J. Korean Math. Soc. **29** (1992), No. 2, pp. 423–436

YOUNG'S LATTICE AND SUBSPACE LATTICE: JUMP NUMBER, GREEDINESS

HYUNG CHAN JUNG

1. Introduction

Suppose we are given a finite number of tasks to be sequenced subject to precedence constraints, that is, a task cannot be scheduled until all of its predecessors have been scheduled. If a task t is scheduled immediately after the task u, then there is a jump (or setup) resulting in a fixed cost if u is not one of t's precedecessors and there is no cost (no setup) if u is one of t's precedecessors. Since the cost of a jump does not depend on where it occurs, the cost of a schedule is completely defined by the structure of the underlying partial order which represents the precedence constraints. The problem is: *schedule the tasks to minimize the number of jumps.* This is the jump number problem of a poset.

Let P be a finite poset and let $|P|$ be the *number of vertices* in P. A *subposet* of P is a subset of P with the induced order. A *chain* C in P is a subposet of P which is a linear order. The *length* of the chain C is $|C|-1$. A poset is *ranked* if every maximal chain has the same length. A *linear extension* of a poset P is a linear order $L = x_1, x_2, \ldots, x_n$ of the elements of P such that $x_i < x_j$ in P implies $i < j$. Let $\mathcal{L}(P)$ be the set of all linear extensions of P. Szpilrajn [13] showed that $\mathcal{L}(P)$ is not empty. Algorithmically, a linear extension L of P can be defined as follows:

1. Choose a minimal element x_1 in P.
2. Given $x_1, x_2, \ldots, x_i$ choose a minimal element from $P\backslash\{x_1, \ldots, x_i\}$ and call this element x_{i+1}.

Let P, Q be two disjoint posets. The *disjoint sum* $P+Q$ of P and Q is the poset on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q

Received November 16, 1991.

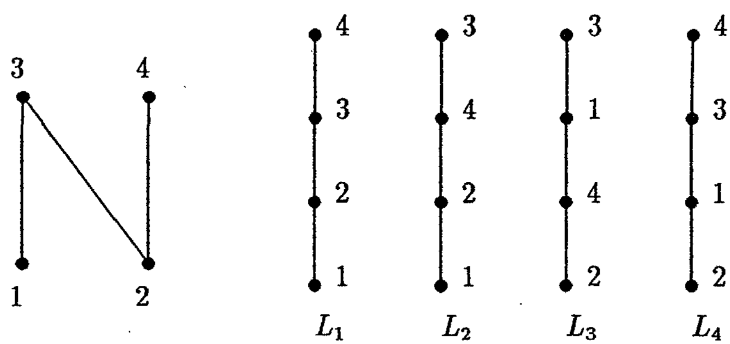


FIGURE 1: THE POSET N AND ITS LINEAR EXTENSIONS.

is obtained from $P+Q$ by adding the relation $x < y$ for all $x \in P$ and $y \in Q$.

Throughout this section, L denotes an arbitrary linear extension of P. Let $a, b \in P$ with $a < b$. Then b *covers* a, denoted $a \prec b$, provided that for any $c \in P$, $a < c \leq b$ implies that $c = b$. A (P, L)-*chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L.

A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* (or *setup*) of P in L if x_i is not comparable to x_{i+1} in P. The jumps induce a decomposition $L = C_1 \oplus \cdots \oplus C_m$ of L into (P, L)-chains $C_1, \ldots, C_m$ where $m = c(L)$ and $(\max C_i, \min C_{i+1})$ is a jump of P in L for $i = 1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of P in L and let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The number $s(P)$ is called the *jump number* of P. If $s(L, P) = s(P)$ then L is called an *optimal linear extension* of P. We denote the set of all optimal linear extensions of P by $\mathcal{O}(P)$.

Let P^d denote the *dual* of the poset P, that is, the poset obtained from P by reversing the order. If L is a linear extension of P, then its dual L^d is a linear extension of P^d.

In Figure 1 only L_3 is optimal.

The *width* $\omega(P)$ of P is the maximal number of elements of an *antichain* (mutually incomparable elements) of P.

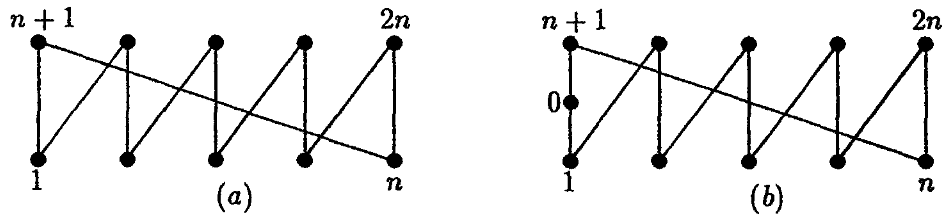


FIGURE 2: (A) CYCLE (B) C.

