J. Korean Math. Soc. **29** (1992), No. 2, pp. 409–422

A RELATIVE MOD K NIELSEN NUMBER

MOO HA WOO AND HYANG KAM CHO

Introduction.

The algebraic aspect of the Nielsen theory deals with the problem of computation for Nielsen number. By many mathematicians, this problem has been solved. B. J. Jiang ([5]) introduced the mod K Nielsen number $N_K(\overline{f})$ of a selfmap $\overline{f} : A \to A$ of a compact polyhedron A, where K is a normal subgroup of the fundamental group $\pi_1(A)$ such that $\overline{f}_\pi(K) \subset K$ for the induced homomorphism $\overline{f}_\pi : \pi_1(A) \to \pi_1(A)$. H. Schirmer ([7]) also introduced the relative Nielsen number $N(f; X, A)$ for a selfmap $f : (X, A) \to (X, A)$ of a compact polyhedral pair.

The purpose of this paper is the introduction of the relative mod K (= kernel of the induced homomorphism $i_\pi : \pi_1(A) \to \pi_1(X)$ of the inclusion map $i : A \to X$) Nielsen number $N_K(f; X, A)$ for the selfmap $f : (X, A) \to (X, A)$ of a compact polyhedral pair using the above two Nielsen numbers. The definition is in §1. In §2, we show that the relative mod K Nielsen number $N_K(f; X, A)$ has basic properties like the relative Nielsen number $N(f; X, A)$.

Now let $q : E \to B$ be a fibration in which E, B and all fibres are compact connected ANR's and let $f : E \to E$ be a fibre preserving map inducing selfmaps f_B on B and f_b on the fibre F_b over some fixed point b in the base. Jiang ([5]) and C. Y. You ([10]) introduced the product formula $N(f) = N(f_B) \cdot N(f_b)$ for the fibration and showed the conditions under which the formula holds. In §3, we study the new product formula and show the conditions that the new formula holds.

Received September 27, 1991.
Partially supported by TGRC-KOSEF Grant.

1. Definitions.

In this section, we introduce the definition of the relative mod K Nielsen number. If $f : (X, A) \to (X, A)$ is a selfmap of a pair of compact polyhedra, then we shall write $\overline{f} : A \to A$ for the restriction of the pair map f to A and write $f : X \to X$ if the condition that $f(A) \subset A$ is immaterial. The homotopies of $f : (X, A) \to (X, A)$ are maps of the form $H : (X \times I, A \times I) \to (X, A)$ and homotopies of $f : X \to X$ are maps of the form $H : X \times I \to X$, where I is the unit interval.

Now let $i : A \to X$ be the inclusion map. Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\overline{f}} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & X. \end{array}$$

Let $i_\pi : \pi_1(A) \to \pi_1(X)$ and $\overline{f}_\pi : \pi_1(A) \to \pi_1(A)$ be the induced homomorphisms. Let K be the kernel of $i_\pi : \pi_1(A) \to \pi_1(X)$. Then $\overline{f}_\pi(K) \subset K$ and $i_{FPC} : FPC_K(\overline{f}) \to FPC(f)$ induced by $i_{FPC} : FPC(\overline{f}) \to FPC(f)$ is well-defined. (See [5].) Therefore every mod K fixed point class of $\overline{f} : A \to A$ is contained in a fixed point class of $f : X \to X$.

Throughout this paper, we always assume that K is the kernel of the induced homomorphism $i_\pi : \pi_1(A) \to \pi_1(X)$.

DEFINITION 1.1. Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra. A fixed point class F of $f : X \to X$ is called a *common mod K fixed point class of f and $\overline{f}$* if F contains an essential mod K fixed point class of $\overline{f} : A \to A$.

It is called an *essential common mod K fixed point class of f and $\overline{f}$* if it is an essential fixed point class of $f : X \to X$ and a common mod K fixed point class of f and $\overline{f}$.

For a pair map $f : (X, A) \to (X, A)$ of compact polyhedra, Schirmer defined a common fixed point class of f and $\overline{f}$ which is a fixed point class of $f : X \to X$ containing an essential fixed point class of $\overline{f} : A \to A$. The number of essential common fixed point classes of f and $\overline{f}$ is denoted by $N(f; \overline{f})$. (See [7].)

DEFINITION 1.2. The number of essential common mod K fixed point classes of f and $\overline{f}$ is denoted by $N_K(f;\overline{f})$.

