.7)

$$\begin{aligned}
\sum_{i\neq 1} u_{ii}^2 \geq & \frac{1}{n-1}(\sum_{i>1} u_{ii})^2 \\
& = \frac{1}{n-1}(\Delta u - u_{11})^2 \\
& \geq \frac{1}{n-1}\left(\xi(u_{11})^2 - \frac{\xi(\Delta u)^2}{\xi - 1}\right) \quad (0 < \xi < 1) \\
& \geq \frac{\xi u_1^4}{(n-1)(\beta-u)^2} + \frac{2\xi u_1^3 \psi' r_1}{\epsilon(n-1)(1+\phi)(\beta-u)} \\
& + \frac{\xi(\psi')^2 u_1^2 r_1^2}{\epsilon^2(n-1)(1+\phi)^2} \\
& - \frac{\xi \eta_1^2 u^2}{(1-\xi)(n-1)}
\end{aligned}$$

Combining (1.5) with (1.7),

$$0 \geq \frac{\xi u_1^2}{(n-1)(\beta-u)^2} - \frac{2(n-1-\xi)\psi' u_1 r_1}{\epsilon(n-1)(1+\phi)(\beta-u)} - \frac{C_1}{1+\phi} - \frac{2(n-1)(\psi')^2 - \xi(\psi')^2 r_1^2}{\epsilon^2(n-1)(1+\phi)^2} - (n-1)K - \frac{\eta_1\beta}{\beta-u} - \frac{\eta_1^2 u^2}{(n-1)u_1^2}\frac{\xi}{1-\xi}. \tag{1.8}$$

It is clear that

$$\frac{1}{n-1}\left(\frac{\alpha^2 u_1^2}{(\beta-u)^2} - \frac{2(2n-2-2\xi)u_1\psi' r_1}{\xi\epsilon(1+\phi)(\beta-u)}\right) \geq -\frac{(2n-2-2\xi)^2(\psi')^2}{\xi^2\epsilon^2\alpha^2(n-1)(1+\phi)^2}. \tag{1.9}$$

Hence ,with (1.9), (1.8) gives

$$0 \geq [\frac{1-\alpha^2}{n-1}\xi](\frac{u_1}{\beta-u})^2 - \frac{[(2n-2-2\xi)^2 - 4\alpha^2\xi^2](\psi')^2}{4\xi\alpha^2\epsilon^2(n-1)(1+\phi)^2} - \frac{2(\psi')^2}{\epsilon^2(1+\phi)^2} - \frac{C_1}{1+\phi} - (n-1)K - \frac{\eta_1\beta}{\beta-u} - \frac{\eta_1^2\xi u^2}{(n-1)(1-\xi)u_1^2}$$

Multiplying through by $(1+\phi)^4\frac{u_1^2}{(\beta-u)^2}$, it is easy to show

$$\frac{(1-\alpha^2)\xi}{n-1}G^2 \leq [\frac{[(2n-2-2\xi)^2 + (8n\xi - 8\xi - 4\xi^2)\alpha^2]}{4\xi\alpha^2\epsilon^2(n-1)}(\psi')^2 + (1+\phi)C_1 + (1+\phi)^2(n-1)K + \frac{\eta_1\beta}{\beta-u}(1+\phi)^2]G + \frac{\eta_1^2\xi}{(n-1)(1-\xi)}(1+\phi)^4\frac{u^2}{(\beta-u)^2}$$

i.e.

$$G \leq \max\{\frac{2(n-1)}{(1-\alpha^2)\xi}[C+(1+H)^2\frac{\beta\eta_1}{\beta-1}], \frac{\eta_1(1+H)^2}{\sqrt{1-\alpha^2}}(\frac{2}{1-\xi})^{1/2}\frac{1}{\beta-1}\}$$

Hence, if $\xi(n,\alpha)$ is the same as above, then our result comes out.

§3. Proofs of Main Theorems 0.1-3

A similar argument to that of [2,5] with the newly-made gradient estimates in [Lemma 1.1-3] gives the conclusions immediately.

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