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HYPERSPACE CONTRACTIBILITY OF TYPE $\sin(\frac{1}{x})$–CONTINUA

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1. Preliminary

Let X be a metric continuum with a metric d. Denoted by 2^X and $C(X)$ the hyperspaces of all nonempty closed subsets and subcontinua of X respectively and endow each with the Hausdorff metric H. A continuous map μ on $C(X)$ into the closed unit interval I is called a whitney map [12] if it satisfies the following conditions: 1. $\mu(x) = 0$ for each $x \in X$, 2. if $A, B \in C(X)$, $A \subset B$, and $A \neq B$, then $\mu(A) < \mu(B)$, and 3. $\mu(X) = 1$. For convenience, we shall fix one such μ throughout. For each point $x \in X$, let $T(x)$ be the set of all elements of $C(X)$ that contain x. Then T is a function on X into $2^{C(X)}$. An element $A \in T(x)$ is said to be admissible at x in X if for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $y \in X$, $d(x,y) < \delta$, there is an element $B \in T(y)$ such that $H(A,B) < \epsilon$. Let $A(x)$ be the set of all elements of $T(x)$ which are admissible at x in X. Then $A : X \to 2^{C(X)}$ is a function [6].

LEMMA 1.1.[6]. *If $B \in A(\xi)$, $C \in A(x)$, and $\xi \in B \cap C$ then $B \cup C \in A(x)$.*

A metric continuum X is said to be T-admissible if, for each $(x,t) \in X \times I$, the following condition is met: for each $A \in A(x) \cap \mu^{-1}(t)$ and $t' \in [t, 1]$, there is an element $B \in A(x) \cap \mu^{-1}(t')$ such that $A \subset B$. It was observed in [8] that T-admissibility is a necessary condition for the contractibility of the hyperspaces of X.

A subset $\mathcal{S}$ of $C(X)$ is monotone-connected if, for each pair A and B of elements of $\mathcal{S}$ with $A \subset B$, there is an arc $\alpha : I \to \mathcal{S}$ joining $A = \alpha(0)$ and $B = \alpha(1)$ such that $\alpha(s) \subset \alpha(t)$ whenever $s \leq t$. If

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$A, B \in C(X)$ and $A \subset B$, we let $T(A,B) = \{C \in C(B) : A \subset C\}$. Then $T(A,B)$ is monotone connected [3].

Let M be a subset of X and $B \in C(X)$ such that $M \subset B$. A fiber function on M into $C(B)$ is a set-valued function $F : M \to C(B)$ such that $\{x\}$, $B \in F(x)$ for each $x \in M$. A fiber function $F : M \to C(B)$ is monotone-connected if $F(x)$ is monotone-connected for each $x \in M$. A monotone-connected, lower semicontinuous fiber function $\alpha : M \to C(B)$ (in the subspace topology) is called a γ-map if $\alpha(x) \subset A(x)$ for each $x \in M$. Let $M = \{x \in X : T(x) \neq A(x)\}$. The set M is called the $\mathcal{M}$-set of X. The points of the complement of M are called k-points of X. It was shown [11] that if $M = \emptyset$ then $C(X)$ is contractible. For $M \neq \emptyset$ let $\overline{M}$ be the closure of M in X. Then we have the following.

THEOREM 1.2.[8]. *For any T-admissible metric continuum X with nonempty $\mathcal{M}$-set M, $C(X)$ is contractible if and only if there exists a γ-map $\alpha : \overline{M} \to C(X)$.*

2. Contractibility of $C(X)$ of type $\sin(\frac{1}{x})$-continua

A continuous map $f : [0,1) \to [0,1]$ is said to be piecewise linear over a sequence V in $[0,1)$ converging to 0 if the restriction map $f \mid [v, v']$ of f is linear for each consecutive pair v, v' of V. And a piecewise linear map over V is called sawtooth if each $v \in V$ is a local extreme point of the map. Let X be the compactification space of the graph of a sawtooth map $f : [0,1) \to [0,1]$ over V with the unit interval as remainder. We reserve $\overline{V} = \{(v, f(v)) : v \in V\}$ for X and call elements of $\overline{V}$ local maximal or minimal points of X.

In [1] Awartani proved that, for each continuous map g of $[0,1)$ onto $[0,1]$, there is a sawtooth map $f : [0,1) \to [0,1]$ such that the compactification spaces in $[0,1] \times [0,1]$ of the graphs of f and g are homeomorphic. Henceforth, we consider only those spaces which are the compactification of graphs of sawtooth maps.

Let X denote the compactification of the graph Y of a sawtooth map with the unit interval $I \times 0 = \tilde{I}$ as its remainder. Then $\tilde{I}$ is non-locally connected because the graph Y is forced to oscillate as it approaches to $\tilde{I}$ and the space X is locally connected at each point of Y. Hence each point Y is a k-point of X and thus if X has a nonempty $\mathcal{M}$-set then it must lie in $\tilde{I}$. Therefore all derived sets being connected are intervals lying in $\tilde{I}$. We investigate these object thoroughly.

Let $\pi_i : [0,1] \times [0,1]$ be the projection maps, $i = 1, 2,$. If $p, q \in Y$, then we write $p \leq q$ if and only if $\pi_1(p) \leq \pi_1(q)$, and the closed arc in Y joining p and q is denoted by $[p, q]$. If $a, b \in \tilde{I}$ we write $a \leq b$ if and only if $\pi_2(a) \leq \pi_2(b)$ and the closed interval in $\tilde{I}$ joining a and b is denoted by $\langle a, b\rangle$ and the half-open interval opened at a by $(a, b\rangle$. Furthermore if ϵ is an number and $p \in \tilde{I}$, $p + \epsilon$ we mean $\pi_2(p) + \epsilon$.

Let $p, q \in Y$ and $p \leq q$. The closed interval $[p, q]$ is called a wedge (respectively spike) if the lowest (highest) points of $[p, q]$ are interior points. If $[p, q]$ is a wedge we write $[p, q]_w$ and if it is a spike we write $[p, q]_s$.

Let $e \in \tilde{I}$. Then e is called essential if it satisfies the following conditions:

(i) there exists a sequence $\{[p_n, q_n]_w\}$ of wedges (or $\{[p'_n, q'_n]_s\}$ of spikes) in Y and a positive number ϵ such that $\lim_{n\to\infty}[p_n, q_n]_w = \langle e, e+\epsilon\rangle(\lim_{n\to\infty}[p'_n, q'_n] = \langle e-\epsilon, e\rangle)$ and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = e + \epsilon(\lim_{n\to\infty} p'_n = \lim_{n\to\infty} q'_n = e - \epsilon)$.

(ii) e is a limit point of a sequence in $\tilde{I}$ satisfying the condition (i).

Let E be the set of all essential points. Since Y is the graph of a sawtooth map (linear over V), the highest (lowest) points of a spike (wedge) occurs at the point of $\overline{V}$. Thus each point $e \in E$ is the limit point of a sequence in $\overline{V}$ of points local maximuum or of points of local minimuum.

Let $0 \leq \epsilon_1 < \epsilon_2 \leq 1$, and let $U(\epsilon_1) = \{(x, y) \in E^2 : y > \epsilon_1\}$ and $U(\epsilon_2) = \{(x, y) \in E^2 : y < \epsilon_2\}$. Then $U(\epsilon_i) \cap X$ is an open set, $i = 1, 2$ and each component of it is an open arc. An arc component C in $U(\epsilon_1) \cap X$ lying in Y is called an arc of type M if both end points of $\overline{C}$ (the closure of C) lie on the horizontal line $y = \epsilon_1$. If C is an arc of type M then $\overline{C}$ contains its maximal points in its interior. An arc component C in $U(\epsilon_2) \cap X$ lying in Y is called an arc of type W if both end points of $\overline{C}$ lie on $y = \epsilon_2$, and hence $\overline{C}$ contains its minimal points in its interior. Thus if C is an arc of type M (or W) then $\overline{C}$ is a spike (wedge). Finally if C is an arc component of $U(\epsilon_1) \cap U(\epsilon_2) \cap X$ lying in Y such that the closed arc $\overline{C}$ has one end point on $y = \epsilon_1$ and the other on $y = \epsilon_2$, then C is called an arc of type N.

Let $\langle a, b\rangle$ be a subinterval of $\tilde{I}, \epsilon > 0$, and $\delta > 0$. Then $U = [0, \delta) \times (a - \epsilon, b + \epsilon) \cap X$ is an open set in X containing $\langle a, b\rangle$, and U is the union of at most countable number of arc components. If $\{C_n\}$ is a sequence of components of $U \cap Y$, then we assign the indices of the sequence according to the natural order relation of the first coordinate of point of each component. Thus if $x \in C_{n+1}$ and $y \in C_n$ then $\pi_1(x) < \pi_1(y)$.

LEMMA 2.1. *Let $e \in \tilde{I}$. e is an essential point if and only if e is the limit point of a sequence $\{w_n\}$ of lowest interior points of arcs $[p_n, q_n]_w$ of type W or the limit point of a sequence $\{m_n\}$ of highest interior points of arcs $[p_n, q_n]_s$ of type M.*

Hence we divide the set $E = \hat{E} \cup \check{E}$, where $\hat{E} = \{e \in E : e = \lim_{n\to\infty} m_n\}$, $\check{E} = \{e \in E : e = \lim_{n\to\infty} w_n\}$. Let $(0,0) = \overline{0}$ and $(0,1) = \overline{1}$. Since the unit interval $\tilde{I}$ is the remainder in the compactification of Y, $\overline{0} \in \check{E}$ and $\overline{I} \in \hat{E}$. It may be that $\hat{E} \cap \check{E} \neq \emptyset$.

LEMMA 2.2. *Let $\langle a_i, b_i\rangle$ be a closed interval in $\tilde{I}$, $i = 1,2$. Then $H(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) = \max\{|a_1 - a_2|, |b_1 - b_2|\}$.*

LEMMA 2.3. *Let $\langle a, b\rangle$ be a closed subinterval in $\tilde{I}$ and let C be an arc component in $U = [0,\epsilon) \times (a-\epsilon, b+\epsilon) \cap X$. Then $H(\overline{C}, \langle a, b\rangle) < \epsilon$ if and only if $H(\pi_2, (\overline{C}), \langle a, b\rangle) < \epsilon$.*

Let $T : X \to C(X)$ be the total fiber map. Since the space X is locally connected at each point of Y, point $x \in Y$ is a k-point. Hence each element of $T(x)$ is admissible at x so that we have $T(x) = A(x)$. If $x \in \tilde{I}$ then some elements of $T(x)$ may not be admissible at x.

PROPOSITION 2.4. *Let $S = \{A \in C(X) : A \supset \tilde{I}\}$. Then $S \subset A(x)$ for each $x \in \tilde{I}$.*

Proof. Let $B \in S$. Suppose $B \backslash \tilde{I} = \emptyset$. Let $\epsilon > 0$. Let $U = [0, \epsilon/2) \times [0,1] \cap X$ be an open set containing $\tilde{I}$. Let $0 < \delta < \epsilon/2$, and y a point of the δ-neighborhood V of x in X. Then U contains only one component X with the following property: C is open in X, $C \supset \tilde{I}$ and $V \subset C$. Hence $\pi_2(\overline{C}) = \tilde{I}$ and $H(\overline{C}, \tilde{I}) < \epsilon$ by (2.3). Suppose $B \backslash \tilde{I} \neq \emptyset$. Let $z \in B \backslash \tilde{I}$. Then choose $0 < \delta < \pi_1(z)/2$. Then if V is the δ-neighborhood of x, then $V \subset B$. Hence for each $y \in V$ we have $y \in B$. Therefore $H(B, B) = 0$. Let $x \in B \in C(X)$. Define $T(x, B) = \{C \in C(B) : x \in C\}$.

