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FUZZY NORMAL SUBGROUPS IN FUZZY SUBGROUPS

D.S. MALIK, JOHN N. MORDESON AND P.S. NAIR

0. Introduction

The theory of fuzzy sets was inspired by Zadeh [10]. Subsequently, Rosenfeld introduced the concept of a fuzzy subgroup of a group [9]. Fuzzy cosets and fuzzy normal subgroups of a group G have been studied in [3, 5, 8]. In [4], the ring of cosets of a fuzzy ideal was constructed. Let A and B be fuzzy subgroups of G such that $B \subseteq A$. The purpose of this paper is to introduce the notion of fuzzy cosets and fuzzy normality of B in A. These ideas differ from those in [3, 5, 8] since there $A = \delta_G$, the characteristic function of G. If B is fuzzy normal in A, then the set of all fuzzy cosets of B in A forms a semigroup under a suitable operation. Structure properties of A/B and A are determined.

Throughout this paper G denotes a group and L denotes a completely distributive lattice. A **fuzzy subgroup** A of G is a fuzzy subset of G (a function of G into L) such that $\forall x, y \in G$, $A(xy^{-1}) \geq \inf\{A(x), A(y)\}$. We let e denote the identity of G and $0, 1$ the least element, the greatest element of L respectively. If X and Y are fuzzy subsets of G, we say that $X \subseteq Y$ if and only if $\forall x \in G$, $X(x) \leq Y(x)$. For any $x \in G$, $t \in L$, we let x_t denote the fuzzy subset of G defined by $\forall y \in G$, $x_t(y) = 0$ if $y \neq x$ and $x_t(y) = t$ if $y = x$. We call x_t a **fuzzy singleton**. B and A always denote fuzzy subgroups of G such that $B \subseteq A$. If $t \in L$, we let $B_t = \{x \in G \mid B(x) \geq t\}$. It follows easily that if $t \in \text{Im}(B)$, then B_t is a subgroup of G. B_t is called a **level subgroup** of G [1]. We let $B_* = B_{B(e)}$. For any fuzzy subgroup A of G we assume that $A(e) > 0$. $\mathbb{N}$ denotes the set of positive integers.

1. Fuzzy cosets and quotient semigroups

We introduce the concept of fuzzy cosets of B in A.

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DEFINITION 1.1. Let X and Y be fuzzy subsets of G. Define the **fuzzy subset** $X \circ Y$ of G by $\forall x \in G$, $(X \circ Y)(x) = \sup\{\inf\{X(y), Y(z)\} | x = yz\}$.

DEFINITION 1.2. Let $x_t \subseteq A$. Then the fuzzy subset $x_t \circ B$ $(B \circ x_t)$ is called a **fuzzy left (right) coset** of B in A with representative x_t.

The notions in [3, 5, 8] deal with $A = \delta_G$ and fuzzy cosets $x_t \circ B$ with $t = 1$.

PROPOSITION 1.3. *Let* $x_t \subseteq A$. *Then* $\forall z \in G$, $(x_t \circ B)(z) = \inf\{t, B(x^{-1}z)\}$ *and* $(B \circ x_t)(z) = \inf\{t, B(zx^{-1})\}$.

Proof. $(x_t \circ B)(z) = \sup\{\inf\{x_t(u), B(v) | z = uv\} = \inf\{t, B(x^{-1}z)\}$ since the supremum is attained when $x = u$. Similarly, $(B \circ x_t)(z) = \inf\{t, B(zx^{-1})\}$.

PROPOSITION 1.4. *Let* $x_t, y_s \subseteq A$. *Then*

(i) $x_t \circ B = y_s \circ B$ *if and only if* $\inf\{t, B(e)\} = \inf\{s, B(y^{-1}x)\}$ *and* $\inf\{s, B(e)\} = \inf\{t, B(x^{-1}y)\}$;

(ii) $B \circ x_t = B \circ y_s$ *if and only if* $\inf\{t, B(e)\} = \inf\{s, B(xy^{-1})\}$ *and* $\inf\{s, B(e)\} = \inf\{t, B(yx^{-1})\}$.