Dilworth [5] showed that $\omega(P)$ equals the minimum number of chains in a partition of P into chains. Since for any linear extension L of P the number of (P, L)-chains is at least as large as the minimum number of chains in a chain partition of P, it follows from Dilworth's theorem that

$$(1) \qquad s(P) \geq \omega(P) - 1.$$

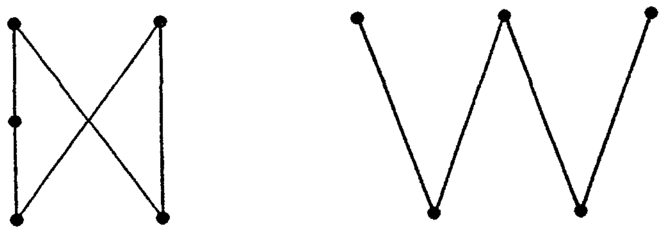
If equality holds in (1), then P is called a *Dilworth poset* or simply a *D-poset.* More discussion about D-posets is given in [6], [12].

A cycle is a partially ordered set with diagram in Figure 2(a). In 1982, Duffus, Rival and Winkler [6] proved that every poset which contains no cycle as a subposet is a D-poset.

A linear extension $L = x_1, x_2, \ldots, x_n$ of P is *greedy* if L can be obtained by applying the following algorithm:

1. Choose a minimal element x_1 of P.
2. Suppose $x_1, \ldots, x_i$ have been chosen. If there is a minimal element of $P \backslash \{x_1, \ldots, x_i\}$ which is greater than x_i then choose x_{i+1} to be this minimal element. If not, choose x_{i+1} to be any minimal element of $P \backslash \{x_1, \ldots, x_i\}$.

In words, L is obtained by *climbing as high as one can.* Let $\mathcal{G}(P)$ be the set of all greedy linear extensions of P. In Figure 1, L_1, L_2, L_3 are greedy linear extensions of the poset N, but L_4 is not greedy. So $\mathcal{O}(N) \subset \mathcal{G}(N)$. In fact, L_3 is a greedy optimal linear extension of N. Since the greedy algorighm above is a particular way of carrying out the algorithm for a linear extension, by induction we obtain [11] that every poset P has a greedy optimal linear extension.



X W

FIGURE 3: THE POSET X AND W.

A poset P is *greedy* if $\mathcal{G}(P) \subseteq \mathcal{O}(P)$, that is, every greedy extension is optimal. In Figure 3, $\mathcal{G}(X) \subset \mathcal{O}(X)$, $\mathcal{O}(W) = \mathcal{G}(W)$. In the above examples, X and W are greedy but N is not greedy.

A poset P is called *series parallel* if it can be constructed from singletons using the operations of disjoint sum $(+)$ and linear sum $(\oplus)$. For example, $(1+1)\oplus(1+1)$, a crown with 4 elements, is a series parallel poset.

In 1979, Cogis and Habib [4] proved that every series parallel poset is greedy. In 1982, Cogis [3] asked for a characterization of greedy posets. The problem remains open. A poset P is *N-free* if P contains no cover-preserving subposet isomorphic to the poset N in Figure 1. Rival [10] showed that every N-free poset is greedy.

In 1985, El-Zahar and Rival [7] proved that if P is a poset which contains no subposet isomorphic to C in Figure 2(b), then $\mathcal{O}(P) \subseteq \mathcal{G}(P)$.

A poset P is said to be *reversible* if $L^d \in \mathcal{G}(P^d)$ for every $L \in \mathcal{G}(P)$. In 1986, Rival and Zaguia [11] showed that a poset P is reversible if and only if $\mathcal{O}(P) = \mathcal{G}(P)$.

In this paper, we study jump number and greediness of some classical posets (Young's Lattice and Subspace Lattice).

2. The Young's lattice $L(m,n)$

Let m, n be positive integers. The *Young's lattice* $L(m,n)$ is a poset defined on

$$\{(a_1, \ldots, a_m) : 0 \leq a_1 \leq \cdots \leq a_m \leq n, \text{ all } a_i\text{'s are integers}\}$$

with the following order relation: $(a_1, \dots a_m) \leq (b_1, \dots, b_m)$ if and only if $a_i \leq b_i$ for $i = 1, \dots, m$.

In this section we obtain an upper bound for the jump number of $L(m,n)$, and also we will study the greediness property of $L(m,n)$.

For $0 \leq i_1 \leq \cdots \leq i_{m-1} \leq n-1$, if $i_{m-1} = n-1$ choose the smallest k such that $i_k = n-1$. We define a chain $C(i_1, \dots, i_{m-1})$ to be

- $\{(i_1, i_2, \dots, i_{m-1}, x_m) : i_{m-1} \leq x_m \leq n\}$ if $i_{m-1} < n-1$,
- $\{(i_1, i_2, \dots, i_k, x_{k+1}, \dots, x_{m-1}, x_m) : n-1 \leq x_{k+1} \leq \cdots \leq x_{m-1} \leq x_m \leq n\}$ if $i_{m-1} = n-1$.

Let $L_0 = \oplus(C(i_1, \dots, i_{m-1}) : 0 \leq i_1 \leq \cdots \leq i_{m-1} \leq n-1$, lex. order).