PROPOSITION 2.5. *Let $\langle a,b\rangle \in T(x,\tilde{I})$, and $\langle a,b\rangle \neq \tilde{I}$. Then $\langle a,b\rangle \in A(x)$ if and only if, for each $\epsilon > 0$, there is $\delta > 0$ such that if C is a componet of the open set $U = [0,\epsilon/2)\times(a-\epsilon/2, b+\epsilon/2)\cap X$ which intersects the δ-neighborhood V of x in X, then $H(\langle a,b\rangle, \pi_2(\overline{C})) < \epsilon$.*

Proof. Suppose $\langle a,b\rangle$ is admissible at x in X. Let $\epsilon > 0$. Then there is $0 < \delta < \epsilon/2$ such that each point y in the δ-neighborhood V of x, there is an element $B \in T(y)$ such that $H(\langle a,b\rangle, B) < \epsilon/4$. Let $x_1, x_2 \in B$ such that $\pi_2(x_1) \geq \pi_2(x)$ and $\pi_2(x_2) \leq \pi_2(x)$ for all $x \in B$. If $\pi_2(x_1) \geq b+\epsilon/4$ then $H(\langle a,b\rangle, B) \geq \epsilon/4$. If $\pi_2(x_2) \leq a-\epsilon/4$, then the distance from $\langle a,b\rangle$ to B would be greater than or equal to $\epsilon/4$. Neither of the cases is possible. Hence $a - \epsilon/4 < \pi_2(x_2) \leq \pi_2(x_1) < b + \epsilon/4$. $(*)$

Now let $w \in B$. Since $\langle a,b\rangle$ is compact there is an element $c \in \langle a,b\rangle$ such that $d(w, \langle a,b\rangle) = d(w,c) \geq \pi_1(w)$. Since $d(w,\langle a,b\rangle) < \epsilon/4$, we have $\pi_1(w) < \epsilon/4)$ $(**)$. Combining $(*)$ and $(**)$, we conclude that $B \subset U$. Let C be the component in U containing $y \in V$. Then $\overline{C} \supset B$. Therefore $\pi_2(B) \subset \pi_2(\overline{C})$ and $\pi_2(\overline{C}) \subset \langle a-\epsilon/2, b+\epsilon/2\rangle$. Therefore we have

$$\begin{aligned} H(\pi_2(\overline{C}), \langle a,b\rangle) &\leq H(\pi_2(\overline{C}), \langle a-\epsilon/2, b+\epsilon/2\rangle) \\ &\quad + H(\langle a-\epsilon/2, b+\epsilon/2\rangle, \langle a,b\rangle) < \epsilon. \end{aligned}$$

Conversely, we may suppose that for each $\epsilon > 0$ there is $\delta > 0$ such that if C is a component of $U = [0,\epsilon/4)\times(a-\epsilon/4, b+\epsilon/4)\cap X$ intersecting the δ-neighborhood V of x in X, then

$$H(\langle a,b\rangle, \pi_2(\overline{C})) < \epsilon/2.$$

If $y \in \tilde{I}$ such that $d(y,x) < \delta < \epsilon/4$, then $B = \langle a-\epsilon/4, b+\epsilon/4\rangle$ is the closure of the components of U (assuming either $a - \epsilon/4 \neq \overline{0}$ or $b+\epsilon/4 \neq \overline{1}$) containing y and $H(\langle a,b\rangle, B) < \epsilon/2$. If $y \in Y\cap V$, let C be the component of U containing y. Since $a-\epsilon/4 \neq \overline{0}$ or $b+\epsilon/4 \neq \overline{1}$, C mush lie in Y. Then for each $m \in \pi_2(\overline{C})$, the horizontal line intersects a point at w of $\overline{C}$. Thus $d(m,\overline{C}) \leq \pi_1(w) \leq \epsilon/4$. Similarly for each $w \in \overline{C}$, we have $d(w,\pi_2(\overline{C})) < \epsilon/4$. Therefore $H(\pi_2(\overline{C}),\overline{C}) \leq \epsilon/4$. And hence $H(\langle a,b\rangle,\overline{C}) \leq H(\langle a,b\rangle, \pi_2(\overline{C})) + H(\pi_2(\overline{C}),\overline{C}) < \epsilon$. Therefore $\langle a,b\rangle$ is admissible at x in X.

PROPOSITION 2.6. *If $\langle x,b\rangle$ is a subcontinuum of $\tilde{I}$ with the end points $x \leq b$ such that $\langle x,b\rangle \in A(x)$, then $T(x,\langle x,b\rangle) \subset A(x)$. Similarly if $\langle a,x\rangle \subset \tilde{I}$ such that $\langle a,x\rangle \in A(x)$, then $T(x,\langle a,x\rangle) \subset A(x)$. Hence if $\langle a,b\rangle \subset \tilde{I}$, $a \leq x \leq b$, such that $\langle a,x\rangle$, $\langle x,b\rangle \in A(x)$ then $T(x,\langle a,b\rangle) \subset A(x)$.*

Proof. We prove first that $T(\overline{0},\tilde{I}) \subset A(\overline{0})$. Let $\langle \overline{0},d\rangle \in T(\overline{0},\tilde{I})$. If $d=1$, then $\langle \overline{0},\overline{1}\rangle = \tilde{I} \in A(\overline{0})$ by (2.4). So we may assume that $d<1$. Let $\epsilon>0$ be a number such that $\epsilon < \frac{1}{2}\min\{d,1-d\}$. Since $\tilde{I} \in A(\overline{0})$, there exists $0<\delta<\epsilon/2$ such that if y is a point of the δ-neighborhood V of $\overline{0}$ then the component C of the open set $U=[0,\epsilon/2)\times[0,1]\cap X$ containing y satisfies $\tilde{I} \subset C$ and $H(\tilde{I},\overline{C})<\epsilon$.

Now let $U_1=[0,\epsilon/2)\times[0,d+\epsilon/2)\cap X$ and let C_1 be the component of U_1 containing y. Then $U_1 \subset U$ and $C_1 \subset C$. If $y \in V \cap \tilde{I}$, then $\overline{C}_1 = \langle \overline{0}, d+\epsilon/2\rangle$ so that $H(\overline{C}_1, \langle \overline{0}, d+\epsilon/2\rangle)<\epsilon$.

Suppose $y \in V\cap Y$. Since $\overline{C}$ contains a maximal element z, $\pi_2(z) = 1 > d+\epsilon/2$, and it also contains y with $0 \leq \pi_2(y) < d+\epsilon/2$, where the horizontal line $y=d+\epsilon/2$ separates $\overline{C}$. Hence the end points of the arc $\overline{C}_1$ must lie on the line $y=d+\epsilon/2$. If w is a minimal point of $\overline{C}_1$, then $0\leq \pi_2(w) \leq \pi_2(y)$. Thus $\pi_2(\overline{C}) = \langle \pi_2(w), d+\epsilon/2\rangle$. Hence $H(\pi_2(\overline{C}_1), \langle \overline{0},d\rangle)<\epsilon$. We have $\langle \overline{0},d\rangle \in A(\overline{0})$ by (2.5). Thus we conclude that $T(\overline{0},\tilde{I}) \subset A(\overline{0})$.

Similarly one can show that $T(\overline{1},\tilde{I}) \subset A(\overline{1})$.

Now suppose $0<x<b\leq 1$ and $\langle x,b\rangle \in A(x)$. We consider the admissibility of $\langle x,d\rangle$ at x in X for $d<b$. Let $\epsilon>0$ be a number such that $\frac{1}{3}\min\{(b-d),x,(d-x)\}$. Since $\langle x,b\rangle \in A(x)$, there exists $0<\delta<\epsilon/2$ such that if y is a point of the δ-neighborhood V of x and C is the component of $U=[0,\epsilon/2)\times(x-\epsilon/2,b+\epsilon/2)\cap X$ containing y, then $H(\langle x,b\rangle,\pi_2(\overline{C}))<\epsilon$. Now let C_1 be the component of $U_1[0,\epsilon/2)\times(x-\epsilon/2,d+\epsilon/2)\cap X$ containing the point y. Then $U_1 \subset U$ and $C_1 \subset C$.

Since $\overline{C}$ contains a point z such that $d+\epsilon/2 < \pi_2(z)$ and $\pi_2(y) < d+\epsilon/2$, the horizontal line $y=d+\epsilon/2$ separates $\overline{C}$. S the arc $\overline{C}_1$ containing y must have at least one end point lying on the line. Let z be a minimal point of $\overline{C}_1$. If z' is a minimal point of $\overline{C}$, then $x-\epsilon/2 \leq \pi_2(z') \leq \pi_2(z) \leq \pi_2(y)$. Hence $\pi_2(\overline{C}_1) = \langle \pi_2(z), d+\epsilon/2\rangle$. Since $d(x,\pi_2(y))<\delta<\epsilon/2$, $H(\pi_2(\overline{C}_1),\langle x,d\rangle) = H(\langle \pi_2(z),d+\epsilon/2\rangle,\langle x,d\rangle) =$

$\max\{|\pi_2(z)-x|, |d+\epsilon/2-d|\} \le \epsilon/2 < \epsilon$. Therefore $\langle x,d\rangle \in A(x)$ by (2.5). We thus conclude that $T(x,\langle x,b\rangle) \subset A(x)$.

The proof of the second assertion is similar to the first one. For the third assertion, we observe that if $\langle a,x\rangle$ and $\langle x,b\rangle$ are admissible at x in X and $a < x < b$ then their union is also admissible at x in X by (1.1).

REMARK. The end points of $\hat{I}$ are k-points. To see it, let $A \in T(\overline{0})$. Then $A \in T(\overline{0},\langle\overline{0},\overline{1}\rangle)$, if $A \subset \hat{I}$. Hence $A \in A(\overline{0})$ by (2.6). If $A \supset \hat{I}$, then $A \in A(\overline{0})$ by (2.4). Similar argument can apply to elements of $T(\overline{1})$.

A nonempty proper subcontinuum K of a metric space Z is an R^2-continuum of Z [2] if there exists an open set U containing K and two sequences $\{C_n^1\}$ and $\{C_2^2\}$ of components of U such that $(\lim_{n\to\infty} C_n^1) \cap (\lim_{n\to\infty} C_n^2) = K$.

In [2] it is proven that if a metric continuum Z contain an R^2-continuum then $C(Z)$ is not contractible.

For the space X with the graph Y of a sawtooth map, no subcontinuum of Y is an R^2-continuum of X. Hence if X has an R^2-subcontinuum, it must be a subcontinuum of $\hat{I}$ or a subcontinuum containing $\hat{I}$. But if $B \in C(X)$, $B \supset \hat{I}$, then each open set containing B has a unique open component containing B properly so that B can not be an R^2-continuum. Suppose $\langle\overline{0},b\rangle$ is a subcontinuum of $\hat{I}$ and $b \in \overline{1}$. Let U be an open set in X containing $\langle\overline{0},b\rangle$. We show that there exists $\epsilon > 0$ such that if $\{C_n\}$ is any sequence of components of U such that $\langle\overline{0},b\rangle \subset \lim_{n\to\infty} C_n$, then $\langle\overline{0},b+\epsilon\rangle \subset \lim_{n\to\infty} C_n$.

Let $\{C_n\}$ be a sequence of components of U such that $\langle\overline{0},b\rangle \subset \lim_{n\to\infty} C_n$. Then, since U is open, there exists $\epsilon > 0$ such that $U' = [\overline{0},\epsilon) \times [\overline{0},b+\epsilon) \cap X \subset U$ with $b+\epsilon < \overline{1}$. Then the horizontal line $y = b+\epsilon$ separates C_n for almost all n. Since $\langle\overline{0},b+\epsilon\rangle \in A(\overline{0})$ by the remark above and $U' \subset U$, there is a sequence $\{C_k'\}$ of arc components of U' of type W each of whose end points lie on the line $y = b+\epsilon$ such that $\overline{C}_k' \subset C_{n_k}$ and $\lim_{n\to\infty} \overline{C}_k' = \langle\overline{0},b+\epsilon\rangle$. Therefore $\langle\overline{0},b+\epsilon\rangle \subset C_n$. This proves that $\langle\overline{0},b\rangle$ can not be an R^2-continuum.

Similar argument applies for showing that $\langle a,\overline{1}\rangle$, $a \neq \overline{1}$, is not an R^2-continuum.

THEOREM 2.7. *A subcontinuum* $\langle a,b\rangle$ *of* $\hat{I}, a \neq \overline{0}, b \neq \overline{1}$, *is an*

R^2-continuum of X if and only if there exist $\epsilon > 0$, two essential points $e_1 \in \check{E}$ and $e_2 \in \hat{E}$, $e_1 \leq e_2$, and two sequences $\{C_n^1\}$ and $\{C_n^2\}$ of components of $U = [0,\epsilon) \times (a-\epsilon, b+\epsilon) \cap X$ of types W and M respectively such that $a = e_1, b = e_2$, and $(\lim_{n\to\infty} C_n^1) \cap (\lim_{n\to\infty} C_n^2) = \langle e_1, e_2 \rangle$.

Proof. Suppose $\langle a,b \rangle$ is an R^2-continuum of X. Let U be an open set containing $\langle a,b \rangle$ and let $\{C_n^1\}$ and $\{C_n^2\}$ be two sequences of components of U such that $(\lim_{n\to\infty} C_n^1) \cap (\lim_{n\to\infty} C_n^2) = \langle a,b \rangle$. We may assume without loss of generality that $C_n^1, C_n^2 \subset Y$ for all n and we let $C^1 = \lim_{n\to\infty} C_n^1$ and $C^2 = \lim_{n\to\infty} C_n^2$.