Proof. (i). $x_t \circ B = y_s \circ B$ if and only if $\forall z \in G$, $(x_t \circ B)(z) = (y_s \circ B)(z)$ if and only if $\forall z \in G$, $\inf\{t, B(x^{-1}z)\} = \inf\{s, B(y^{-1}z)\}$. Suppose that $x_t \circ B = y_s \circ B$. Then letting $z = x$ and then $z = y$, we obtain $\inf\{t, B(e)\} = \inf\{s, B(y^{-1}x)\}$ and $\inf\{t, B(x^{-1}y)\} = \inf\{s, B(e)\}$. Conversely, suppose that the conditions concerning the infimum hold. Let $z \in G$. Then $(x_t \circ B)(z) = \inf\{t, B(x^{-1}z)\} = \inf\{t, B(x^{-1}yy^{-1}z)\} \geq \inf\{t, \inf\{B(x^{-1}y), B(y^{-1}z)\}\} = \inf\{\inf\{t, B(x^{-1}y)\}, B(y^{-1}z)\} = \inf\{\inf\{s, B(e)\}, B(y^{-1}z)\} = \inf\{s, \inf\{B(e), B(y^{-1}z)\}\} = \inf\{s, B(y^{-1}z)\} = (y_s \circ B)(z)$. Similarly $x_t \circ B \subseteq y_s \circ B$. Thus $x_t \circ B = y_s \circ B$.

(ii). The proof is similar to that of (i).

COROLLARY 1.5. *Let* $x_t, y_t \subseteq A$. *If* $B(y^{-1}x) = B(e)$, *then* $x_t \circ B = y_t \circ B$.

Proof. Since $B(x^{-1}y) = B(y^{-1}x) = B(e)$, $\inf\{t, B(e)\} = \inf\{t, B(x^{-1}y)\} = \inf\{t, B(y^{-1}x)\}$. Hence by Proposition 1.4(i), $x_t \circ B = y_t \circ B$.

PROPOSITION 1.6. *Let $x_t, y_t \subseteq A$. Then the following conditions are equivalent.*

(i) $x_t \circ B = y_t \circ B$;
(ii) $(y^{-1}x)_t \circ B = e_t \circ B$;
(iii) $(x^{-1}y)_t \circ B = e_t \circ B$.

Proof. By Proposition 1.4, $x_t \circ B = y_t \circ B$ if and only if $\inf\{t, B(e)\} = \inf\{t, B(y^{-1}x)\}$ and $\inf\{t, B(e)\} = \inf\{t, B(x^{-1}y)\}$. The latter conditions are equivalent to (ii) and (iii).

PROPOSITION 1.7. *Let $x, y \in G$ and $s, t \in [0, A(e)]$. Suppose that $B(e) = A(e)$. Then*

(i) $x_t \circ B = y_s \circ B$ *if and only if* $t = \inf\{s, B(y^{-1}x)\}$, $s = \inf\{t, B(x^{-1}y)\}$;
(ii) $x_t \circ B = y_t \circ B$ *if and only if* $(y^{-1}x)_t \subseteq B$;
(iii) $x_t \circ B = y_s \circ B$ *if and only if* $t = s \leq B(x^{-1}y)$;
(iv) $x_t \circ B = x_s \circ B$ *if and only if* $t = s$.

Proof. (i). By Proposition 1.4, $x_t \circ B = y_s \circ B$ if and only if $t = \inf\{s, B(y^{-1}x)\}$ and $s = \inf\{t, B(x^{-1}y)\}$.

(ii). By (i), $x_t \circ B = y_t \circ B$ if and only if $t = \inf\{t, B(y^{-1}x)\}$, $t = \inf\{t, B(x^{-1}y)\}$ if and only if $B(y^{-1}x) \geq t$, $B(x^{-1}y) \geq t$.

(iii). By (i) and the fact that $B(y^{-1}x) = B(x^{-1}y)$, $x_t \circ B = y_s \circ B$ if and only if $t = s \leq B(x^{-1}y)$.

(iv). The result here is immediate from (iii).

The proof of the next result is immediate from Proposition 1.7(iii).

COROLLARY 1.8. *Let $s, t \in [0, A(e)]$. Suppose that $B(e) = A(e)$. If $t \neq s$, then $\{x_t \circ B \mid x_t \subseteq A\} \cap \{y_s \circ B \mid y_s \subseteq A\} = \phi$.*

DEFINITION 1.9. B is said to be **fuzzy normal** in A if and only if $\forall x_t \subseteq A$, $x_t \circ B = B \circ x_t$.

PROPOSITION 1.10. *Let $x_t, y_s \subseteq A$. If B is fuzzy normal in A, then* $(x_t \circ B) \circ (y_s \circ B) = (xy)_r \circ B$ where $r = \inf\{t, s\}$.