LEMMA 2.1. *L_0 is a linear extension of $L(m,n)$.*

Proof. Let $x \in C(i_1, \dots, i_{n-1})$ and $y \in C(j_1, \dots, j_{n-1})$ be distinct elements such that $x < y$ in $L(m,n)$. By definition of linear extension it is sufficient to show that $x < y$ in L_0.

If $(i_1, \dots, i_{n-1}) = (j_1, \dots, j_{n-1})$, then x and y belong to the same chain and thus $x < y$ in L_0.

Suppose $(i_1, \dots, i_{n-1}) \neq (j_1, \dots, j_{n-1})$. Let p be the smallest k such that $i_k \neq j_k$. Since $x < y$ in $L(m,n)$, we get $i_p < j_p$. If $(i_1, \dots, i_{n-1}) < (j_1, \dots, j_{n-1})$ in lexicographic order, then $x < y$ in L_0. If $(i_1, \dots, i_{n-1}) \not< (j_1, \dots, j_{n-1})$, then there exists $q > p$ such that $i_q > j_q$. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Since $i_p < j_p \leq j_q < i_q \leq n-1$ and $i_q \leq x_q$, we have $x_p < y_p \leq y_q < x_q$. Thus $x_p < y_p$ but $x_q \not< y_q$, and hence $x \not< y$ in $L(m,n)$ which is a contradiction.

COROLLARY 2.2.

$$s(L(m,n)) \leq \binom{m+n-2}{m-1} - 1.$$

Proof. Since $c(L_0) = |\{(i_1, \dots, i_{m-1}) : 0 \leq i_1 \leq \cdots \leq i_{m-1} \leq n-1\}|$,

$$c(L_0) = \binom{m+n-2}{m-1} \text{ and thus } s(L(m,n)) \leq \binom{m+n-2}{m-1} - 1.$$

It is easy to check $s(L(2,n)) = n-1$, that is, equality holds in Corollary 2.2. Consider $L(3,4)$. Let $C_1 = C(0,0)$, $C_2 = C(0,1)$, $C_3 = C(1,1)$, $C_4 = C(0,2)$, $C_5 = C(1,2)$, $C_6 = C(2,2)$, $C_7 = \{(i,3,3) : i = 0,1,2,3\}$, $C_8 = \{(i,3,4) : i = 0,1,2,3\}$, $C_9 = \{(i,4,4) : i = 0,1,2,3,4\}$. Let $\ell = \oplus_{i=1}^{9} C_i$. Then ℓ is a linear extension of $L(3,4)$. Thus $s(L(3,4)) \leq 8$, that is, equality does not hold in Corollary 2.2.

By (1), we get $\omega(L(m,n)) - 1 \leq s(L(m,n))$. But this is not a good lower bound. If m or n is, 1, then equality holds. On the other hand, $\omega(L(2,4)) = 3$ while $s(L(2,4)) = 3$.

PROPOSITION 2.3. *$L(m,n)$ is greedy if and only if m,n satisfy one of the followings: (1) $m = 1$ or $n = 1$, (2) $m \leq 2$ and $n \leq 2$.*

Proof. Suppose that m,n satisfy (1) or (2). Since $L(1,n)$ is a chain, $L(1,n)$ is greedy. Since $L(n,1)$ is isomorphic to $L(1,n)$, $L(n,1)$ is greedy. By direct calculation, $s(L(2,2)) = 1$. Note that every greedy linear extension of $L(2,2)$ has two chains. Thus $L(2,2)$ is greedy.

Now assume that either $m \geq 3$ and $n \geq 2$ or $m \geq 2$ and $n \geq 3$ holds. Let $C_1 = \{(0,0),(0,1),(1,1)\}$, $C_2 = \{(0,2),(0,3)\}$, $C_3 = \{(1,2),(1,3)\}$, $C_4 = \{(2,2),(2,3),(3,3)\}$. Let $L = C_1 \oplus C_2 \oplus C_3 \oplus C_4$. L is a greedy linear extension of $L(2,3)$. Since $s(L(2,3)) = 2$, L is not an optimal linear extension of $L(2,3)$. Hence $L(2,3)$ is not greedy. Since $L(3,2)$ is isomorphic to $L(2,3)$, $L(3,2)$ is not greedy.

$L(m,n)$ contains either $L(2,3)$ or $L(3,2)$. Without loss of generality, assume that $L(m,n)$ contains $L(2,3)$. We can construct a greedy linear extension of $L(m,n)$ which is not optimal. Let

$$C_1 = \{(0,\ldots,0),(0,\ldots,0,1),(0,\ldots,0,1,1)\},$$
$$C_2 = \{(0,\ldots,0,i) : 2 \leq i \leq n\},$$
$$C_3 = \{(0,\ldots,0,1,i) : 2 \leq i \leq n\},$$

$\ell_0 = \oplus C(i_1,\ldots,i_{m-1})$ over all $(i_1,\ldots,i_{m-1})$ satisfying $0 \leq i_1 \leq \cdots \leq i_{m-1} \leq n-1$ and $(i_1,\ldots,i_{m-1}) \neq (0,\ldots,0,\ell)$ for $\ell = 0,1$, and arranged lexicographically. Let $L = C_1 \oplus C_2 \oplus C_3 \oplus \ell_0$, then L is a greedy linear extension of $L(m,n)$. Now we get

$$c(L) = \binom{m+n-2}{m-1} + 1 > \binom{m+n-2}{m-1}.$$

By Corollary 2.2, L is not an optimal linear extension of $L(m,n)$. Hence $L(m,n)$ is not greedy.

