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CAUSAL STRUCTURES OF LORENTZIAN MANIFOLD

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1. Introduction

In the general theory of relativity, the space-time universe is considered as a collection of all events which has a four dimensional differentiable manifold structure. In the theory of singularities, black holes, etc. causal structure of the space-time plays the important roles. Moreover, the uncertainties involved in the measurement which is implied by the quantum uncertainty principle are taken into account by the causality structures-Woodhause causality axiom [13].

The various cauality structures have been developed and investigated their relations as well [1, 3, 5, 7, 11, 12, 13] in the recent developments concerning the relations among the various casuality structure. Levichev [7] partially showed their relations for the homogeneous case.

As a model of the space-time, the homogeneous Lorentzian manifold is very restricted one, but it is physically and mathematically effective up to Lie group manifolds and Lie algebra.

In this paper we take a space-time as the homogeneous 4-dimensional Lorentzian manifold with the signature $(- + + +)$, and show more detailed relations of the causal structures and some properties of the causalities.

2. Preliminaries and main results

The Lorentzian manifold M is a paracompact Hausdorff manifold with one negative signature, and the homogeneous space-time M is a 4-dimensional Lorentzian manifold such that the group G of all isometric motions of M is transitive on it. We assume that M is connected, noncompact, and G acts leftly on M. It clearly preserves time orientation. The notations one;

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$I_x^+(I_x^-)$; chronological future(past) set at x
$J_x^+(J_x^-)$; casual future(past) set at x
$\uparrow I_x^-(\downarrow I_x^+)$; chronological common future(past) set of $I_x^-(I_x^+)$
$J_s^+(x)(J_s^-(x))$; Seifert casual future(past) set at x
$A_x^+(A_x^-)$; almost casual future(past) set at x
$\overline{A}(A^\circ)$; closure(interior) of A

The definitions and the terminologies are refferred to [2, 4, 8, 9, 10]. Let us fix the point 0 in M and $f(0) = x$ for f in G. Then

$$f(I_0^+) = I_x^+, f(I_0^-) = I_x^-, f(J_0^+) = J_x^+ \text{ and } f(J_0^-) = J_x^- \text{ [7].}$$

If G is a simply transive subgroup of the group on M, we may identify it with M and the point 0 with the identity of the Lie group M. Levichev [7] showed that J_0^+, J_0^-, I_0^+ and I_0^- are subsemigroup (I_0^+ and I_0^- may not contain the identity), $J_0^- = (J_0^+)^-$, $I_0^- = (I_0^+)^-$, $J_x^+ = x \cdot J_0^+$, $J_x^- = x \cdot J_0^-$, $I_x^+ = x \cdot I_0^+$ and $I_x^- = x \cdot I_0^-$ for all x in M.

We also know that $A_0^{+0} = \uparrow I_0^-$ and $A_0^{-0} = \downarrow I_0^+$ [1].

Thus, by the above properties, the following can be easily proved;

PROPOSITION 1.

$$f(A_0^+) = A_x^+, f(A_0^-) = A_x^-.$$
$$f(\overline{I_0^+}) = \overline{I_x^+}, f(\overline{I_0^-}) = \overline{I_x^-}, f(\overline{J_0^+}) = \overline{J_x^+}, f(\overline{J_0^-}) = \overline{J_x^-}$$
$$f(\uparrow I_0^-) = \uparrow I_x^-, f(\downarrow I_0^+) = \downarrow I_x^+, f(J_s^+(0)) = J_s^+(x), f(J_s^-(0)) = J_s^-(x)$$
for all x in M and f in G.

As like the uncertainty principle, the fact that a physical experiment can never be made sure to have performed at a precise event of the space-time, it leads to think of an event as the limit of a converging sequence of neighborhoods.

Thus Woodhause [13] gives an axiom;

an event x almost causally precedes another event y, denoted by xAy, if for all $x \in I_x^-$, $I_y^+ \subset I_z^+$; or equivalently, if for all $z \in I_y^+$, $I_x^- \subset I_z^-$. Thus axiom can be able to analyze resionably causal influence under the uncertainty principle.