LEMMA 1.3. *If* $f:(X,A)\to(X,A)$ *is a pair map of compact polyhedra, then a common mod* K *fixed point class of* f *and* $\overline{f}$ *is a common fixed point class of* f *and* $\overline{f}$.

Proof. Let F be a common mod K fixed point class of f and $\overline{f}$. Then F contains an essential mod K fixed point class $\overline{F}_K$ of $\overline{f}:A\to A$. But $\overline{F}_K$ is a disjoint union of ordinary fixed point classes of $\overline{f}$ and index $(\overline{f},\overline{F}_K)=\sum_i$ index $(\overline{f},\overline{F}_i)$, where $\overline{F}_K=\cup_i\overline{F}_i$ and $\overline{F}_i$ is an ordinary fixed point class of $\overline{f}$ for each i. Since $\overline{F}_K$ is essential, there exists at least one essential fixed point class of $\overline{f}$. If $\overline{F}_i$ is such an essential fixed point class of $\overline{f}$, then $\overline{F}_i\subset\overline{F}_K\subset F$. So F is a common fixed point class of f and $\overline{f}$.

Since (X,A) is a compact polyhedral pair and $0\le N_K(f;\overline{f})\le N(f;\overline{f})$, $N_K(f;\overline{f})$ is a finite integer.

The following lemmas give a condition to be $N_K(f;\overline{f})=N(f;\overline{f})$.

LEMMA 1.4. *Let* $f:(X,A)\to(X,A)$ *be a pair map of compact polyhedra with* $\overline{f}_\pi(K)\subset J(\overline{f})$, *where* $J(\overline{f})$ *is the Jiang subgroup of* $\overline{f}:A\to A$. *Then a common mod* K *fixed point class of* f *and* $\overline{f}$ *coincides with a common fixed point class of* f *and* $\overline{f}$.

Proof. Since $\overline{f}_\pi(K)\subset J(\overline{f})$, any two ordinary fixed point classes of $\overline{f}$ in a given mod K fixed point class of $\overline{f}$ have the same index. (See [5].) It suffices to show that a common fixed point class of f and $\overline{f}$ is a common mod K fixed point class of f and $\overline{f}$. Let F be a common fixed point class of f and $\overline{f}$. Then there exists an essential fixed point class $\overline{F}$ of $\overline{f}$ such that $\overline{F}\subset F$. Since $\overline{F}$ is contained in a mod K fixed point class $\overline{F}_K$ of $\overline{f}$, $\overline{F}_K$ is essential. Since $\overline{F}_K\subset F$, F is a common mod K fixed point class of f and $\overline{f}$.

LEMMA 1.5. *Let* $f:(X,A)\to(X,A)$ *be a pair map of compact polyhedra. Suppose every ordinary fixed point class in a given inessential mod* K *fixed point class of* $\overline{f}:A\to A$ *is inessential. Then a common mod* K *fixed point class of* f *and* $\overline{f}$ *coincides with a common fixed point class of* f *and* $\overline{f}$.

Proof. Let F be a common fixed point class of f and $\overline{f}$. Then F contains an essential fixed point class $\overline{F}$ of $\overline{f}$. By hypothesis, $\overline{F}$ is contained in an essential mod K fixed point class $\overline{F}_K$ of $\overline{f}$. Since $\overline{F}_K \subset F$, F is a common mod K fixed point class of f and $\overline{f}$.

In [7], Schirmer defined the relative Nielsen number $N(f;X,A)$ to be $N(f)+N(\overline{f})-N(f;\overline{f})$ for a compact polyhedral pair map $f:(X,A)\to(X,A)$.

Now we will give the following definition.

DEFINITION 1.6. Let $f:(X,A)\to(X,A)$ be a pair map of compact polyhedra. The *relative mod K Nielsen number* $N_K(f;X,A)$ is defined by

$$N_K(f;X,A)=N(f)+N_K(\overline{f})-N_K(f;\overline{f}),$$

where $N_K(\overline{f})$ is the number of essential mod K fixed point classes of $\overline{f}:A\to A$.

Hence $N_K(f;X,A)$ is a finite integer ≥ 0.

If $X=A$, then $N_K(f;X,A)=N(f)=N(f;X,A)$. If K is the trivial group, then $N_K(f;X,A)=N(f;X,A)$.