3. The subspace lattice $L_p(n)$

Let p be a prime number and F_p be the Galois field of p elements. Let $L_p(n)$ be the lattice of subspaces of the n dimensional vector space $F_p^{(n)}$ over F_p ordered by inclusion. In general it is very difficult to find $s(L_p(n))$. In this section we study $s(L_p(n))$ for $n = 3, 4$. Let $L_p(n)^k$ be the set of all k dimensional subspaces of $F_p^{(n)}$, i.e., the k-th level of $L_p(n)$. It is well known [1] that the following properties hold in $L_p(n)$:

1. For $k \geq 1$, $|L_p(n)^k|$ equals the Gaussian coefficient $\binom{n}{k}_p$ where

$$\binom{n}{0}_p = 1, \ \binom{n}{k}_p = \frac{(p^n - 1)(p^{n-1} - 1)\cdots(p^{n-k+1} - 1)}{(p^k - 1)(p^{k-1} - 1)\cdots(p - 1)}.$$

2. $|L_p(n)| = G_{n,p}$ where $G_{n,p}$ is the Galois number, i.e.,

$$G_{n,p} = \sum_{k=0}^{n} \binom{n}{k}_p.$$

3. For $k = 0, 1, \ldots, n$,

$$\binom{n}{k}_p = \binom{n}{n-k}_p.$$

4. The Gaussian coefficient is unimodal, i.e.,

   (a) if n is even then

$$\binom{n}{0}_p < \binom{n}{1}_p < \cdots < \binom{n}{n/2}_p > \cdots > \binom{n}{n}_p,$$

   (b) if n is odd then

$$\binom{n}{0}_p < \cdots < \binom{n}{(n-1)/2}_p = \binom{n}{(n+1)/2}_p > \cdots > \binom{n}{n}_p.$$

Let $A \in L_p(n)$. Let A_g denote a basis of A. Let x be any element of $F_p^{(n)}$ such that $x \notin A$. Denote $\langle A_g \cup \{x\} \rangle$ to be the subspace generated by $A_g \cup \{x\}$.

LEMMA 3.1. *If $L \in \mathcal{L}(L_p(n))$, then every $(L_p(n), L)$-chain C has length at most one.*

Proof. Let $X = \sup C$ and $Y = \inf C$. We may choose bases X_g and Y_g so that $Y_g \subseteq X_g$. Suppose that $|C| \geq 3$, so that $|X_g \backslash Y_g| \geq 2$. Then there exist $y_1, y_2 \in X_g \backslash Y_g$ such that $\langle Y_g \cup \{y_1\}\rangle \in C$ and $y_2 \in X \backslash \langle Y_g \cup \{y_1\}\rangle$. This implies that $\langle Y_g \cup \{y_2\}\rangle \neq \langle Y_g \cup \{y_1\}\rangle$. But $\langle Y_g \cup \{y_2\}\rangle \notin C$. Hence $\langle Y_g \cup \{y_2\}\rangle > X$ or $\langle Y_g \cup \{y_2\}\rangle < Y$ in L, contradicting $Y \leq \langle Y_g \cup \{y_2\}\rangle \leq X$.

A $(k+1)$-dimensional subspace covers $\frac{p^{k+1}-1}{p-1}$ k-dimensional subspaces, while a k-dimensional subspace is covered by $\frac{p^{n-k}-1}{p-1}$ $(k+1)$-dimensional subspaces.

We may regard $L_p(n-1)$ as the sublattice of $L_p(n)$ obtained by considering only those n-tuples whose n-th coordinate is 0. No subspace of $L_p(n) \backslash L_p(n-1)$ is contained in a subspace of $L_p(n-1)$.

Let $e_1, \ldots, e_n$ be the standard basis of $F_p^{(n)}$. Let

$$M_p(k+1) = L_p(k+1) \backslash (L_p(k) \cup L_p(k)^*)$$

where $L_p(k)^* = \{\langle X \cup \{e_{k+1}\}\rangle : X \in L_p(k)\}$. Given a linear ordering $L = x_1, x_2, \ldots$ of $L_p(k)$, let $L^* = \overline{x_1}, \overline{x_2}, \ldots$ where $\overline{x_i} = \langle x_i \cup \{e_{k+1}\}\rangle$.

Define a *good extension* L_n of $L_p(n)$ inductively as follows:

1. Let $L_1 = C_1 \in \mathcal{O}(L_p(1))$ where $C_1 = \{\langle 0\rangle, \langle e_1\rangle\}$.
2. Given L_k, let $L_{k+1} = L_k \oplus \ell_{k+1} \oplus L_k^*$ where $\ell_{k+1} \in \mathcal{O}(M_p(k+1))$.