First we show that the R^2-continuum $\langle a,b \rangle$ is properly contained in C^1. Suppose $C^1 = \langle a,b \rangle$. Then there exists $\epsilon > 0$ such that $\overline{U}(\epsilon) = [0,\epsilon] \times [a-\epsilon, b+\epsilon] \cap X$ is contained in U. Furthermore there is a positive integer k such that $C_n^1 \subset U(\frac{\epsilon}{2}) = [0, b+\epsilon/2) \times (a-\epsilon/2, b+\epsilon/2) \cap X$ for all $n \geq k$. So we have $C_n^1 \subset U(\frac{\epsilon}{2}) \subset \overline{U}(\epsilon) \subset U$ for $n \geq k$. Sine each C_n^1 is a component of U, the end points of $\overline{C}_n^1$ must lie on $\overline{U} \backslash U$. On the other hand, for each $n \geq k$, $C_n^1 \subset U(\frac{\epsilon}{2})$ so that $\overline{C}_n^1 \subset \overline{U}(\frac{\epsilon}{2})$. But $(\overline{U} \backslash U) \cap \overline{U}(\frac{\epsilon}{2}) = \phi$. This is contradiction. Therefore $C^1 \neq \langle a,b \rangle$. Similar argument applies to show that $C^2 \neq \langle a,b \rangle$.

Let $a' \in C^1 \backslash \langle a,b \rangle$ and $b' \in C^2 \backslash \langle a,b \rangle$. Suppose $a' < a$ we show that $b' > b$ (the argument for $a' > b$ implies $b' < a$ is similar). If $b' < b$, then $\langle b', b \rangle \subset C^2$. Since $a' < a$, we also have $\langle a', b \rangle \subset C^1$. Combining those two, we have $\langle a', b \rangle \cap \langle b', b \rangle \subset C^1 \cap C^2$. But this is impossible. Therefore $b' > b$.

Let us assume that $a' < a$ for each $a' \in C^1 \backslash \langle a,b \rangle$ and $b < b'$ for each $b' \in C^2 \backslash \langle a,b \rangle$. Let $a_0 \in C^1 \backslash \langle a,b \rangle$ and $b_0 \in C^2 \backslash \langle a,b \rangle$ be fixed. Choose $\epsilon > 0$ such that $a_0 < a - \epsilon < a$ and $b < b + \epsilon < b_0$, and

$$U_1 = [0,\epsilon) \times (a-\epsilon, b-\epsilon) \cap X \subset U.$$

Then the condition $a' < a$ for all $a' \in C^1 \backslash \langle a,b \rangle$ implies that there is a subsequence $\{C_{n_i}^1\}$ of $\{C_n^1\}$ such that if x_i is a maximal point of $\overline{C}_{n_i}^1$ (i.e. $\pi_2(x_i) \geq \pi_2(x)$ for all $x \in \overline{C}_{n_i}^1$) then $\pi_2(x_i) < b + \epsilon$. Since $a_0 < a - \epsilon < a$ there exists a positive integer k such that each $C_{n_i}^1$ intersects the line $y = a - \epsilon$ for $i \geq k$.

Now let A_i be the arc component of U_1 containing the point x_i, $i \geq k$. Then $A_i \subset C^1_{n_i}$ for each $i \geq k$ so that x_i is a maximal point of $\overline{A}_i$. It is easily seen that each $\overline{A}_i$ intersects the line $y = a - \epsilon$. And hence x_i is an interior point of $\overline{A}_i$. This means that each A_i is an arc of type M with its maximal point x_i in its interior and whose both end points lie on $y = a - \epsilon$. It is clear that $\lim_{i\to\infty} C^1_{n_i} = \langle a - \epsilon, b\rangle$, and $\lim_{i\to\infty} x_i = b$. Hence $b \in \hat{E}$.

Similar argument can be applied by using the condition that $b < b'$ for each $b' \in C^2\backslash\langle a, b\rangle$ to show that there is a subsequence $\{B_i\}$ of $\{C^1_n\}$ of type W with lowest point $y_i \in B_i$ such that $\lim_{i\to\infty} B_i = \langle a, b + \epsilon\rangle$ and $\lim_{i\to\infty} y_i = a$, $a \in \check{E}$. Thus we have $\lim_{i\to\infty} B_i \cap \lim_{i\to\infty} A_i = \langle a, b\rangle$ such that $a \in \check{E}$ and $b \in \hat{E}$. Converse is obvious.

COROLLARY 2.8. *If $e \in \check{E} \cap \hat{E}$, then $\{e\}$ is an R^2-continuum of X.*

COROLLARY 2.9. *Let $e_1 \in \check{E}$ and $e_2 \in \hat{E}$ and $e_1 \leq e_2$. Suppose there are points $x, y \in \hat{I}\backslash E$ which satisfy the following:*

(i) $x < e_1 \leq e_2 < y$

(ii) (a) $\langle x, e_2\rangle \in A(x)$ *but* $\langle x, z_1\rangle \notin A(x)$ *for some* z_1 *such that* $e_2 < z_1 < y$ *and* $(e_2, z_1) \cap E = \emptyset$, *and*

(b) $\langle e_1, y\rangle \in A(y)$ *but* $\langle z_2, y\rangle \notin A(y)$ *for some* z_2 *such that* $x < z_2 < e_1$ *and* $(z_2, e_1) \cap E = \emptyset$.

Then $\langle e_1, e_2\rangle$ *is an* R^2*-continuum of* X.

Proof. We shall find an open set U and two sequences $\{C_n\}$ and $\{D_n\}$ of arc components of U of types M and W respectively such that $\lim_{n\to\infty} C_n \cap \lim_{n\to\infty} D_n = \langle e_1, e_2\rangle$. Since $\langle x, z_1\rangle \notin A(x)$, there exists $\epsilon_1 > 0$ such that for each $\delta_n = \frac{1}{n}$, there exists $x_n, d(x_n, x) < \frac{1}{n}$ such that $H(\langle x, z_1\rangle, T(x_n)) \geq \epsilon_1$. Similarly there exist $\epsilon_2 > 0$ and $y_n, d(y_n, y) < \frac{1}{n}$ such that $H(\langle z_2, y\rangle, T(y_n)) \geq \epsilon_2$. Let $\epsilon = \frac{1}{2} \cdot \min\{\epsilon_1, \epsilon_2, d(z_1, E), d(z_2, E)\}$, and let $U = [0, \epsilon) \times (x - \epsilon, y + \epsilon) \cap X$.

Let $P = [0, \epsilon) \times [e_2 + \epsilon, z_1] \cap X$. Since $(e_2, z_1) \cap E = \emptyset$ we may assume without loss of generality that P does not contain any point $v \in \overline{V}$.

Let C'_n be the component of U containing x_n for each $n = 1, 2, \ldots$. Then by the condition (i) (a) we have $\langle x, e_2\rangle \in A(x)$ implies each C'_n contains an element $A_n \in T(x_n)$ such that $H(\langle x, e_2\rangle, A_n) < \epsilon$ and $\langle x, z_1\rangle \notin A(x)$ implies $H(\langle x, e_2\rangle, B_n) > \epsilon$ for each $B_n \in T(x_n)$.

Consider the open set $U_1 = [0, \epsilon) \times (x - \epsilon, e_2 + \epsilon) \cap X$. For each n with $\frac{1}{n} < \epsilon$, let C_n be the arc component of U_1 such that $x_n \in C_n$. We may assume without loss of generality that $C_n \cap C_m = \emptyset$ for $m \neq n$. Then $C_n \subset C'_n$ for each n.

Let m_n be a maximal point of $\overline{C}_n$. We will show that m_n is an interior point of $\overline{C}_n$. Suppose m_n lies on the line $y = e_2 + \epsilon$. Then $\overline{V} \cap P = \emptyset$ implies that $m_n \notin \overline{V}$. This means that m_n is not a point of local maximuum. Because $P \cap \overline{V} = \emptyset$, the component C'_n must intersect the line $y = z_1$ at a point z. This would imply that C'_n contains the subcontinuum $[x_n, z] \in T(x_n)$ such that $H(\langle x, z_1 \rangle, [x_n, z]) < \epsilon$, which is a contradiction. Thus we conclude that m_n is below the line $y = e_2 + \epsilon$, so that m_n is a point of C_m. Hence C_n is an arc of type M. Therefore the end points of C_n must lie on the line $y = x - \epsilon$.

Since $H(\overline{C}_n, \langle x, e_2 \rangle) < \epsilon$ for almost all n and $\{m_n\}$ is a sequence of maximal vertices of C'_n's, we may assume that $m_n \to e_2$. Then it is easy to verify that $\lim_{n \to \infty} = (x, e_2\rangle$.

In similar manner, one can find a sequence $\{D_n\}$ of component of U of type W whose end points lie on $y = y + \epsilon$ and the sequence $\{w_n\}$ of minimal points of D_n converging to e_1 such that $\lim_{n \to \infty} D_n = \langle e_1, y + \epsilon)$. Therefore by (2.7), $\langle e_1, e_2 \rangle$ is an R^2-continuum.

If $\check{E} \cap \hat{E} \neq \emptyset$, then the set E of essential points of X contains an R^2-continuum by (2.8) and hence $C(X)$ is not contractible [2]. In order to avoid some unnecessary technical consideration, we assume that $\check{E} \cap \hat{E} = \emptyset$.

Furthermore, we assume that E is finite and we give the natural order on E.

PROPOSITION 2.10. *Suppose $\langle a, b \rangle$ is a subinterval of $\tilde{I}$ such that $\langle a, b \rangle \cap E = \emptyset$. Then $T(x, \langle a, b \rangle) \subset A(x)$ for each $x \in \langle a, b \rangle$. Moreover if a and b are two consecutive elements of E then $T(x, \langle a, b \rangle) \subset A(x)$ for each $a < x < b$.*

Proof. Let $\epsilon > 0$ be such that $\epsilon < \min\{\frac{b-a}{2}, H(\langle a, b \rangle)\}$, where H is the Hausdorff metric for 2^X. Let $U = [0, \epsilon/2) \times (a - \epsilon/2, b + \epsilon/2) \cap X$. Since $\langle a - \epsilon/2, b + \epsilon/2 \rangle \cap E = \emptyset$, all but finite number of arc components A_n of U have the property that one end point of $\overline{A}_n$ lies on $y = a - \epsilon/2$ and the other lies on $y = b + \epsilon/2$. Therefore each $\overline{A}_n$ is an arc of type N for almost all n such that a maximal point of $\overline{A}_n$ lies on the line

$y = b+\epsilon/2$ and a minimal point of $\overline{A}_n$ lies on the line $y = a-\epsilon/2$. Thus if $\delta < \epsilon/2$ and $d(y,x) < \delta, x \in \langle a,b\rangle$, then $H(\langle a,b\rangle, \overline{A}_n) < \epsilon$ for $y \in A_n$. Therefore $\langle a,b\rangle \in A(x)$. By similar argument one can show that if $\langle a',b'\rangle$ is a subcontinuum of $\langle a,b\rangle$ and $a' \leq x \leq b'$, then $\langle a',b'\rangle \in A(x)$.

For the second part, let $a_n, b_n \in \langle a,b\rangle$ and $a_n < x < b_n$ and $a_n \to a$ and $b_n \to b$. Then by compactness of $A(x)$, $\langle a_n,b_n\rangle \in A(x)$, $n = 1,2\dots$, we have $\langle a,b\rangle \in A(x)$. Therefore $T(x,\langle a,b\rangle) \subset A(x)$ for each $a < x < b$.

PROPOSITION 2.11. *Let e_1, e_2 and e_3 be three consecutive elements of E such that $e_1 < e_2 < e_3$.*

(i) *Suppose $e_2 \in \check{E}$. Then*

(a) *$T(e_2,\langle e_2,e_2\rangle) \subset A(e_3)$ and hence $T(x,\langle e_1,e_3\rangle) \subset A(x)$ for all $e_1 < x \leq e_2$.*

(b) *for any $a < e_2$ and $e_2 \leq x < e_3$ we have $\langle a,x\rangle \notin A(x)$.*

(ii) *Suppose $e_2 \in \hat{E}$. Then*

(a) *$T(e_2,\langle e_1,e_2\rangle) \subset A(e_2)$ and hence $T(x,\langle e_1,e_3\rangle) \subset A(x)$ for all $e_2 \leq x < e_3$,*

(b) *for any $b > e_2$ and $e_1 < x \leq e_2$ we ahve $\langle x,b\rangle \notin A(x)$.*

Proof. (i). (a). Let $B \in T(e_2,\langle e_2,e_3\rangle)$. Then $B = \langle e_2,y\rangle$ for some $y, e_2 \leq y \leq e_3$. Assume that $e_2 < y < e_3$. Let $\epsilon > 0$. Choose $\epsilon' = \min\{\frac{\epsilon}{2}, \frac{y-e_2}{3} \frac{e_2-e_2}{3}\}$. Then the closed interval $\langle e_2-\epsilon', y+\epsilon'\rangle$ in $\hat{I}$ contains only one element of E, namely e_2. Let $U = [0,\epsilon') \times (e_2 - \epsilon', y+\epsilon') \cap X$ be an open set containing B. If U has an infinite number of arc components. C_n, each of which has its maximal element, say $x_n \in C_n$, in its interior then the sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to an element $e \in \hat{E}$. This would mean that $e_1 < e < e_3$ which is impossible. So let us assume that, for convenience, U does not contain any arc component which has its maximal point its interior. Similar argument applies to deduce to have U containing no arc component having its minimal point in its interior lying above or on the line $y = e_2 + \epsilon$. Thus each component of U is either an arc of type W whose minimal point lies below the line $y = e_2 + \epsilon'$ and whose end points lie on the line $y = y + \epsilon'$ or an arc of type N whose one end point lies on the line $y = y + \epsilon'$ and the other one on the line $y = e_2 - \epsilon'$.