Proof. $\circ$ is associative and $B \circ B = B$, [5, p.134], [6, p.32].

THEOREM 1.11. *Let $A/B = \{x_t \circ B \mid x_t \subseteq A,\ x \in G\}$. Suppose that B is fuzzy normal in A. Then $(A/B, \circ)$ is a semigroup with identity. If $B(e) = A(e)$, then A/B is completely regular, i.e. A/B is a union of (disjoint) groups.*

Proof. If $x_t \circ B,\ y_s \circ B \in A/B$, then clearly $(xy)_r \circ B \in A/B$ where $r = \inf\{t, s\}$. Clearly $e_{A(e)}$ is the identity of A/B. By [5, p.134], $\circ$ is associative. For fixed $t \in [0, A(e)]$, let $(A/B)^{(t)} = \{x_t \circ B \mid x_t \subseteq A,\ x \in G\}$. Then $(A/B)^{(t)}$ is closed under $\circ$, $e_t \circ B$ is the identity of $(A/B)^{(t)}$, and $(x^{-1})_t \circ B$ is the inverse of $x_t \circ B$. Hence $(A/B)^{(t)}$ is a group. Clearly $A/B = \cup_{t\in[0,A(e)]}(A/B)^{(t)}$.

EXAMPLE 1.12. Let $G = \{e, a, b, c\}$ be the Klein four-group. Define the fuzzy subsets A and B of G by $A(e) = A(a) = 1,\ A(b) = A(c) = \frac{3}{4}$ and $B(e) = B(a) = 1,\ B(b) = B(c) = \frac{1}{2}$. Then A and B are fuzzy subgroups of G such that $B \subseteq A$ and B is fuzzy normal in A. Now $e_1 \circ B$ is the identity of A/B, but $e_{\frac{3}{4}} \circ B$ does not have an inverse. Hence A/B is not a group.

2. Structure of quotient semigroups of fuzzy subgroups

We investigate the structure of A/B.

THEOREM 2.1. *B is fuzzy normal in A if and only if $\forall t \in [0, B(e)]$, B_t is normal in A_t.*

Proof. Suppose that B is fuzzy normal in A. Let $t \in [0, B(e)]$. Let $x \in A_t$ and $b \in B_t$. Then $x_t \circ B = B \circ x_t$. Hence $(x_t \circ B)(bx) = (B \circ x_t)(bx)$. Thus $\inf\{t, B(x^{-1}bx)\} = \inf\{t, B(bxx^{-1})\} = \inf\{t, B(b)\} = t$. Hence $B(x^{-1}bx) \geq t$. Thus $x^{-1}bx \in B_t$. Conversely, suppose that B_t is normal in A_t $\forall t \in [0, B(e)]$. Let $x_t \subseteq A$ and $z \in G$. Suppose that $t \leq B(e)$. Since B_t is normal in A_t, $x^{-1}z \in B_t$ if and only if $zx^{-1} \in B_t$. Now $(x_t \circ B)(z) = \inf\{t, B(x^{-1}z)\}$ and $(B \circ x_t)(z) = \inf\{t, B(zx^{-1})\}$. If $x^{-1}z \in B_t$, then $(x_t \circ B)(z) = (B \circ x_t)(z)$. Suppose that $x^{-1}z \notin B_t$. Then $zx^{-1} \notin B_t$ and $t > 0$. Thus $(x_t \circ B)(z) = B(x^{-1}z)$ and $(B \circ x_t)(z) = B(zx^{-1})$. It sufficies to show that $B(x^{-1}z) = B(zx^{-1})$. Let $B(x^{-1}z) = m < t$ and $B(zx^{-1}) = n < t$. Now $x^{-1}z \in B_m$ and $x \in A_t \subseteq A_m$. Since B_m is normal in A_m and $x \in A_m$, $zx^{-1} \in B_m$. Thus $B(zx^{-1}) = n \geq m$. Similarly $m \geq n$ and so $m = n$. Now suppose that $t > B(e)$. Then $(x_t \circ B)(z) = \inf\{t, B(x^{-1}z)\} = B(x^{-1}z)$ and similarly $(B \circ x_t)(z) = B(zx^{-1})$. Since $B(x^{-1}z) < t$ and $B(zx^{-1}) < t$, $B(x^{-1}z) = B(zx^{-1})$ as previously shown. Thus $x_t \circ B = B \circ x_t \ \forall x_t \subseteq A$. Hence B is fuzzy normal in A.