THEOREM 3.2. *Let L_n be a good extension of $L_p(n)$. Then $L_n \in \mathcal{L}(L_p(n))$.*

Proof. For any $X \in L_p(n-1)$, there doesn't exist $Y \in L_p(n) \backslash L_p(n-1)$ such that $X > Y$. Thus $\mathcal{L}(L_p(n))$ has an element of the form $L_{n-1} \oplus L$ where $L \in \mathcal{L}(L_p(n) \backslash L_p(n-1))$. Now consider $L_p(n) \backslash L_p(n-1)$. For any $Y \in L_p(n) \backslash (L_p(n-1) \cup L_p(n-1)^*)$, there does not exist $Z \in L_p(n-1)^*$ such that $Y > Z$. This implies that $\ell_n \oplus L_{n-1}^* \in \mathcal{L}(L_p(n) \backslash L_p(n-1))$ where $\ell_n \in \mathcal{O}(M_p(n))$. Hence $L_n = L_{n-1} \oplus \ell_n \oplus L_{n-1}^* \in \mathcal{L}(L_p(n))$.

CONJECTURE 3.3. *The good extension L_n of $L_p(n)$ is an optimal linear extension.*

By direct calculation, we can obtain $s(L_p(1)) = 0$ and $s(L_p(2)) = p$. That is, $L_1 = C_1 = \{\langle 0\rangle, \langle e_1\rangle\} \in \mathcal{O}(L_p(1))$ and $L_2 = L_1 \oplus \ell_2 \oplus L_1^* \in \mathcal{O}(L_p(2))$ where $\ell_2 = \oplus_{i=1}^{p-1}\{\langle e_1 + i \cdot e_2\rangle\}$.

Let $N(i, L)$ be the number of one-element chains of L which belong to $L_p(n)^i$.

PROPOSITION 3.4. *For $n \geq 2$, let $L \in \mathcal{L}(L_2(n))$. Then $N(1, L) \geq n - 1$.*

Proof. For any $L \in \mathcal{L}(L_2(n))$, we can rearrange the $(L_2(n), L)$-chains in L so that all one-element $(L_2(n), L)$-chains $\langle u_1\rangle, \ldots, \langle u_{N(1,L)}\rangle$ which belong to $L_2(n)^1$ come before all the two-element $(L_2(n), L)$-chains which begin with an element in $L_2(n)^1$. Since any two-dimensional subspace X covers 3 one-dimensional subspaces and $X\backslash\{0\}$ has 3 elements, the sum of two different elements in $X\backslash\{0\}$ is the other element in $X\backslash\{0\}$. Without loss of generality we may assume that the first $(L_2(n), L)$-chain in L is $\{\langle 0\rangle, \langle e_1\rangle\}$ since the first $(L_2(n), L)$-chain in L has two elements. Thus

$$|\{a_0 \cdot e_1 + a_1 \cdot u_1 + \cdots + a_{N(1,L)} \cdot u_{N(L,1)}\}\backslash\{0\}| \geq |L_2(n)^1|.$$

So $2^{N(1,L)+1} - 1 \geq 2^n - 1$. Hence we get $N(1, L) \geq n - 1$.

COROLLARY 3.5. *For $n \geq 2$, let $L \in \mathcal{L}(L_2(n))$. Then $N(n-1, L) \geq n - 1$.*

Proof. Consider $L_2(n)^d$.

APPLICATION 3.6. *$s(L_2(3))$ and $s(L_2(4))$.*

For the simplicity of notation, let $ij = e_i + e_j$ and $i, j = e_i, e_j$.

[1] We determine $s(L_2(3))$. Let L be any linear extension of $L_2(3)$. With $n = 3$ we get, by Proposition 3.4 and Corollary 3.5, $N(1, L) \geq 2$ and $N(2, L) \geq 2$. And thus L has at least 4 one-element $(L_2(3), L)$-chains. By Lemma 3.1, every $(L_2(3), L)$-chain which has more than one element has two elements. So L has at most $(G_{3,2} - 2 - 2)/2 = 6$ two-element $(L_2(3), L)$-chains. Thus $s(L_2(3)) \geq 9$. The good extension L_3 of $L_2(3)$, i.e., $L_3 = L_2 \oplus \ell_3 \oplus L_2^*$ where $\ell_3 = \{\langle 123\rangle\} \oplus \{\langle 13\rangle, \langle 2, 13\rangle\} \oplus \{\langle 23, \rangle\langle 1, 23\rangle\} \oplus \{\langle 13, 23\rangle\}$, has 10 chains. Hence $s(L_2(3)) = 9$.

[2] We determine $s(L_2(4))$. Let L be any linear extension of $L_2(4)$. With $n = 4$ we get, by Proposition 3.4 and Corollary 3.5, $N(1, L) \geq 3$ and $N(3, L) \geq 3$. So any L has at least 6 one-element chains in $L_2(4)^1$ and $L_2(4)^3$.