Let $\delta < \epsilon'$ and $y \in X$ such that $d(y, e_2) < \delta$. Let C be a component of U and $y \in C$. If C is of type W with its minimal point m, then $e_2 - \epsilon' < \pi_2(m) < e_2 + \epsilon$. And hence $H(\langle e_2, y\rangle, \pi_2(\overline{C})) = H(\langle e_2, y\rangle, \langle \pi_2(m), y + \epsilon'\rangle) < \epsilon$. If C is an arc of type N, then $\pi_2(\overline{C}) = \langle e_2 - \epsilon', y + \epsilon'\rangle$ and so $H(\langle e_2, y\rangle, \overline{C}) < \epsilon$ by (2.3). This proves that $B \in A(e_2)$.

If $y = e_3$, then the compactness of $A(e_2)$ provides $\langle e_2, e_3\rangle \in A(e_2)$.

For the second part of (a), let $e_1 < x < e_2$. Then by (2.10) we have $T(x, \langle e_1, e_2\rangle) \subset A(x)$. Now suppose $B \in T(x, \langle e_1, e_3\rangle)$ such that $B = \langle b, c\rangle = \langle b, e_2\rangle \cup \langle e_2, c\rangle$ where $e_1 \le b \le x < e_2 \le c \le e_3$. Then $\langle b, e_2\rangle \in A(x)$ by (2.10) and $\langle e_2, c\rangle \in A(x)$ by the first part of (a). Hence by (1.1), we have $B \in A(x)$.

(b). Let $a < e_2$ and $e_2 \le x < e_3$.

Let $\epsilon > 0$ such that $\epsilon < \frac{1}{3}\min\{(e_2 - a), (x - e_2), (e_3 - x)\}$. Let $U = [0, \epsilon) \times (a - \epsilon, x + \epsilon) \cap X$. Since e_2 is the only essential point between e_1 and $x + \epsilon$ and $e_2 \in \check{E}$, there exists a sequence $\{C_n\}$ of arc components of U of type W such that $\lim_{n\to\infty} \overline{C}_n = \langle e_2, x + \epsilon\rangle$. Thus if $d(y, x) < \delta < \epsilon/2$, and $y \in C_n$, then $H(\langle e_2, x\rangle \pi_2(\overline{C}_n)) < \epsilon$ for almost all n. This implies that $H(\langle a, x\rangle, \pi_2(\overline{C}_n)) > 2\epsilon$ for almost all n. Therefore $\langle a, x\rangle \notin A(x)$. Argument for part (ii) is similar to that of (i).

COROLLARY 2.12. *Suppose $e_1 < e_2 < \cdots < e_n$ are n consecutive elements of E.*

(i) *If $e_i \in \check{E}$ for $i = 1, 2, \ldots, n$, then $T(x, \langle e_2, e_n\rangle) \subset A(x)$, $e_1 \le x \le e_2$.*

(ii) *If $e_i \in \hat{E}$ for $i = 1, 2, \ldots, n$, then $T(x, \langle e_1, e_{n-1}\rangle) \subset A(x)$, $e_{n-1} \le x \le e_n$.*

PROPOSITION 2.13. *The space X has nonempty $\mathcal{M}$-set if and only if the set E of essential points has more than two elements.*

Proof. If E contains only two elements, they must be the end points of $\tilde{I}$ so that $\overline{0} \in \check{E}$ and $\overline{1} \in \hat{E}$. Thus $T(x, \hat{I}) \subset A(x)$ for each $x \in \hat{I}$. Hence by the remark above $x \in \hat{I}$ is a k-point of X. This means that X has the empty $\mathcal{M}$-set. Conversely, suppose $e \in E$ which is not an end point of $\hat{I}$. Suppose $e \in \check{E}$. Let $x \in \hat{I}$ such that $e < x$ and $\langle e, x\rangle$ contain no essential point other than e. Then $\langle a, x\rangle \notin A(x)$ for $a < e$, by part (i) of (2.11). Hence x is not a k-point. If $e \in \hat{E}$ then choose $x < e$ so that $\langle x, e\rangle$ contains no essential point other than e. Then $\langle x, b\rangle \notin A(x)$ for $e < b$ by (2.11). Hence x is not a k-point of X.

In either case X has points x which are not k-points. Thus X has its nonempty $\mathcal{M}$-set.

REMARK. Let $x \in \hat{I}$ and $A \in T(x)$. Then either $A \subset \tilde{I}$ or $A \supset \hat{I}$. If $A \supset \hat{I}$, then $A \in A(x)$ by (2.4). Hence we have that a point $x \in \hat{I}$ is not a k-point of X if and only if there is $C \in T(x, \hat{I})$ such that $C \notin A(x)$.

PROPOSITION 2.14. *Suppose $e_1 < e_2$ are two consecutive essential points of X. Suppose there is a point $y_0, e_1 < y_0 < e_2$, such that y_0 is a point of the $\mathcal{M}$-set M of X. Then the open interval (e_1, e_2) is entirely contained in M.*

Proof. In view of the remark above, let $\langle b_0, b_1\rangle \in T(y_0, \hat{I})$ such that $\langle b_0, b_1\rangle \notin A(y_0)$.

Let $\langle b_0, b_1\rangle = \langle b_0, y_0\rangle \cup \langle y_0, b_1\rangle$. Then at least one of these subintervals is not admissible at y_0. Suppose $\langle b_0, y_0\rangle \notin A(y_0)$. Then $T(y_0, \langle e_1, e_2\rangle) \subset A(y_0)$ by (2.10) and $b_0 < y_0$ imply $b_0 < e_1$. This means that for each $x, e_1 < x \leq y_0$, $\langle b_0, x\rangle \notin A(x)$. Because $\langle b_0, x\rangle \in A(x)$ would imply $\langle b_0, x\rangle \cup \langle x, y_0\rangle \in A(x)$, and $\langle x, y_0\rangle \in A(x)$. Hence each $x, e_1 < x \leq y_0$, is an element of the $\mathcal{M}$-set M of X. Now suppose $y_0 < x < c_2$. We show that $x \in M$ by showing $\langle b_0, x\rangle \notin A(x)$. Since $e_1 < y_0 < x < e_2$ and no other essential point is between e_1 and e_2, and $\langle b_0, y_0\rangle \notin A(y_0)$, we choose $\epsilon > 0$ such that $\epsilon < \frac{1}{2}\min\{(e_2 - x), (x - y_0)\}$ and which satisfies the following conditions: the open set $U_1 = [0, \epsilon/2) \times (y_0 - \epsilon/2, x + \epsilon/2) \cap U$ does not intersect the set $\overline{V} = \{v \in V : v \text{ is a local extreme point}\}$, and for every $0 < \delta_n < \epsilon/2, \delta_n \to 0$, there is $y_n \in Y$, $d(y_n, y_0) < \delta_n$, and a component C_n in $U_2 = [0, \epsilon/2) \times (b_0 - \epsilon/2, y_0 + \epsilon/2) \cap Y$ containing y_n such that $H(\langle b_0, y_0\rangle, \pi_2(\overline{C})) > \epsilon$. Let $p_n \in \overline{C}_n$ be a maximal point of $\overline{C}_n$ and let $z_n \in \overline{C}_n$ be a minimal point of $\overline{C}_n$. Then $|y_0 - \pi_2(y_n)| < \delta_n < \epsilon/2$ and $\pi_2(y_n) \leq \pi_2(p_n) \leq y_0 + \epsilon/2$ imply $|y_0 - \pi_2(p_n)| \leq \epsilon/2$. Also $H(\langle b_0, y_0\rangle, \pi_2(\overline{C}_n)) = H(\langle b_0, y_0\rangle, \langle \pi_2(z_n), \pi_2(p_n)\rangle) = \max\{|b_0 - \pi_2(z_n)|, |y_0 - \pi_2(p_n)|\} > \epsilon$. Therefore we have $|b_0 - \pi_2(z_n)| > \epsilon$. This means that z_n is above the line of $y = b_0 + \epsilon/2$. Hence $z_n \in \overline{V}$. That is z_n is a minimal point lying in the interior of $\overline{C}_n$. Therefore $\overline{C}_n$ is an arc of type W whose both end points lie on the line $y = y_0 + \epsilon/2$.

Now let $U_3 = [0, \epsilon/2) \times (b_0 - \epsilon/2, x + \epsilon/2) \cap X$. Then U_3 is an open set and contains U_2. Let C'_n be the component of U_3 containing C_n. We note that the intersection of the line $y = y_0 + \epsilon/2$ and U_1 contains at most finite number of elements of $\overline{V}$; otherwise $y_0 + \epsilon/2$ would be an essential point. So we assume that $(U_1 \cap C'_n) \cap \overline{V} = \phi$, for each n. If C'_n has a point z such that $\pi_2(z) < y_0 + \epsilon/2$, then C'_n would contain an arc joining z to one of the end point of $\overline{C}_n$ which lies on the line $y = y_0 + \epsilon/2$. This would mean that C'_n contains a local maximal point $v \in \overline{V}$ which is above the line $y = y_0 + \epsilon/2$. This is impossible. Thus we must conclude that $\overline{C}'_n$ is an arc of type W whose both end points lie on $y = x + \epsilon/2$. Since $\{\overline{C}'_n\}$ has converging subsequence, we may assume that $\{\overline{C}'_n\}$ converges to a closed interval in $\hat{I}$. Thus $d(x, \overline{C}_n) \to 0$. Since $\pi_2(\overline{C}'_n) = \langle \pi_2(z_n), x + \epsilon/2 \rangle, H(\langle b_0, x \rangle, \pi_2(\overline{C}'_n)) = H(\langle b_0, x \rangle, \langle \pi_2(z_n), x + \epsilon/2 \rangle) = \max\{|b_0 - \pi_2(z_n)|, \epsilon/2\} = |b_0 - \pi_2(z_n)| > \epsilon$. This proves that $\langle b_0, x \rangle \notin A(x)$. Hence $x \in M$.

COROLLARY 2.15. *Suppose $e_1 < e_2$ are two consecutive essential points of X. If the open interval (e_1, e_2) contains a k-point then every point of (e_1, e_2) is a k-point.*

COROLLARY 2.16. *If M is the $\mathcal{M}$-set of X, then the components of M are nondegenerate.*

Proof. Let E be the set of essential points of X. Suppose $x \in M \backslash E$. Then the component of M containing x is nondegenerate by (2.14). Suppose $z \in M \cap E$. Since the end points of $\hat{I}$ are k-points by the remark after (2.6), we assume that z is not an end point of $\hat{I}$. Let $e_1, e_2 \in E$ such that $e_1 < z < e_2$ and $\langle e_1, e_2 \rangle \cap E = \{e_1, z, e_2\}$. If $z \in \hat{E}$, we consider the closed interval $\langle e_1, z \rangle$. let $z < b < e_2$. Then for each $e_1 \leq x \leq z$, $\langle x, b \rangle \notin A(x)$ by (b) of part (ii) of (2.11). Hence $x \in M$. Thus $\langle e_1, z \rangle \subset M$. If $z \in \check{E}$, then we consider $\langle z, e_2 \rangle$ and a point $e_1 < a < z$. Then for each $z \leq x \leq e_2$, $\langle a, x \rangle \notin A(x)$ by (b) of part (i) of (2.11). Hence $x \in M$ and $\langle z, e_2 \rangle \subset M$.

PROPOSITION 2.17. *Let M_α be a component of the $\mathcal{M}$-set M of X. Then there exist essential points $a, b \in E$ with $a \in \check{E}$ and $b \in \hat{E}$ such that $\overline{M}_\alpha = \langle a, b \rangle$.*

Proof. Since M_α is connected, let $a, b \in \tilde{I}$ such that $\overline{M}_\alpha = \langle a, b \rangle$.

Since the lower end point $\overline{0} \in \check{E}$ is a k-point, we may assume that $a \neq \overline{0}$.