Let $A_x^+ = \{y \in M \,|\, xAy\}$ and $A_x^- = \{y \in M \,|\, yAx\}$. Woodhause gives the principle of causality for a space-time such that if for any x, y in M, xAy and yAx, then it must be $x = y$.

We say it Woodhause causality. Thus we have following property under our space-time.

PROPOSITION 2. *$x \cdot A_0^+ = A_x^+$ for all x in M.*

Proof. Let $xm \in x \cdot A_0^+$ and $m \in A_0^+$. Then, for all $q \in I_0^-$, $I_m^+ \subset I_q^-$. That is, every neighborhood N_0 of 0 contains events which chronologically precedes some events in any neighborhood N_m of m.

Pick an event p in I_m^+ and q in I_0^-, and let chronological futhre curves α, β such that $\alpha(0) = m, \alpha(1) = p, \beta(0) = q$ and $\beta(1) = p = \alpha(1)$. Consider chronological future curves $\gamma(t) = x\alpha(t)$ and $\eta(t) = x\beta(t)$. Then

$$\gamma(0) = x\alpha(0) = xm, \quad \gamma(1) = x\alpha(1) = xp,$$
$$\eta(0) = x\beta(0) = xq, \quad \eta(1) = x\beta(1) = xp = \gamma(1),$$
$$\text{and} \quad xq \in xI_0^- = I_x^-.$$

This shows $xm \in A_x^+$. That is, $xA_0^+ \subset A_x^+$.

Conversely, let $m \in A_x^+$ and $q \in I_x^-$.

Since $I_x^- = xI_0^-$, $q = xq_1$, for some q_1 in I_0^-.

Thus there exist chronological future (timelike geodesic) curves α, β such that $\alpha(0) = m$, $\alpha(1) = p$, $\beta(0) = q = xq_1$, and $\beta(1) = p = \alpha(1)$.

If s is an event in A_0^+, then for q_1 in I_0^- and p_1 in I_s^-, there exist chronological future (timelike geodesic) curves α_1, β_1, such that

$$\alpha_1(0) = s, \alpha_1(1) = p_1, \beta_1(0) = q_1 \text{ and } \beta_1(1) = p_1 = \alpha_1(1).$$

Let $\gamma(t) = x\alpha_1(t)$, and $\delta(t) = x\beta_1(t)$. Thus by the uniqueness of timelike geodesic from xq_1, $\beta = \delta$. Therefore $\beta(1) = \alpha(1)$. That is, $p = \beta(1) = \delta(1) = xp_1$ and $\alpha(1) = p = xp_1 = \delta(1) = \alpha_1(1) = \gamma(1) \in I_m^+$. Since I_m^+ is open and $\gamma(1) = \alpha(1) \in I_m^+$, $\alpha = \gamma$ by the same reason. Thus $m = \alpha(0) = \gamma(0) = xs \in xA_0^+$. This shows $A_x^+ \subset xA_0^+$. The proof is complete.

The causalities of space-time were developed in the various ways and the relations of there were partially showed in many situation [1, 3, 7, 11, 12, 13] as follows;

(1) globally hyperbolic $\Longrightarrow$ (2) causally simple $\Longrightarrow$ (3) causally continuous $\Longrightarrow$ (4) stably causal $\Longrightarrow$ (5) woodhause causal $\Longrightarrow$ (6)

strongly causal $\Longrightarrow$ (7) distinguishing $\Longrightarrow$ (8) future distinguishing $\Longrightarrow$ (9) past distinguishing $\Longrightarrow$ (10) causal $\Longrightarrow$ (11) chronological $\Longrightarrow$ (12) non-compact.

The converse relations of these also have partially been showed under the reflectingness or other cases [1, 5, 7, 11].

The followings were showed in [1, 7].