Suppose that there exist $L_0 \in \mathcal{L}(L_2(4))$ which has 6 one-element $(L_2(4), L)$-chains in $L_2(4)^1$ and $L_2(4)^3$. Then

$$N(2, L_0) = \binom{4}{2}_2 - 2(\binom{4}{1}_2 - 3 - 1) = 13.$$

By Lemma 3.1, every $(L_2(4), L_0)$-chain which has more than one element has two elements. So L_0 has $(G_{4,2} - 3 - 13 - 3)/2 = 24$ two-element $(L_2(4), L_0)$-chains. Thus $s(L_0, L_2(4)) = 3 + 13 + 3 + 24 - 1 = 42$. So $s(L_2(4)) \geq 42$. Let

$$\begin{aligned}
\ell_\alpha &= \{\langle 14\rangle, \langle 14, 23\rangle\} \oplus \{\langle 124\rangle, \langle 3, 124\rangle\} \oplus \{\langle 24\rangle, \langle 24, 13\rangle\} \\
&\oplus \{\langle 134\rangle, \langle 2, 134\rangle\} \oplus \{\langle 234\rangle, \langle 1, 234\rangle\} \oplus \{\langle 34\rangle, \langle 12, 34\rangle\}, \\
\ell_\beta &= \{\langle 1, 24\rangle\} \oplus \{\langle 1, 34\rangle\} \oplus \{\langle 14, 2\rangle\} \oplus \{\langle 14, 234\rangle\} \oplus \{\langle 23, 134\rangle\} \\
&\oplus \{\langle 13, 234\rangle\} \oplus \{\langle 24, 134\rangle\} \oplus \{\langle 34, 124\rangle\} \oplus \{\langle 134, 234\rangle\}, \\
\ell_\gamma &= \{\langle 2, 34\rangle, \langle 1, 2, 34\rangle\} \oplus \{\langle 3, 24\rangle, \langle 1, 3, 24\rangle\} \oplus \{\langle 3, 14\rangle, \langle 2, 3, 14\rangle\}, \\
\ell_\delta &= \{\langle 23, 24\rangle, \langle 1, 23, 24\rangle\} \oplus \{\langle 13, 14\rangle, \langle 2, 13, 14\rangle\} \oplus \\
&\{\langle 12, 14\rangle, \langle 3, 12, 14\rangle\}, \\
\ell_4 &= \{\langle 1234\rangle\} \oplus \ell_\alpha \oplus \ell_\beta \oplus \ell_\gamma \oplus \ell_\delta \oplus \{\langle 21, 23, 24\rangle\}.
\end{aligned}$$

Then $\ell_4 \in \mathcal{O}(M_2(4))$. Define $L_4 = L_3 \oplus \ell_4 \oplus L_3^*$. This L_4 is a good extension of $L_2(4)$ and has 43 $(L_2(4), L_4)$-chains. Hence $s(L_2(4)) = 42$.

Note that every element in $M_p(n)^k$ is a $(k+1)$-dimensional subspace in $M_p(n)$ and every element in $L_p(n-1)^{*k}$ is a $(k+1)$-dimensional subspace in $L_p(n-1)^*$. Note also that $L_p(n)^i = L_p(n-1)^i \cup M_p(n)^{i-1} \cup L_p(n-1)^{*i-1}$.

APPLICATION 3.7. *Upper Bounds of $s(L_p(3))$ and $s(L_p(4))$.*

[1] Consider $L_p(3)$.

Let

$$A = \{\langle e_1 + a_2 \cdot e_2 + a_3 \cdot e_3\rangle : 1 \le a_i \le p-1\},$$
$$B = \{\langle e_1 + a_3 \cdot e_3\rangle : 1 \le a_3 \le p-1\},$$
$$C = \{\langle e_2 + a_3 \cdot e_3\rangle : 1 \le a_3 \le p-1\},$$
$$M = M_p(3)^1 \backslash [\{\langle\{e_2\} \cup Y\rangle : Y \in B\} \cup \{\langle\{e_1\} \cup Z\rangle : Z \in C\}].$$

Define

$$\ell(A) = \oplus_{X\in A}\{X\},$$
$$\ell(B) = \oplus_{Y\in B}\{Y, \langle\{e_2\} \cup Y\rangle\},$$
$$\ell(C) = \oplus_{Z\in C}\{Z, \langle\{e_1\} \cup Z\rangle\},$$
$$\ell(M) = \oplus_{Z\in M}\{Z\},$$
$$\ell_3 = \ell(A) \oplus \ell(B) \oplus \ell(C) \oplus \ell(M).$$

Now define

(2) $$L_3^0 = L_2 \oplus \ell_3 \oplus L_2^*$$

where

(3) $L_2 = \{\langle 0\rangle, \langle e_1\rangle\} \oplus (\oplus\{\langle e_1 + a \cdot e_2\rangle\} : 1 \le a \le p-1) \oplus \{\langle e_2\rangle, \langle e_1, e_2\rangle\}$.