Suppose $a \notin E$. Then there are elements $e_1, e_2 \in E$ such that $e_1 < a < e_2$ and $\langle e_1, e_2 \rangle \cap E = \{e_1, e_2\}$. Then $(e_1, e_2) \cap M_\alpha \neq \phi$. Hence $(e_1, e_2) \subset M$ by (2.14). Since M_α is a component of M, we have $(e_1, e_2) \subset M_\alpha$. But this would mean that M_α must contain elements $y \in (e_1, a)$. This is a contradiction. Hence the point a must be an essential point. But $a \in E$ implies that $a \in M$ by (2.11). Therefore $a \in M_\alpha$.

Now suppose $a \in \hat{E}$. Let $e_1 \in E$ such that $e_1 < a$ are two consecutive elements of E. Let $e_1 < x \leq a < a'$. Then $\langle x, a' \rangle \notin A(x)$ by (b) of part (ii) of(2.11). Therefore each point of $(e_1, a\rangle$ is a point of M. This implies that $(e_1, a\rangle \cup M_\alpha$ is a connected subset of M which contradicts the fact that M_α is a component of M. Therefore the point a must be an element of $\check{E}$.

Since the upper end point $\overline{1}$ of $\hat{I}$ is an essential point belong to $\hat{E}$ which is also a k-point, we may assume that $b < \overline{1}$. Then an argument similar to the above can be applied to get $b \in \hat{E} \cap M_\alpha$.

COROLLARY 2.18. (i) *If M_α is a component of M and $\langle e_1, e_2 \rangle = \overline{M}_\alpha$ such that $e_i \neq \overline{0}, \overline{1}, i = 1, 2$. Then M_α is closed.*

(ii) *If M_α and M_β are two distinct components of M, then $\overline{M}_\alpha \cap \overline{M}_\beta = \emptyset$.*

We define the collection $\mathcal{M}_n$ of the n^{th} derived sets as follows:

Let $\mathcal{M}_0 = \{\overline{M}_\alpha : M_\alpha$ is a component of $\{x \in X : T(x) \neq A(x)\}\}$. Suppose $\mathcal{M}_n$ is defined and $\mathcal{M}_n \neq \emptyset$. Then we define $\mathcal{M}_{n+1} = \{\overline{N}_\alpha : N_\alpha$ is a component of $\{x \in \overline{N}_\beta : T(x, \overline{N}_\beta) \neq A(x) \cap C(\overline{N}_\beta), \overline{N}_\beta \in \mathcal{M}_n\}$.

PROPOSITION 2.19. *Let $\overline{N} \in \mathcal{M}_k$ for some $k > 0$. Let $\langle a, b \rangle = \overline{N}$ such that $a \in \check{E}$ and $b \in \hat{E}$, and let $M = \{x \in \overline{N} : T(x, \overline{N}) \neq A(x) \cap C(\overline{N})\}$. Then $m \neq \emptyset$ if and only if $\overline{N}$ containis more than two essential points.*

Proof. The proof is indentical to that of (2.13) if one replace $\tilde{I}$ by $\overline{N}$ and k-point x by x satisfying $T(x, \overline{N}) = A(x) \cap C(\overline{N})$.

PROPOSITION 2.20. *Let $\overline{N} \in \mathcal{M}_k$ for some $k > 0$. Let $\langle a, b \rangle = \overline{N}$ such that $a \in \check{E}$ and $b \in \hat{E}$, and let*

$$M = \{x \in \overline{N} : T(x, \overline{N}) \neq A(x) \cap C(\overline{N})\}.$$

(1). *Suppose* $e_1 < e_2$ *are two consecutive essential points of* X *lying in* $\overline{N}$ *such that there is a point* $y_0, e_1 < y_0 < e_1$ *such that* $y_0 \in M$. *Then the open interval* (e_1, e_2) *is entirely contained in* M.

(2). *If* $M \neq \emptyset$ *then the components of* M *are nondegenerate.*

(3). *If* M_α *is a components of* M, *then there exist essential points* $a \in \check{E}$ *and* $b \in \hat{E}$ *such that* $\overline{M_\alpha} = \langle a, b\rangle$. *And furthermore if* M_α *and* M_β *are two distinct components of* M *then* $\overline{M}_\alpha \cap \overline{M}_\beta = \emptyset$.

The proofs of (1), (2), and (3) are identical to those of (2.14), (2.16) and (2.17).

PROPOSITION 2.21. *Let* $\overline{N} \in \mathcal{M}_k$ *for some* $k > 0$. *Let* $M = \{x \in \overline{N} : T(x, \overline{N}) \neq A(x) \cap C(\overline{N})\}$ *and let* $e_0 < e_1 < \cdots < e_{n+1}$ *be the set of essential points lying in* $\overline{N}$ *such that* $\langle e_0, e_{n+1}\rangle = \overline{N}$. *Then*

(i) *if there is a point* $x \in (e_0, e_1)$ *such that* $x \in M$, *then there is* $e_j \in \hat{E} \cap \overline{N}, 1 \leq j \leq n$ *such that* $T(x, \langle x, e_j\rangle) = A(x) \cap C(\langle x, e_j\rangle)$ *and* $\langle x, b\rangle \notin A(x)$ *for any* $b, e_j < b \leq e_{n+1}$. *Similarly*

(ii) *if there is a point* $x \in (e_n, e_{n+1}) \cap M$, *then there is an element* $e_i \in \check{E} \cap \overline{N}$, $1 \leq i \leq n$, *such that* $T(x, \langle e_i, x\rangle) = A(x) \cap C(\langle e_i, x\rangle)$ *and* $\langle a, x\rangle \notin A(x)$ *for any* $a, e_0 \leq a < e_i$.

Proof. Since the proof of (ii) is similar to that of (i), we prove only (i). Let $D = \{c \in \overline{N} : T(x, \langle x, c\rangle) = A(x) \cap C(\langle x, c\rangle)\}$. Then $T(x, \langle x, e_1\rangle) = A(x) \cap C(\langle x, e_1\rangle)$ by (2.6) implies that $D \neq \emptyset$. Let $d = \max D$. Suppose $\{c_n\}$ is a sequence in D such that $c_n \to d$. Then $\langle x, c_n\rangle \in A(x)$ for each n. So by compactness of $A(x), \langle x, d\rangle \in A(x)$.

If $e_n < d \leq e_{n+1}$, then $T(d, \langle d, e_{n+1}\rangle) = A(d) \cap C(\langle d, e_{n+1}\rangle)$ by (2.6). This together with $T(x, \langle x, d\rangle) = A(x) \cap C(\langle x, d\rangle)$ imply $T(x, \langle x, e_{n+1}\rangle) = A(x) \cap C(\langle x, e_{n+1}\rangle)$ by (1.1). This means that $T(x, \langle e_0, e_{n+1}\rangle) = A(x) \cap C(\langle e_0, e_{n+1}\rangle)$, which contradicts the fact that $x \in M$. Therefore $e_j \leq d \leq e_{j+1}$ for some $1 < j < n$. If $e_j < d < e_{j+1}$, then choose a point b such that $d < b < e_{j+1}$. Then $\langle d, b\rangle \in A(d)$, so that the conditions $\langle x, d\rangle \in A(x)$ and $\langle d, b\rangle \in A(d)$ yield $\langle x, b\rangle \in A(x)$ by (1.1). And hence $T(x, \langle x, b\rangle) = A(x) \cap C(\langle x, b\rangle)$ which contradicts the choice of d. So we must assume that d is an essential point, say $d = e_j$. If $e_j \in \check{E}$ then $\langle e_j, c\rangle \in A(e_j)$, for $e_j < c < e_{j+1}$, by (2.11) so that $\langle x, c\rangle \in A(x)$. This means that $T(x, \langle x, c\rangle) = A(x) \cap C(\langle x, c\rangle)$, which is a contradiction again. Thus e_j must be an element of $\hat{E}$.

PROPOSITION 2.22. *Let $\langle a,b\rangle$ be a closed interval in $\hat{I}$. Let $e_0 < e_1 < \cdots < e_{n+1}$ be the set of all essential points lying in $\langle a,b\rangle$. Let $e_i < x_0 < e_{i+1}$.*

(i) *If $T(x_0,\langle x_0,b\rangle) \neq A(x_0) \cap C(\langle x_0,b\rangle)$, then there exists $e_j \in \hat{E} \cap \langle a,b\rangle, e_{i+1} \leq e_j$ such that*

(a) $T(x_0,\langle x_0,e_j\rangle) = A(x_0) \cap C(\langle x_0,e_j\rangle)$,

(b) $\langle x_0,b'\rangle \notin A(x_0)$ *for* $e_j < b' \leq b$,

(c) *(a) and (b) imply that $T(x,\langle e_i,e_j\rangle) = A(x) \cap C(\langle e_i,e_j\rangle)$ for any $x, e_i < x < e_{i+1}$ and $\langle x,b'\rangle \notin A(x), e_j < b' \leq b$. Similarly*

(ii) *if $T(x_0,\langle a,x_0\rangle) \neq A(x_0) \cap C(\langle a,x_0\rangle)$, then there exists $e_k \in \check{E} \cap \langle a,b\rangle, e_k \leq e_i$, such that*

(a) $T(x_0,\langle e_k,x_0\rangle) = A(x_0) \cap C(\langle e_k,x_0\rangle)$,

(b) $\langle a',x_0\rangle \notin A(x_0)$ *for* $a \leq a' < e_k$.

(c) $T(x,\langle e_k,e_{i+1}\rangle = A(x) \cap C(\langle e_k,e_{i+1}\rangle), e_i < x < e_{i+1}$ *and* $\langle a',x\rangle \notin A(x)$ *for* $a \leq a' < e_k, e_i < x < e_{i+1}$.

Proof. We only give proof of (i). The proof (ii) is similar.

(a) and (b). Let $d = \max\{c \in \langle a,b\rangle : T(x_0,\langle x_0,c\rangle) = A(x_0) \cap C(\langle x_0,c\rangle)\}$. Then by the same proof as that of (i) of (2.21), $d = e_j \in \hat{E} \cap \langle a,b\rangle, e_{i+1} \leq e_j$ and $\langle x_0,b'\rangle \notin A(x_0)$ for $e_j < b' \leq b$.

(c) First assume that $x_0 < x < e_{i+1}$. Let $\epsilon > 0$ be chosen so that $\epsilon < \frac{1}{2}\min\{(e_{i+1}-x),(x_0-e_i)\}$. Since e_i and e_{i+1} are consecutive pair, we may assume without loss of generality that the open set $U_0 = [0,\epsilon) \times (x_0-\epsilon, x+\epsilon) \cap X$ does not intersect the set $\overline{V}$ of local extrema.

Since $\langle x_0,e_j\rangle \in A(x_0)$ and $\langle x_0,b'\rangle \notin A(x_0)$ for $e_j < b'$ the set $\{A_n\}$ of arc components of $U_1 = [0,\epsilon) \times (x_0-\epsilon, e_j+\epsilon) \cap X$ must satisfy the followings: $H(\overline{A}_n, \langle x_0-\epsilon, e_j+\epsilon\rangle) \leq \epsilon$, all but finite number of $\overline{A}_n$'s are arcs of type N or W, and $\langle x_0,b'\rangle \notin A(x_0)$ implies that $\{\overline{A}_n\}$ has a subsequence $\{\overline{A}_{n_i}\}$ of arcs of type M such that the end points of each $\overline{A}_{n_i}$ lie on the line $y = x_0-\epsilon$, the maximal points z_{n_i} of $\overline{A}_{n_i}$ lie below the line $y = e_j+\epsilon$, and $\overline{A}_{n_i} \to \langle x_0-\epsilon, e_j\rangle$, and if z_{n_i} is a minimal interior point of $\overline{A}_{n_i}$, then z_{n_i} lies above the line $y = x+\epsilon$ for almost all i.

Now let B_j be an arc component of $U_2 = [0,\epsilon) \times (x-\epsilon, e_j+\epsilon) \cap X$. Since $U_2 \subset U_1, B_j \subset A_{n_j}$ for some n_j. Let $y_j \in B_j$ such that $d(y_j,x) < \epsilon$, and let C_j be the unique arc in $\overline{A}_{n_j}$ joining y_j to a maximal point of $\overline{A}_{n_j}$. Since one end point of B_j must lie on $y = x - \epsilon$ and there is no local extreme in U_0, we see that $B_j \supset C_j$. Thus $H(\langle x, e_j\rangle, \overline{B}_j) \leq \epsilon$. This implies that

$$T(x, \langle x, e_j\rangle) = A(x) \cap C(\langle x, e_j\rangle).$$

If $e_i < x < x_0$, then $\langle x, x_0\rangle \in A(x)$ by (2.10). So that $\langle x, x_0\rangle \cup \langle x_0, e_j\rangle = \langle x, e_j\rangle \in A(x)$ by (1.1). Therefore $T(x, \langle e_i, e_j\rangle) = A(x) \cap C(\langle e_i, e_j\rangle)$. Now suppose there is b', $e_j < b' \leq b$ such that $\langle x, b'\rangle \in A(x)$ for some $x, e_i < x < e_{i+1}$. Applying the same reasoning as above, $\langle x, b'\rangle \in A(x)$ would imply $\langle x_0, b'\rangle \in A(x_0)$ which is a contradiction. Thus the proposition is proved.