PROPOSITION 2.2. *Suppose that* $0 \leq t \leq B(e)$, $x_s \subseteq A$, *and* $t \leq s$. *Then* $(x_s \circ B)_t = xB_t$ *and* $(B \circ x_s)_t = B_t x$.

Proof. $y \in (x_s \circ B)_t$ if and only if $(x_s \circ B)(y) \geq t$ if and only if $\inf\{s, B(x^{-1}y)\} \geq t$ if and only if $B(x^{-1}y) \geq t$ if and only if $x^{-1}y \in B_t$ if and only if $y \in xB_t$.

THEOREM 2.3. *Let* $t \in [0, B(e)]$. *Suppose that* B *is fuzzy normal in* A. *Then* $A_t/B_t \simeq (A/B)^{(t)}$.

Proof. By Theorem 2.1, B_t is normal in A_t. Define the mapping $f : A_t \to (A/B)^{(t)}$ by $\forall x \in A_t$, $f(x) = x_t \circ B$. Then clearly f is a homomorphism of A_t onto $(A/B)^{(t)}$. Now $x \in \operatorname{Ker} f$ if and only if $f(x) = e_t \circ B$ if and only if $x_t \circ B = e_t \circ B$ if and only if $x_t \subseteq B$ (by Proposition 1.7(ii)) if and only if $x \in B_t$. Hence $\operatorname{Ker} f = B_t$.

If B is fuzzy normal in A and $B(e) = A(e)$, then structure properties of A/B can be determined from those of A_t/B_t, $t \in [0, A(e)]$, since $A/B = \cup_{t \in [0, A(e)]} (A/B)^{(t)}$ by Theorem 1.12 and $(A/B)^{(t)} \simeq A_t/B_t$ by Theorem 2.3.

For the remainder of the section we assume that G is commutative and $L = [0, 1]$. Then B is fuzzy normal in A. We say that A is **bounded** over B if $\exists n \in \mathbb{N}$ such that $\forall x_t \subseteq A$, $(x_t)^n \subseteq B$. Then it can be shown easily that A is bounded over B if and only if A_t/B_t is uniformly bounded $\forall t \in [0, A(e)]$. Hence if A is bounded over B, then A_t/B_t is a direct product of cyclic groups $\forall t \in [0, A(e)]$ by [2, Theorem 17.2, p.88].

As another example, suppose that C is a fuzzy subgroup of G such that $C \subseteq A$ and $A = B \otimes C$, the **fuzzy direct product** of B and C, i.e. $A = B \circ C$ and $\forall x \in G$, $(B \cap C)(x) = 0$ [7, Definition 4.1]. Then $A_t = B_t \odot C_t \ \forall t \in (0, A(e)]$ by [7, Corollary 4.7.]. Thus $A/B \simeq (\cup_{t \in (0, A(e)]} C_t) \cup \{e_0 \circ B\}$. We now give some conditions for B to be a fuzzy direct factor of A.

Let $\mathcal{F}(A)$ denote the set of all fuzzy subgroups C of G such that $C \subseteq A$ and $C(e) = A(e)$. Let $C^* = \{x \in G \mid C(x) > 0\}$. Then C^* is a subgroup of G if $L = [0,1]$.

We say that B is **compatible** in A if and only if $A(e) = B(e)$ and $\forall s,t \in (0, A(e)]$, $s \leq t$, $A_s = A_t B_s$ and $A_t \cap B_s = B_t$.

In [7, Example 4.3], it is shown that if B is **divisible**, i.e. $\forall x_t \subseteq B$ with $t > 0$ and $\forall n \in \mathbb{N}\ \exists y_t \subseteq B$ such that $(y_t)^n = x_t$ [7, Definition 2.1], then it need not be the case that B is a fuzzy direct factor of A. However in Corollary 2.6, we show that if B is compatible in A and B is divisible, then B is a fuzzy direct factor of A. Thus compatibility is a straightening factor which allows results in the crisp case to be carried over to the fuzzy case.

THEOREM 2.4. *The following conditions are equivalent*

(i) $A(e) = B(e)$ *and there* $\exists$ *subgroup* H *of* G *such that* $\forall t \in (0, A(e)]$, $A_t = B_t \otimes H$;

(ii) $\exists C \in \mathcal{F}(A)$ *such that* $A = B \otimes C$ *and* $C^* = C_*$;

(iii) B *is compatible in* A *and* $\exists C \in \mathcal{F}(A)$ *such that* $A_* = B_* \otimes C_*$.