Note that $N(1, L_3^0) = N(2, L_3^0) = p^2 - p$. By the same technique as in [1] of Application 3.6, we get

$$c(L_3^0) = 2(p^2 - p) + \frac{G_{3,p} - 2(p^2 - p)}{2} = 2p^2 + 2.$$

Hence

(4) $$s(L_p(3)) \le 2p^2 + 1.$$

[2] Consider $L_p(4)$. Let

$$A = \{\langle e_1 + a_2 \cdot e_2 + a_3 \cdot e_3 + a_4 \cdot e_4\rangle : 1 \le a_i \le p-1\},$$
$$B_1 = \{\langle e_1 + a \cdot e_4, e_3\rangle : 1 \le a \le p-1\},$$
$$B_2 = \{\langle e_2 + a \cdot e_4, e_3\rangle : 1 \le a \le p-1\},$$
$$B_3 = \{\langle e_3 + a \cdot e_4, e_2\rangle : 1 \le a \le p-1\},$$
$$B_4 = \{\langle e_1 + a \cdot e_4, e_2 + b \cdot e_4\rangle : 1 \le a, b \le p-1\},$$
$$B_5 = \{\langle e_1 + a \cdot e_4, e_3 + b \cdot e_4\rangle : 1 \le a, b \le p-1\},$$
$$B_6 = \{\langle e_2 + a \cdot e_4, e_3 + b \cdot e_4\rangle : 1 \le a, b \le p-1\}$$
$$C = \{\langle e_1 + a \cdot e_4, e_2 + b \cdot e_4, e_3 + c \cdot e_4\rangle : 1 \le a, b, c \le p-1\},$$

For each $X \in A^c$ where $A^c = M_p(4)^0 \backslash A$, choose a $T_X \in L_p(3)^1$ such that $X + T_X \in A$. Define

$$\ell(A) = \oplus_{X_a \in A}\{X_a\},$$
$$\ell(A^c) = \oplus_{X \in A^c}\{X, \langle X, T_X\rangle\},$$
$$\ell(M) = (\oplus\{Y_0\} : Y_0 \in M_p(4)^1 \backslash \cup_{i=1}^6 B_i \backslash \{\langle X, T_X\rangle : X \in A^c\}),$$
$$\ell(B_1) = \oplus_{Y_1 \in B_1}\{Y_1, \langle\{e_2\} \cup Y_1\rangle\},$$
$$\ell(B_2) = \oplus_{Y_2 \in B_2}\{Y_2, \langle\{e_1\} \cup Y_2\rangle\},$$
$$\ell(B_3) = \oplus_{Y_3 \in B_3}\{Y_3, \langle\{e_1\} \cup Y_3\rangle\},$$
$$\ell(B_4) = \oplus_{Y_4 \in B_4}\{Y_4, \langle\{e_3\} \cup Y_4\rangle\},$$
$$\ell(B_5) = \oplus_{Y_5 \in B_5}\{Y_5, \langle\{e_2\} \cup Y_5\rangle\},$$
$$\ell(B_6) = \oplus_{Y_6 \in B_6}\{Y_6, \langle\{e_1\} \cup Y_6\rangle\},$$
$$\ell(C) = \oplus_{Z \in C}\{Z\},$$
$$\ell_4 = \ell(A) \oplus \ell(A^c) \oplus \ell(M) \oplus [\oplus_{i=1}^6 \ell(B_i)] \oplus \ell(C). \tag{5}$$

Then $c(\ell_4) = p^4 + 3p^3 - 6p^2 + 5p - 3$. Now define

$$L_4^0 = L_3^0 \oplus \ell_4 \oplus (L_3^0)^*$$

where L_3^0 is defined in (2). Note that $N(1, L_4^0) = N(3, L_4^0) = p^3 - 2p^2 + 2p - 1$. The L_4^0 has $p^4 + 3p^3 - 2p^2 + 5p + 1$ $(L_p(4), L_3^0)$-chains. Thus

$$s(L_p(4)) \le p^4 + 3p^3 - 2p^2 + 5p. \tag{6}$$

[3] In general, equality does not hold in (4) and (6). If $p = 2$, then we get equality in (4) and (6). Let L_2 be defined as in (3) and ℓ_4 as in (5). We construct $L_\alpha \in \mathcal{L}(L_3(3))$ such that $s(L_\alpha, L_3(3)) = 18$. Let

$$\ell_1^e = \{\langle e_1 + e_2 + e_3\rangle\} \oplus \{\langle e_1 + 2e_2 + 2e_3\rangle\} \oplus \{\langle e_1 + e_2 + 2e_3\rangle\},$$
$$\ell_2^e = \{\langle e_2 + e_3\rangle, \langle e_1, e_2 + e_3\rangle\} \oplus \{\langle e_1 + 2e_3\rangle, \langle e_2, e_1 + 2e_3\rangle\}$$
$$\oplus \{\langle e_1 + 2e_2 + e_3\rangle\}, \langle e_1 + e_2, e_2 + e_3\rangle\}$$
$$\oplus \{\langle e_2 + 2e_3\rangle, \langle e_1, e_2 + 2e_3\rangle\} \oplus \{\langle e_1 + e_3\rangle, \langle e_2, e_1 + e_3\rangle\},$$
$$\ell_3^e = \{\langle e_1 + e_2, e_2 + 2e_3\rangle\} \oplus \{\langle e_1 + 2e_2, e_2 + e_3\rangle\}$$
$$\oplus \{\langle e_1 + 2e_2, e_2 + 2e_3\rangle\}.$$