PROPOSITION 2.23. *Suppose $\langle e_i, e_j\rangle$ is an R^2-continuum of X. Then there are two essential points a and b, $a < e_i < e_j < b$ such that the closed interval $\langle a, b\rangle$ is contained in some element $\overline{N}_n$ of $\mathcal{M}_n$ for each $n = 0, 1, 2, \ldots$.*

Proof. Since e_i and e_j are not the end points of $\hat{I}$, let $a, b \in E$ such that $a < e_i < e_j < b$ and $\langle a, e_i\rangle \cap E = \{a, e_i\}$ and $\langle e_j, b\rangle \cap E = \{e_j, b\}$.

First we show that $\langle e_i, e_j\rangle$ is entirely contained in the $\mathcal{M}$-set M of X. Let $x \in \langle e_i, e_j\rangle$. Since $\langle e_i, e_j\rangle$ is an R^2-continuum, there exists $\epsilon < \frac{1}{2}\min(e_i - a), (e_j - e_i), (b - e_j)\}$ such that the open set $U = [0, \epsilon/2) \times (e_i - \epsilon/2, e_j + \epsilon/2) \cap X$ contains two sequences $\{A_n\}$ and $\{B_n\}$ of arc components of type M and W respectively such that $\lim_{n\to\infty} \overline{A}_n = \langle e_i - \epsilon/2, e_j\rangle$ and $\lim_{n\to\infty} \overline{B}_n = \langle e_i, e_j + \epsilon/2\rangle$ by (2.7). Furthermore both end points of each $\overline{A}_n$ lie on the line $y = e_i - \epsilon/2$ for almost all n, and both end points of each $\overline{B}_n$ lie on the line $y = e_j + \epsilon/2$ for almost all n. Let $U_1 = [0, \epsilon/2) \times (a - \epsilon/2, e_j + \epsilon/2) \cap X$. Then $\langle a, x\rangle \subset U_1$ and $U \subset U_1$. Since end points of $\overline{B}_n$ lie on the line $y = e_j + \epsilon/2$ and $U \subset U_1$, B_n's are components of U_1. Let $y \in B_n$ and $d(x, y) < \epsilon/2$ and let x_n be the lowest point of B_n such that $d(e_i, x_n) < \epsilon/2$. If A is a subcontinuum containing y and $H(\langle a, x\rangle, A) < \epsilon/2$, then $A \subset U_1$. Since B_n is a component of U_1 as well and $y \in B_n$, $A \subset B_n$. If a' is a lowest point of A then $\pi_2(x_n) \leq \pi_2(a')$ and hence $|a - \pi_2(a')| \geq |a - \pi_2(x_n)| \geq \frac{3}{4}\epsilon$. Thus by (2.2) $(\langle a, x\rangle, A) \geq \frac{3}{4}\epsilon$. This contradicts the asuumption that $H(\langle a, x\rangle, A) < \epsilon/2$. So $x \in M$. Therefore $T(x, \langle a, b\rangle) \neq A(x) \cap C(\langle a, b\rangle)$, for $x \in \langle e_i, e_j\rangle$.

Now let $x \in (a, e_i)$. Choose $\epsilon' = \frac{1}{2}\min\{\epsilon, (x-a)\}$. We take the sequence $\{A_n\}$ of arc components of U of type M. Let $x_n \in A_n$ be a maximal interior point of $\overline{A}_n$ which converges to e_j. Consider the open set $U_2 = [0, \epsilon/2) \times (x - \epsilon', e_j + \epsilon/2) \cap X$. Since there is no essential point in $\langle x-\epsilon', e_i - \epsilon/2)\rangle$, we may assume that the components C_n of U_2 containing A_n has both of its end points lie on the line $y = x - \epsilon'$. Then such C_n is an arc of type M having x_n as its maximal interior point. Let $U_3 = [0, \epsilon/2) \times (x - \epsilon', b + \epsilon/2) \cap X$. Then $U_3 \supset U_2$ ane each C_n is a components of U_3. By the same argument applied above, we see now that $\langle a, b\rangle \notin A(x) \cap C(\langle a, b\rangle)$. Therefore $T(x, \langle a, b\rangle) \neq A(x) \cap C(\langle a, b\rangle), x \in \langle a, e_i\rangle$. Similarly one can show that for each $x \in (e_j, b), T(x, \langle a, b\rangle) \neq A(x) \cap C(a, b\rangle)$. Therefore we conclude that $T(x, \langle a, b\rangle) \neq A(x) \cap C(\langle a, b\rangle)$ for each $a < x < b$.

Now let $\overline{N}_n \in \mathcal{M}_n$ such that $\langle a, b\rangle \subset \overline{N}_n$ and let $M = \{x \in \overline{N}_n : T(x, \overline{N}_n) \neq A(x) \cap C(\overline{N}_n)\}$. The established condition $T(x\langle a, b\rangle) \neq A(x) \cap C(\langle a, b\rangle)$ for each $a < x < b$ implies that the open interval (a, b) is contained in M. Hence, if N is the component of M containing (a, b), then $\langle a, b\rangle \subset \overline{N} \in \mathcal{M}_{n+1}$.

PROPOSITION 2.24. *Suppose X does not contain any R^2-continuum. Let $\overline{N}_1 \in \mathcal{M}_i$ and $\overline{N} \in \mathcal{M}_{i-1}$ such that $\overline{N}_1 \subset \overline{N}$. Suppose $x \in \overline{N}_1 \in A(x)$. Then $T(\overline{N}_1, \overline{N}) = \{A \in C(\overline{N}) : A \supset \overline{N}_1\} \subset A(x)$.*

Proof. Let $\langle a_1, b_1\rangle = \overline{N}_1$ and $\langle a, b\rangle = \overline{N}$ with $a, a_1 \in \check{E}$ and $b, b_1 \in \hat{E}$. Let e_i be the element of E which is immediate predecessor of a_1, let $e_j \in E \cap \overline{N}$ be the immediate successor of b_1. Then we have $a \leq e_1 \leq a_1 < b_1 \leq e_j \leq b$.

There are three cases to consider: $a = a_1, b = b_1$ or $a < a_1 < b_1 < b$. We prove for the third case and leave the other cases for the reader.

Let $\epsilon > 0$ be chosen such that $\epsilon < \frac{1}{2}\min\{(a_1 - e_i), (e_j - b_1)\}$. Let $U = [0, \frac{\epsilon}{2}) \times (a_1 - \epsilon, b_1 + \epsilon) \cap X$. Then by (2.5) there exists $\delta > 0$ such that if C is a components of U which intersects the δ-neighborhood $\mathcal{O}$ of x, then $H(\langle a_1, b_1\rangle, \pi_2(\overline{C})) < \epsilon$ and hence by (2.3) $H(\langle a_1, b_1\rangle, \overline{C}) < \epsilon$.

Let $\{C_n\}$ be the set of all arc component of U, each of which intersects $\mathcal{O}$. This set can not contain an infinite sequence of arcs of type M at the same time containing an infinite number of arcs of type W. Otherwise $\overline{N}$ would have an R^2-continuum. So we suppose that $\{C_n\}$ contains a subsequence $\{C_{n_i}\}$ of arcs of type M. Then the end points of $\overline{C}_{n_i}$ must lie on the line $y = a_1 - \epsilon$. Let x_i be a maximal point of $\overline{C}_{n_i}$ which converges to b_1. Choose x so that $a_1 - \epsilon < x_1 < a_1$ and let $y = b_1 + \frac{9\epsilon}{10}$. Since $\overline{N}_1$ is a derived set of $\overline{N}$, by (2.20), $T(x,\overline{N}) = A(x) \cap C(\overline{N})$. Hence $\langle x,y\rangle \in A(x)$. But $H(\langle x,y\rangle, B_{n_i}) \geq H(\langle x,y\rangle, \langle a-\epsilon, \pi_2(x_i)\rangle) \geq \frac{9}{10}\epsilon$ for any subcontinuum $B_{n_i} \subset C_{n_i}$. This means that $\langle x_1,y\rangle$ can not be approximated by a sequence $\{B_{n_i}\}$ of subcontinua, $B_{n_i} \subset C_{n_i}$, which contradicts the admissibility of $\langle x,y\rangle$ at x. So we may assume that $\{C_n\}$ is the set of arc components of U of type N, each of which has one end point lying on $y = a_1 - \epsilon$ and the other one on the line $y = b_1 + \epsilon$. Now let $A \in T(\overline{N}_1, \overline{N})$ such that $A \subset (a_1, -\epsilon, b_1 + \epsilon)$. Let z_1 and z_2 be the lowest and highest point of A respectively. Then, since each $\overline{C}_n$ is an arc with one end point on $y = a_1 - \epsilon$ and the other one on $y = b_1 + \epsilon$, there are $x_n, y_n \in \overline{C}_n$ such $\pi_2(x_n) = \pi_2(z_1)$ and $\pi_2(y_n) = \pi_2(z_2)$. Let B_n be the arc in $\overline{C}_n$ joining x_n and y_n. Then $B_n \to A$. Therefore $A \in A(x)$.

Now let $D \in T(\overline{N}_1, \overline{N})$. Assume that $D\backslash(e_i, e_j) \neq \emptyset$. Let $A \in T(\overline{N}_1, \overline{N})$ such that $A \subset (a_1 - \epsilon, b_1 + \epsilon)$ and $A \subset D$ and $A\backslash\langle a_1, b_1\rangle \neq \emptyset$. Let $x_1 \in A\backslash\langle a_1, b_1\rangle$. Then $T(x_1, \overline{N}) = A(x_1) \cap C(\overline{N})$ so that $A \in A(x_1)$ and $D \in A(x_1)$. Since $A \in A(x)$ by above, $x_1 \in A \cap D$ implies $D \in A(x)$ by (1.1). This proves the proposition.

Let us call a consecutive pair e_i, e_{i+1} with $e_i < e_{i+1}$ of essential points of X open (or closed) if $e_i \in \hat{E}$ and $e_{i+1} \in \check{E}$($e_i \in \check{E}$ and $e_{i+1} \in \hat{E}$).

PROPOSITION 2.25. *Suppose $e_0 < e_1 < \cdots < e_n$ is the set of all essential points in $\overline{N} \in \mathcal{M}_s$. If $\overline{N}$ does not contain any open consecutive pair of essential points, then it contains a unique closed consecutive pair $e_k < e_{k+1}$ such that $e_i \in \check{E}$ for $0 \leq i \leq k$ and $e_i \in \hat{E}$ for each $k < i \leq n$.*

Proof. Since $\overline{N} \in \mathcal{M}_s$, we have $\overline{N} = \langle e_0, e_n\rangle$ with $e_0 \in \check{E}$ and $e_n \in \hat{E}$ by (2.20).

Let $k = \text{mix}\{j : e_j \in \overline{N} \cap \check{E}\}$. Then $e_k \in \check{E}$ and $e_j \in \hat{E}$ for all $j < k$.

If there is $i < k$ such that $e_i \in \hat{E}$, let $m = \text{mix}\{i : e_i \in \overline{N} \cap \hat{E}, i < k\}$. Then $e_m \in \hat{E}$ and $e_m < e_k$. Thus $e_{m+1} \in \check{E}$, $m+1 \leq k$. This would mean that e_m and e_{m+1} form an open consecutive pair which contradicts the hypothesis. Therefore $a_i \in \hat{E}$ implies $i > k$. Hence $e_{k+1} \in \hat{E}$ and e_k and e_{k+1} form a unique closed consecutive pair.

Let us denote the cardinality of $E \cap A$ by $|E \cap A|$.

PROPOSITION 2.26. *Suppose X does not contain any R^2-continuum. Suppose $\overline{N} \in \mathcal{M}_i$ which does not contain open consecutive pair of essential points. Then for each component N_α of $\{x \in \overline{N} : T(x, \overline{N}) \neq A(x) \cap C(\overline{N})\}$, we have $|\overline{N}_\alpha \cap E| < |\overline{N} \cap E|$ and $\overline{N}_\alpha$ does not contain any open consecutive pair.*

Proof. We assume that $\overline{N}$ contains more than two essential points. Otherwise $\overline{N}$ would have empty derived set. Let $e_0 < e_1 < \cdots < e_n$ be the set of all essential points in $\overline{N}$. Then $\overline{N} = \langle e_0, e_n \rangle$ with $e_0 \in \check{E}$ and $e_n \in \hat{E}$ by (2.20).