Proof. (i) $\Longrightarrow$ (ii). Define the fuzzy subset C of G by $C(x) = A(e)$ if $x \in H$ and $C(x) = 0$ otherwise. Then C is a fuzzy subgroup of G and $C^* = H = C_*$. Now $H \subseteq A_*$ and so $C \subseteq A$. Since $C^* = C_*$, $C_t = C^* = C_*\ \forall t \in (0, A(e)]$. Hence $A_t = B_t \otimes C_t \forall t \in (0, A(e)]$. Thus $A = B \otimes C$ by [7, Corollary 4.7].

(ii) $\Longrightarrow$ (iii). Now $A(e) = B(e) = C(e)$ and $A_t = B_t \otimes C_t \forall t \in (0, A(e)]$ by [7, Corollary 4.7]. Also $C^* = C_t = C_* \forall t \in (0, A(e)]$. Hence $A_* = B_* \otimes C^*$. In fact, $A_t = B_t \otimes C_* \forall t \in (0, A(e)]$. Now $B_t \subseteq A_t \cap B_s = (B_t \otimes C_t) \cap B_s = B_t$ for $s \leq t$. Thus $B_t \cap A_t$ for $s \leq t$. Also $A_s = B_s \otimes C_* \subseteq B_s B_t C_* = B_s A_t \subseteq A_s$. Hence $A_s = A_t B_s$.

(iii) $\Longrightarrow$ (i). Since $C^* \subseteq A_*$ and $A(e) = C(e)$, $C^* = C_*$. $\forall s \in (0, A(e)]$, $A_s = A_* B_s = C^* \otimes B_s \subseteq C^* \otimes B^*$. Thus $A^* \subseteq C^* \otimes B^*$. Hence $A^* = C^* \otimes B^*$. By [7, Theorem 4.2], it suffices to show that $A = B \circ C$. Since $C_* = C^*$ and $A(e) = C(e)$, $\mathrm{Im}(C) = \{0, A(e)\}$. Thus $\mathrm{Im}(B) \cap \mathrm{Im}(C) \subseteq \{0, A(e)\}$. Hence $A = B \circ C$ by [7, Theorem 4.5].

If A and B are fuzzy subgroups of G such that $B \subseteq A$, we say that B is **pure** in A of and only if $\forall x_t \subseteq B$ with $t > 0$, $\forall n \in \mathbb{N}$, $\forall y_t \subseteq A$, $(y_t)^n = x_t$ implies that $\exists b_t \subseteq B$ such that $(b_t)^n = x_t$ [7, Definition 3.1].

COROLLARY 2.5. *Suppose that* B *is compatible and pure in* A

(i) *If* A_*/B_* *is a direct product of cyclic groups, then* B *is a fuzzy* *direct factor of A;*

(ii) *If B is bounded, then B is a fuzzy direct factor of A.*

Proof. (i). Since B is pure in A, B_* is pure in A_* [7, Proposition 3.2]. Hence $\exists$ a subgroup H of A_* such that $A_* = B_* \otimes H$, [2, Theorem 28.2, p.120]. Since B is compatible in A, $A_t = B_t \otimes H \forall t \in (0, A(e)]$. The desired result now follows from Theorem 2.4.

(ii). B_* is pure in A_* and B_* is bounded. Hence $\exists$ subgroup H of A_* such that $A_* = B_* \otimes H$, [2, Theorem 27.5, p.118]. Then remainder of the proof is as in the proof of (i).

COROLLARY 2.6. *Suppose that B is compatible in A. If B is divisible, then B is a direct factor of A.*

Proof. By [7, Proposition 2.2], B_* is divisible. Hence $\exists$ a subgroup H of A_* such that $A_* = B_* \otimes H$. The remainder of the proof is as in the proof of Corollary 2.5(i).

In the above Corollaries, we have that $A_t = B_t \otimes C_* \forall t \in (0, A(e)]$. Hence $(A/B)\backslash\{e_0 \circ B\}$ is isomorphic to an uncountable number of groups each isomorphic to C_*. In Corollary 2.5(i), C_* is a direct product of cyclic groups. We also have that $A = B \otimes C$ where $C(x) = A(e) \forall x \in C^*$.

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Department of Mathematics and Computer Science
Creighton University
Omaha, Nebraska 68178, USA