Let

$$L_\alpha = L_2 \oplus \ell_1^e \oplus \ell_2^e \oplus \ell_3^e \oplus L_2^*.$$

Then $s(L_\alpha, L_3(3)) = 18$. So $s(L_3(3)) \leq 18$. So equality does not hold in (4). Similiarly, we can construct a linear extension $L_\beta = L_\alpha \oplus \ell_4 \oplus L_\alpha^* \in \mathcal{L}(L_3(4))$. Since $s(L_\beta, L_3(4)) = 157$, equality does not hold in (6).

PROPOSITION 3.8. *$L_p(n)$ is greedy if and only if $n = 1, 2$. Furthermore, $L_p(n)$ is reversible, i.e., $\mathcal{G}(L_p(n)) = \mathcal{O}(L_p(n))$, if and only if $n = 1, 2$.*

Proof. $L_p(1)$ is obviously greedy. If $L \in \mathcal{G}(L_p(2))$, then only the first $(L_p(2), L)$-chain and the last $(L_p(2), L)$-chain have two elements, and the other $(L_p(2), L)$-chains have one element. Since $s(L, L_p(2)) = 1 + p - 1 + 1 - 1 = p$, $L \in \mathcal{O}(L_p(2))$.

Suppose $n \geq 3$. Let L be the linear extension of $L_p(3)$ defined by

$$L = L_2 \oplus \{\langle e_1 + e_3 \rangle\} \oplus (\oplus\{\langle e_1 + a \cdot e_2 + b \cdot e_3 \rangle\} : 1 \leq a, b \leq p - 1) \\ \oplus \ell_3^0 \oplus \{\langle e_2, e_1 + e_3 \rangle\} \oplus L_2^*$$

where L_2 is a good extension of $L_p(2)$ and

$$\ell_3^0 \in \mathcal{O}(M_p(3) \backslash \cup_{1 \leq a, b \leq p-1} \langle e_1 + a \cdot e_2 + b \cdot e_3 \rangle \backslash \langle e_1 + e_3 \rangle \backslash \langle e_2, e_1 + e_3 \rangle).$$

Then $L \in \mathcal{G}(L_p(3))$ but $L \notin \mathcal{O}(L_p(3))$. Thus $L_p(3)$ is not greedy. Since $L_p(n) \supset L_p(3)$, we can construct $\ell \in \mathcal{G}(L_p(n) \backslash L_p(3))$. Then $L \oplus \ell \in \mathcal{G}(L_p(n))$ where L is defined as above. But $L \oplus \ell \notin \mathcal{O}(L_p(n))$. This shows that $L_p(n)$ is not greedy for $n \geq 3$.

For $n = 1, 2$ we have $\mathcal{O}(L_p(n)) \subset \mathcal{G}(L_p(n))$. Thus from the above result we obtain that $\mathcal{G}(L_p(n)) = \mathcal{O}(L_p(n))$ if and only if $n = 1, 2$.

References

1. M. Aigner, *Combinatorial Theory*, Springer Verlag (1979).
2. M. Chein and M. Habib, *The jump number of dags and posets: an introduction*, Ann. Disc. Math. **9** (1980), 189–194.
3. O. Cogis, *Problem 4.6*, in *Ordered Sets* (I. Rival, ed.), D. Reidel, Dordrecht, 814 (1982).
4. ——— and M. Habib, *Nombre de sauts et graphes serie-paralleles*, RAIRO. Inform. Theor. **13** (1979), 3–18.

5. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. **51** (1950), 161–166.
6. D. Duffus, I. Rival, and P. Winkler, *Minimizing setups for cycle-free ordered sets*, Proc. Amer. Math. Soc. **85** (1982), 509–513.
7. M. H. ElZahar and I. Rival, *Greedy linear extensions to minimize jumps*, Disc. Applied Math. **11** (1985), 143–156.
8. Sneh Gulati, *A Study of Sperner theory and its application of $L(m,n)$-The Young's lattice*, master's thesis, Dept. of Math., Univ. of South Carolina (1987).
9. Bernt Lindstrom, *A Partition of $L(3,n)$ into Saturated Symmetric Chains*, Europ. J., Combinatorics **1** (1980), 61–63.
10. I. Rival, *Optimal linear extensions by interchanging chains*, Proc. Amer. Math. Soc. **89** (1983), 387–394.
11. ——— and N. Zaguia, *Constructing greedy linear extensions by interchanging chains*, Order **3** (1986), 107–121.
12. M. M. Syslo, *Remarks on Dilworth partially ordered sets*, in Proceedings of the WG'85 (H. Noltemeir, ed.), Trauner Verlag, Linz, (1985), 355–362.
13. E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fund. Math. **16** (1930), 386–389.

Department of Mathematics Education
KonKuk University
Seoul 133–701, Korea