Since $\overline{N}$ does not contain open consecutive pair of essential points, let $e_k \in \check{E}$ and $e_{k+1} \in \hat{E}$ be the unique closed consecutive pair provided by (2.25). Let $e_0 < x < e_1$ and $e_{n-1} < y < e_n$. Let i and j be the smallest and largest indices provided by (2.21) and (2.22) respectively such that $T(x, \langle e_0, e_j \rangle) = A(x) \cap C(\langle e_0, e_j \rangle)$ and $\langle x, b \rangle \notin A(x)$ for $e_j < b$, and $T(y, \langle e_i, e_n \rangle) = A(y) \cap C(\langle e_i, e_n \rangle)$ and $\langle a, y \rangle \notin A(y) a < e_i$. Then by (2.12) and (2.25) we have $i \leq k$ and $j \geq k+1$. Hence by (2.25) again $e_i \in \check{E}$ and $e_j \in \hat{E}$. Suppose $e_i \neq e_0$ and $e_j \neq e_n$. Then by (2.9) $\langle e_i, e_j \rangle$ is an R^2-continuum. Since $\overline{N}$ does not contain any R^2-continuum, we must have either $e_i = e_0$ or $e_j = e_n$.

Suppose $e_i = e_0$. Then $T(y, \langle e_0, e_n \rangle) = A(y) \cap C(\langle e_0, e_n \rangle)$ for each $e_{n-1} < y \leq e_n$. Thus $N_\alpha \cap (e_{n-1}, e_n) = \phi$. Therefore $e_n \notin \overline{N}_\alpha$. Hence $|\overline{N}_\alpha \cap E| < |\overline{N} \cap E|$.

If $e_j = e_n$ the argument is the same. The second part of the proposition is obvious. Thus we proved the proposition.

COROLLARY 2.27. *Let $\langle a, b \rangle$ be an interval and let $e_0 < \cdots < e_n$ be the set of all essential points in $\langle a, b \rangle$ such that $a < e_0$ and $e_n < b$. Suppose $\langle e_0, e_n \rangle$ does not contain any R^2-continuum, and contains no open consecutive pair of essential elements. Let $a < z_1 < e_0$ and $e_n < z_2 < b$. Then either $T(z_1, \langle z_1, z_2 \rangle) = A(z_1) \cap C(\langle z_1, z_2 \rangle)$ or $T(z_2, \langle z_1, z_2 \rangle) = A(z_2) \cap C(\langle z_1, z_2 \rangle)$.*

Proof. Let $e_0 < x < e_1$ and $e_{n-1} < y < e_n$. Then by (2.25) $e_0 \in \check{E}$ and $e_n \in \hat{E}$. And by (2.26), either $T(x, \langle e_0, e_n \rangle) = A(x) \cap C(\langle e_0, e_n \rangle)$ or $T(y, \langle e_0, e_n \rangle) = A(y) \cap C(\langle e_0, e_n \rangle)$. Since $e_0 \in \check{E}$, $\langle z_1, x \rangle \in A(z_1)$ by (2.11). Similarly $e_n \in \hat{E}$ implies $\langle y, z_2 \rangle \in A(z_2)$. Combining these with above, we have $T(z_1, \langle z_1, e_n\rangle) = A(z_1) \cap C(\langle z_1, e_n\rangle)$ or $T(z_2, \langle e_0, z_2\rangle) = A(z_2) \cap C(\langle e_0, z_2\rangle)$. Since $\langle e_0, e_n\rangle$ is not an R^2-continuum, we must have either $T(z_1, \langle z_1, z_2\rangle) = A(z_1) \cap C(\langle z_1, z_2\rangle)$ or $T(z_2, \langle z_1, z_2\rangle) = A(z_2) \cap C(\langle z_1, z_2\rangle)$.

Let $\overline{N} \in \mathcal{M}_i$ for some i. Let $e_0 < e_1 < \cdots < e_n$ be the set of essential points in $\overline{N} = \langle e_0, e_n\rangle$. Suppose $\overline{N}$ contains open consecutive pairs of essential points. Let $e(i,0) < e(i,1)$ be the i^{th} open consecutive pair in $\overline{N}$ such that $e(i,0) \in \hat{E}$ and $e(i,1) \in \check{E}$. We have linear ordering $e(i,0) < e(i,1) < e(i+1,0) < e(i+1,1), i = 1,2,\ldots$. If $\overline{N}$ contains k number of open consecutive pairs,we let, for convenience, $e_0 = e(0,1) \in \check{E}$, $e_n = e(k+1,0) \in \hat{E}$. Then there are $(k+1)$ number of intervals $P_i = \langle e(i,1), e(i+1,0)\rangle$ in $\overline{N}$, each of which contains no open consecutive pair, for $i = 0,1,\ldots,k$, and hence each P_i contains a unique closed consecutive pair, denoted by $e(i,\vee) < e(i,\wedge)$ between $e(i,1)$ and $e(i+1,0)$. Thus, for each $P_i, i = 0,1,\ldots,k$, we have

$$e(i,1) \leq e(i,\vee) < e(i,\wedge) \leq e(i+1,0).$$

Let U_0 be the open interval (e_0, e_1) and U_{k+1} be the open interval (e_{n-1}, e_n). And for each $i = 1,2,\ldots,k$, let $U_i = (e(i,0), e(i,1))$ be the open interval between the i^{th} open consecutive pair. We fix a point $z_i \in U_i$ for each $i = 0,1,2,\ldots,k+1$. Then each P_i is contained in the interior of the closed interval $\langle z_i, z_{i+1}\rangle$, so that we have the natural ordering of P_i with the assigned index i.

PROPOSITION 2.28. *Suppose X does not contain any R^2-continuum. Suppose $\overline{N} \in \mathcal{M}_i$ such that $\overline{N}$ contains open consecutive pairs of essential points. Then, for each components N_1 of the set $\{x \in \overline{N} : T(x,\overline{N}) \neq A(x) \cap C(\overline{N})\}$, we have $|\overline{N}_1 \cap E| < |\overline{N} \cap E|$.*

Proof. Let $\{P_0, P_1, \ldots, P_{k+1}\}$, $z_i \in U_i$ be the same as defined above for $\overline{N}$. We patch up inductively consecutive elements of $\{P_i\}$ so that at the end each derived set N_1 of N contains at least one less essential point than N. For each consecutive pair P_i and P_{i+1} with containing intervals $\langle z_i, z_{i+1}\rangle$ and $\langle z_{i+1}, z_{i+2}\rangle$ respectively, $i = 1,2,\ldots,k$, we have the following conditions by (2.27).

I. (i) $T(z_i, \langle z_i, z_{i+1}\rangle) = A(z_i) \cap C(\langle z_i, z_{i+1}\rangle)$ or
(ii) $T(z_{i+1}, \langle z_i, z_{i+1}\rangle) = A(z_{i+1}) \cap C(\langle z_i, z_{i+1}\rangle)$.
II. (i) $T(z_{i+1}, \langle z_{i+1}, z_{i+2}\rangle) = A(z_{i+1}) \cap C(\langle z_{i+1}, z_{i+2}\rangle)$ or

(ii) $T(z_{i+2}, \langle z_{i+1}, z_{i+2}\rangle) = A(z_{i+2}) \cap C(\langle z_{i+1}, z_{i+2}\rangle)$.

Then we have the following four cases to consider:

Case I. I (i) and II (i).

Let $A \in T(z_i, \langle z_i, z_{i+2}\rangle)$. Then either $A \subset \langle z_i, z_{i+1}\rangle$ or $A = \langle z_i, z_{i+1}\rangle \cup \langle z_{i+1}, b\rangle$ for some $b \in \langle z_{i+1}, z_{i+2}\rangle$. Since $\langle z_i, z_{i+1}\rangle \in A(z_i)$ and $\langle z_{i+1}, b\rangle \in A(z_{i+1})$, by (1.1) we have $A = \langle z_i, b\rangle \in A(z_i)$. Therefore we have $T(z_i, \langle z_i, z_{i+2}\rangle) \subset A(z_i)$.

Case 2. I (ii) and II (ii).

In this case we have $T(z_{i+2}, \langle z_i, z_{i+2}\rangle) \subset A(z_{i+2})$. The proof is similar to that of Case 1.

Case 3. I (ii) and II (ii).

Since $T(z_{i+1}, \langle z_i, z_{i+1}\rangle) \subset A(z_{i+1})$ and $T(z_{i+1}, \langle z_{i+1}, z_{i+2}\rangle) \subset A(z_{i+1})$, we see immediately that $T(z_{i+1}, \langle z_i, z_{i+2}\rangle) \subset A(z_{i+1})$.

Case 4. I (i) and II (ii).

For $T(z_i, \langle z_i, z_{i+1}\rangle) \subset A(z_i)$, we extend the set $\langle z_i, z_{i+1}\rangle$ according to (2.22) to $\langle z_i, c\rangle, z_{i+1} < c < z_{i+2}$, such that c is a largest element to satisfy $T(z_{i+1}, \langle z_{i+1}, c\rangle) \subset A(z_{i+1})$. Then $T(z_i, c\rangle) \subset A(z_i)$. Similarly we extend the set $\langle z_{i+1}, z_{i+2}\rangle$ to $\langle d, z_{i+2}\rangle$, $z_i < d < z_{i+1}$ such that d is the smallest element for which $T(z_{i+1}, \langle d, z_{i+1}\rangle) \subset A(z_{i+1})$. Then we have $T(z_{i+2}, \langle d, z_{i+2}\rangle) \subset A(z_{i+2})$. If $d \neq z_i$ and $c \neq z_{i+2}$, then $d \in \check{E}$ and $c \in \hat{E}$ and $\langle d, c\rangle$ would be an R^2-continuum, so we must have either $d = z_i$ or $c = z_{i+2}$. That is $T(z_i, \langle z_i, z_{i+2}\rangle) \subset A(z_i)$ or $T(z_{i+2}, \langle z_i, z_{i+2}\rangle) \subset A(z_{i+2})$. So we conclude that, for each consecutive pair P_i, P_{i+1}, we have reduced to three cases as follow:

$\mathcal{P}_1$: (i) $T(z_i, \langle z_i, z_{i+2}\rangle) \subset A(z_i)$ or

(ii) $T(z_{i+2}, \langle z_i, z_{i+2}\rangle) \subset A(z_{i+2})$ or

(iii) $T(z_{i+1}, \langle z_i, z_{i+2}\rangle) \subset A(z_{i+1})$.

Now we assume that for each consecutive m-tuple $P_i, P_{i+1}, \ldots,$ P_{i+m-1}, with the interval $\langle z_i, z_{i+m}\rangle$, we have

$\mathcal{P}_m$: (i) $T(z_i, \langle z_i, z_{i+m}\rangle) \subset A(z_i)$, or

(ii) $T(z_{i+m}, \langle z_i, z_{i+m}\rangle) \subset A(z_{i+m})$, or

(iii) $T(z_j, \langle z_i, z_{i+m}\rangle) \subset A(z_j)$, $i < j < i+m$.

We attach the interval $\langle z_{i+m}, z_{i+m+1}\rangle$ containing P_{i+m} to the interval of P_m with the following given conditions.

(a) $T(z_{i+m}, \langle z_{i+m}, z_{i+m+1}\rangle) \subset A(z_{i+m})$ or

(b) $T(z_{i+m+1}, \langle z_{i+m+1}\rangle) \subset A(z_{i+m+1})$.

There are six cases to be considered:

(i) and (a) imply $T(z_i, \langle z_i, z_{i+m+1}\rangle) \subset A(z_i)$.

(ii) and (b) imply $T(z_{i+m+1}, \langle z_i, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$.

(ii) and (a) imply $T(z_{i+m}, \langle z_i, z_{i+m+1}\rangle \subset A(z_{i+m})$.

(iii) and (a) with $T(z_j, \langle z_i, z_{i+m}\rangle) \subset A(z_j)$ imply that $T(z_j, \langle z_j, z_{i+m}\rangle) \subset A(z_j)$. Combine this with (a), we have $T(z_j, \langle z_j, z_{i+m+1}\rangle) \subset A(z_j)$.

Therefore $T(z_j, \langle z_j, z_{i+m+1}\rangle) \subset A(z_j)$.

(i) and (b). There is a largest element c in $\langle z_{i+m}, z_{i+m+1}\rangle$ such that $T(z_i, \langle z_i, c\rangle) \subset A(z_i)$. Also there is a smallest element d in $\langle z_i, z_{i+m}\rangle$ such that $T(z_{i+m+1}, \langle d, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$. If $d \neq z_i$ and $c \neq z_{i+m+1}$, then $d \in \check{E}$ and $c \in \hat{E}$ such that $\langle d, c\rangle$ would be an R^2-continuum. Thus we conclude that either $T(z_i, \langle z_i, z_{i+m+1}\rangle) \subset A(z_i)$ or $T(z_{i+m+1}, \langle z_i, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$.