PROPOSITION A. *If a homogeneous space-time is future(past) distinguishing then it is past(future) distinguishing.*

PROPOSITION B. *If a space-time is reflecting, then (4)–(9) are equivalent.*

PROPOSITION C. *In a homogeneous space-time (3)–(9) are equivalent.*

We say that J_K^+ *and* J_K^- *are causal future and past of the set* K *[4].*

PROPOSITION 3. *If* M *is a homogeneous space-time, and* $J_0^+(J_0^-)$ *is closed, then* M *is causally simple.*

Proof. Let K be a compact subset of M. Then, since M is homogeneous, KJ_0^+ is closed in M.

By proposition 1,

$$KJ_0^+ = \{kJ_0^+ \mid k \in K\} = \{J_k^+ \mid k \in K\} = \bigcup_{k \in K} J_k^+ = J_K^+.$$

Similarly, J_K^- is closed. This completes the proof.

PROPOSITION 4. *If a null geodesically complete or naturally reductive, homogeneous space-time satisfies the generic condition, and the nonnegative Ricci tensor for all null vectors, then (3)–(11) are equivalent.*

Proof. It can be proved from Proposition A,B,C, and Proposition 6.4.6 [4]. In the case of a naturally reductive Homogeneous space-time, the space-time is complete. Thus it can be showed by the same method.

Let F be a funtion which assigns to each x in M an open set $F(x)$ in M. F is called inner continuous if for any x and compact set $K \subset F(x)$, there is a neighborhood U of x such that $K \subset F(z)$ for all z in U. F is outer continuous if for any x and any compact set $K \subset M - \overline{F(x)}$, there is a neighborhood U of x such that $K \subset M - \overline{F(z)}$ for all z in U.

DEFINITION. Let F be a funtion which assigns to each x in M on closed set $F(x)$ in M. F is C-outer continuous if for any x and any compact set $K \subset M - \overline{F(x)}$, there is a neighborhood U of x such that $K \subset M - \overline{F(z)}$ for all z in U.

Clearly, there is no relation between these kinds of continuities in general.

The several causal sets as set-function have partially been showed the inner or outer continuity of the sets under the different situations [5, 6].

Our space-time M may be assumed homogeneous one.

PROPOSITION 5. $\uparrow I^-(\downarrow I^+)$ *is inner continuous.*

Proof. It suffices to show the case of $\uparrow I^-$ at 0 by Proposition 1. We know that $\uparrow I^-$ is open. Suppose K is a compact subset of $\uparrow I^-$. Since K is closed, there exists an open subset V such that $K \subset V \subset \uparrow I_0^-$. Let $U = \{f \in G \mid f(K) \subset V\}$. Then U is an open neighborhood of the identity of G, and may be $U = U^{-1}$. Thus $f(K) \subset \uparrow I_0^-$ for all f in U. $V \subset f(\uparrow I_0^-)$ for all f in U, and $K \subset f(\uparrow I_0^-)$ for all f in U.

Let $W = \{f(0) \mid f \in U\}$. Then W is an open neighborhood of 0, and $K \subset \uparrow I_x^-$ for all x in W by Proposition 1 and the above. This shows that $\uparrow I^-$ is continuous. Similarly $\downarrow I^+$ is inner continuous.

PROPOSITION 6. $A^+(A^-)$ *is C-outer continuous.*

Proof. It suffices to show the case of A^+ at 0 by Proposition 1. We know that A_0^+ is closed. Let K be a compact subset of $M - A_0^+$. Then there exists an open subset U such that $K \subset U \subset M - A_0^+$.

By the homogenuity of M, there exists an open subset V of G such that $f(K) \subset U$ for all f in V, and V may be equal to V^{-1}. Clearly V is a neighborhood of the identity of G.

Thus, $K \subset M - \cup_{f \in V} f(A)$ if and only if $\cup_{f \in V} f(K) \subset M - A$.

Let $W = \{f(0) \mid f \subset V\}$. Then W is an open subset of M containing 0, and $K \subset M - A_0^+$ for each x in W. This shows that A^+ is C-outer continuous. Similarly A^- is C-outer continuous.

PROPOSITION 7. $I^+(I^-)$ *is outer continuous.*

Proof. It can similarly be proved.

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