(iii) and (b). Since $T(z_j, \langle z_i, z_{i+m}\rangle) \subset A(z_j)$ implies $T(z_j, \langle z_j, z_{i+m}\rangle) \subset A(z_j)$, there is a largest element c in $\langle z_{i+m}, z_{i+m+1}\rangle$ such that $T(z_j, \langle z_j, c\rangle) \subset A(z_j)$. Also there is an element $d \in \langle z_i, z_{i+m}\rangle$ such that $T(z_{i+m+1}, \langle d, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$. But if $d \leq z_j$, then $T(z_j, \langle z_i, z_j\rangle) \subset A(z_j)$ and $T(z_{i+m+1}, \langle d, z_{i+m+1}\rangle)$ would imply $T(z_{i+m+1}, \langle z_i, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$. If d is a smallest element in $\langle z_j, z_{i+m}\rangle$ such that $T(z_{i+m+1}, \langle d, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$, then $c = z_{i+m+1}$. Otherwise $\langle d, c\rangle$ would be an R^2-continuum. Thus we have either $T(z_j, \langle z_i, z_{i+m+1}\rangle) \subset A(z_j)$ or $T(z_{i+m+1}, \langle z_i, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$. Thus, for each consecutive $(m+1)$-truple $P_i, P_{i+1}, \ldots, P_{i+m}$ with the interval $\langle z_i, z_{i+m+1}\rangle$, at least one of the following must be true:

$\mathcal{P}_{m+1}$: (i) $T(z_i, \langle z_i, z_{i+m+1}\rangle) \subset A(z_i)$

(ii) $T(z_{i+m+1}, \langle z_i, z_{i+m+1}\rangle) \subset A(z_{i+m+1})$

(iii) $T(z_j, \langle z_i, z_{i+m+1}\rangle) \subset A(z_j)$, for some $j, i < j < i+m+1$.

Now suppose $\overline{N} = \langle e_0, e_{n+1}\rangle \in \mathcal{M}_i$ contains k number of open consecutive pair of essential points. Let $m = k$ and $i = 0$ in $\mathcal{P}_{m+1}$. Let $M = \{x \in \overline{N} : T(x, \overline{N}) \neq A(x) \cap C(\overline{N})\}$. Let N_1 be a component of M.

Case a. $T(z_0, \langle z_0, z_{k+1}\rangle) \subset A(z_0)$.

We apply (2.6) to get $T(x, \langle e_0, e_1\rangle) \subset A(x), e_0 < x < e_1$, and $T(y, \langle e_{n-1}, e_n\rangle) \subset A(y)$ for $e_{n-1} < y < e_n$. So we apply (1.1) to get $T(x, \overline{N}) \subset A(x)$, $x \in U_0$. Therefore $U_0 \subset \overline{N} \backslash M$. Hence $e_0 \notin \overline{N}_1$.

Case b. $T(z_{k+1}, \langle z_0, z_{k+1}\rangle) \subset A(z_{k+1})$.

Argument is the same as (i). $e_n \notin \overline{N}_1$.

Case c. $T(z_j, \langle z_0, z_{k+1}\rangle) \subset A(z_j)$ for some $0 < j < k+1$.

In this case we have $T(x, \langle e(j,0), e(j,1)\rangle) \subset A(x)$, for $x \in U_j$. So that $T(x, \langle e(j,0), e(j,1)\rangle \subset A(x)$, for $x \in U_j$. Thus $T(x, \overline{N}) \subset A(x)$ for each $x \in U_j$. Therefore $e(j,0), e(j,1) \notin \overline{N}_1$.

In any event we have $|\overline{N}_1 \cap E| < |\overline{N} \cap E|$.

THEOREM 2.29. *Suppose X has the finite set of essential points. Then $C(X)$ is contractible if and only if X does not contain any R^2-continuum.*

Proof. If X contain an R^2-continuum, then $C(X)$ is not contractible [2].

Suppose X does not contain an R^2-continuum. If X has the empty $\mathcal{M}$-set, then X has property k and ehce $C(X)$ is contractible [11]. Let us assume that X has nonempty $\mathcal{M}$-set M. Since E is finite, the end points of each element of $\mathcal{M}_i$ are elements of E and the elements of $\mathcal{M}_i$ are pairwise disjoint by (2.20) and each $\mathcal{M}_i$ is finite. Furthermore, by successive application of (2.26) and (2.28), there is an integer n such that $\mathcal{M}_n \neq \emptyset$ and $\mathcal{M}_{n+1} = \emptyset$.

First we prove that if $N \in \mathcal{M}_i$ and $T(x,N) = A(x) \cap C(N)$ for each $x \in N$, then the set-valued map $\alpha_N : N \to C(N)$ defined by $\alpha_N(x) = T(x,N)$, $x \in N$, is a γ-map.

Clearly $\{x\}, N \in \alpha_N(x)$ for each $x \in N$. The monotone-connectedness of $\alpha_N(x)$ follows from [3]. Now let $\epsilon > 0$ and $A \in \alpha_N(x)$. Let $\delta = \frac{\epsilon}{2}$, and $y \in N$ with $d(x,y) < \delta$. Since N is a closed arc, the arc B having x and y as its end points lies in N. Then by the hypothesis and (1.1) we have $A \cup B \in T(y,N) \subset A(y)$. Also $H(A, A\cup B) < \epsilon$. This proves that α_N is lower semicontinuous at x. Hence α_N is a γ-map.

We define a set-valued map α_n on the union of the elements of $\mathcal{M}_n$ whose restriction on each element of $\mathcal{M}_n$ is a γ-map and extend it inductively to a set-valued map α_0 on the $\mathcal{M}$-set M of X into $2^{C(M)}$ whose restriction on each element M_i of $\mathcal{M}_0$ is a γ-map into $2^{C(M_i)}$.

Since $\mathcal{M}_n \neq \emptyset$ and $\mathcal{M}_{n+1} = \emptyset$, each element N of $\mathcal{M}_n$ satisfies the condition that $T(x,N) = A(x) \cap C(N)$, for each $x \in N$. Let $\mathcal{M}_n = \{N_1, N_2, \ldots, N_k\}$. We define the set-valued map α_n as follows: for each $i = 1, 2, \ldots, k$, let $\alpha_n(x) = T(x,N)$ for each $x \in N$. Then α_N is a γ-map on each N_i. Since the set $\mathcal{M}_n$ is finite and the elements of $\mathcal{M}_n$ are disjoint and closed, the lower semicontinuity of α_n on each N_i

provides the lower semicontinuity of α_n on $\cup_{i=1}^k N_i$.

Let $K \in \mathcal{M}_{n-1}$. If K is an element such that $T(x,K) = A(x) \cap C(K)$ for each $x \in K$, then define $\alpha_{n-1}(x) = \alpha_K(x)$, for each $x \in K$. If K is an element such that $T(x,K) \neq A(x) \cap C(K)$ for some $x \in K$, let $\{N_1, N_2, \ldots, N_k\}$ be the set of elements of $\mathcal{M}_n$ such that $N_i \subset K$ for $i = 1, 2, \ldots, k$ and define

$$\alpha_{n-1}(x) = \begin{cases} \alpha_n(x) \cup P(N_i, K) & \text{if } x \in N_i,\ i = 1, 2, \ldots, k \\ T(x,K) & \text{if } x \in K \setminus \cup_{i=1}^k N_i \end{cases}$$

If $x \in K$ such that $\alpha_{n-1}(x) = \alpha_K(x)$, then clearly $\alpha_{n-1} : K \to C(K)$ is a γ-map. If $x \in N_i$, and $\alpha_{n-1}(x) = \alpha_n(x) \cup P(N_i, K)$, then the monotone-connectedness of $\alpha_n(x)$ with N_i as its a maximal element along with the monotone-connectedness $P(N_i, K)$ by [3] with N_i as its minimal element provides the monotone-connectedness of $\alpha_{n-1}(x)$. Also $P(N_i, K) \subset A(x)$ for each $x \in N_i$ by (2.24).

Since α_n is lower semicontinuous at eac $x \in N_i$ and $P(N_i, K)$ is a constant factor of $\alpha_{n-1}(x)$ at each $x \in N_i$, we see that $\alpha_{n-1} : N_i \to C(K)$ is a γ-map. Suppose that x is a limit point of $K \setminus \cup_{j=1}^k N_j$ such that $x \in N_i$ for some i. Let $\epsilon > 0$ and $A \in \alpha_{n-1}(x)$. Then the lower semicontinuity of α_{n-1} at $x \in N_i$ (α_{n-1} restricted on N_i) implies that there exists $\delta_1 > 0$ such that if $y \in N_i$, $d(x,y) < \delta_1$, then there exists an element $B \in \alpha_{n-1}(y)$ such that $H(A,B) < \epsilon$. Let $\delta_2 > 0$ such that $\delta_2 < \frac{\epsilon}{2}$ and suppose $y \in K \setminus \cup_{j=i}^k N_j$ and $d(x,y) < \delta_2$. Let B be an arc in K having x and y as its end points. Then $H(A, A \cup B) < \epsilon$. Also $y \in K \setminus \cup_{j=1}^k N_j$ implies that $A \cup B \in A(y) \cap C(K)$. Therefore if $\delta = \min\{\delta_1, \delta_2\}$ and y is a point of the δ-neighborhood of x in K, then there exists an element $C \in \alpha_{n-1}(y)$ such that $H(A,C) < \epsilon$. This proves the lower semicontinuity of α_{n-1} at x. The lower semicontinuity of α_{n-1} at each point of the open set $K \setminus \cup_{j=1}^k N_j$ in K is rather obvious.

Now we assume that, for $0 < i < n$, we have a lower semicontinuous set-valued map α_i on the union of elements of $\mathcal{M}_i$ such that α_i restricted on each $N \in \mathcal{M}_i$ is a γ-map from N into $C(N)$. Let $K \in \mathcal{M}_{i-1}$. If K is such that $T(x,K) = A(x) \cap C(K)$ for each $x \in N$, and let $\alpha_{i-1}(x) = \alpha_K(x)$ for each $x \in K$. Then α_{i-1} is a γ-map on K. If $T(x,K) \neq A(x) \cap C(K)$ for some $x \in K$, let $\{N_1, N_2, \ldots, N_k\}$ be the set of all elements of $\mathcal{M}_i$ such that $N_i \subset K$, $i = 1, 2, \ldots, k$ and define

$$\alpha_{i-1}(x) = \begin{cases} \alpha_i(x) \cup P(N_i, K) & \text{if } x \in N_i,\ i = 1, 2, \ldots, k \\ T(x, K) & \text{if } x \in K \setminus \cup_{j=1}^{k} N_j \end{cases}.$$

Then the argument showing α_{i-1} to be a γ-map on K is identical with that of α_{n-1}.

Since α_{i-1} restricted on each $K \in \mathcal{M}_{i-1}$ is a γ-map of K into $C(K)$ and elements of $\mathcal{M}_{i-1}$ are closed and disjoint, α_{i-1} is lower semicontinuous on the union of the elements of $\mathcal{M}_{i-1}$.

Let $i = 1$. Then we have a set-valued map α_0 on the union of elements of $\mathcal{M}_0$ such that α_0 restried on each element $M_i \in \mathcal{M}_0$ is a γ-map on M_i into $C(M_i)$.

For each $M_i \in \mathcal{M}_0$, let $T(M_i, I) = \{C \in C(I) : M_i \subset C\}$. Then by applying the same technique as in (2.24), we see that $T(M_i, I) \subset A(x)$ for each $x \in M_i$. We now define a γ-map on the $\mathcal{M}$-set M of X into $C(I)$ by $F(x) = \alpha_0(x) \cup T(M_i, I)$ if $x \in M_i$. Then F is a γ-map.

For the T-admissibility of X, let us first define a set $T(I, X) = \{C \in C(X) : I \subset C\}$. Then $T(I, X)$ is monotone-connected [3] and $T(I, X) \subset A(x)$ for each $x \in I$ by (2.4). So, for $x \in M$, we have a monotone-connected set $F(x) \cup T(I, X) \subset A(x)$. Therefore $\mu(F(x) \cup T(I, X)) = [0, 1]$. If $x \in X \setminus M$, then x is a k-point of X. So that $T(x, X) = A(x)$. The monotone-connectedness of $T(x, X)$ and $\{x\}$, $X \in T(x, X)$ imply that $\mu(T(x, X)) = [0, 1]$. Therefore X is T-admissible. Hence by (1.2) we conclude that $C(X)$ is contractible.

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