THEOREM 1.7. *Let $f:(X,A)\to(X,A)$ be a pair map of compact polyhedra. Then we have $N_K(f;X,A)\leq N(f;X,A)$.*

Proof. Let $F_1,F_2,\ldots,F_k,\ F_{k+1},\ldots,F_m,F_{m+1},\ldots,F_n$ be essential fixed point classes of $f:X\to X$, where $0<k\leq m\leq n$ are positive integers. Of these, let $F_1,F_2,\ldots,F_k,F_{k+1},\ldots,F_m$ be common fixed point classes of f and $\overline{f}$ and let $F_1,F_2,\ldots,F_k$ be common mod K fixed point classes of f and $\overline{f}$. For each $i=1,\ldots,m$, let c_i be the number of essential fixed point classes of $\overline{f}:A\to A$ which are contained in F_i and let d be the number of essential fixed point classes of $\overline{f}:A\to A$ which are contained in inessential fixed point classes of $f:X\to X$.

Then we have $N(\overline{f})=c_1+c_2+\cdots+c_k+c_{k+1}+\cdots+c_m+d$ and $N_K(\overline{f})\leq c_1+c_2+\cdots+c_k+d$. Thus since $c_j\geq 1$ for $k+1\leq j\leq m$, $N(\overline{f})\geq N_K(\overline{f})+(m-k)$ and hence

$$N(\overline{f})-N(f;\overline{f})\geq N_K(\overline{f})-N_K(f;\overline{f}).$$

We have the conclusion.

THEOREM 1.8. *If* $K \subset \cup_n Ker\overline{f}^n_\pi$ *, then we have* $N_K(f;X,A) = N(f;X,A)$.

Proof. If $K \subset \cup_n \mathrm{Ker}\overline{f}^n_\pi$, then a mod K fixed point class of $\overline{f}$ coincides with an ordinary one. Then $N_K(\overline{f}) = N(\overline{f})$ and hence $N_K(f;\overline{f}) = N(f;\overline{f})$.

THEOREM 1.9. *Let* $f:(X,A) \to (X,A)$ *be a pair map of compact polyhedra. Then we obtain*

(i) $N(f) \leq N_K(f;X,A)$ *and* $N_K(\overline{f}) \leq N_K(f;X,A)$,

(ii) $N_K(f;X,A) = \begin{cases} N_K(\overline{f}) & \textit{if } N(f)=0 \\ N(f) & \textit{if } N_K(\overline{f})=0. \end{cases}$

Proof. By the definition of $N_K(f;\overline{f})$, $N_K(f;\overline{f}) \leq N_K(\overline{f})$ implies

$$\begin{aligned} N_K(f;X,A) &= N(f) + [N_K(\overline{f}) - N_K(f;\overline{f})] \\ &\geq N(f). \end{aligned}$$

From $N_K(f;\overline{f}) \leq N(f)$, we have

$$\begin{aligned} N_K(f;X,A) &= N_K(\overline{f}) + [N(f) - N_K(f;\overline{f})] \\ &\geq N_K(\overline{f}). \end{aligned}$$

If $N(f) = 0$ or $N_K(\overline{f}) = 0$, then $N_K(f;\overline{f}) = 0$ and hence (ii) holds.

If we consider the following example, then we find that $N(f;X,A)$ is strictly larger than $N_K(f;X,A)$.

EXAMPLE 1.10. Let $C = \{(x,y) \in R^2 : (x-2)^2 + y^2 = 1\}$ and $D = \{(x,y) \in R^2 : x^2+y^2 \leq 1\}$. Let X be their union $X = C \vee D$ with a point $(1,0)$ in common and A be the figure eight as the boundary of X. Let $f:(X,A) \to (X,A)$ be the self map satisfying $f((x,y) = ((x,y)-(2,0))^3 + (2,0)$ if $(x,y) \in C$ and $f((x,y)) = (x,-y)$ if $(x,y) \in Bd(D)$. Then $\Phi(f) \cap A = \{(-1,0),(1,0),(3,0)\}$, where $\Phi(f)$ is the set of fixed points of $f : X \to X$. Since $\pi_1(X)$ is isomorphic to the integer group Z, K is not the trivial group. Then $N(f) = N(f;\overline{f}) = 2$, $N(\overline{f}) = 3$ and $N_K(\overline{f}) = N_K(f;\overline{f}) = 1$. Thus $2 = N_K(f;X,A) < N(f;X,A) = 3$.

But if we consider the pair map $f:(X,C) \to (X,C)$, then clearly the kernel of the induced homomorphism $j_\pi : \pi_1(C) \to \pi_1(X)$ by the inclusion map $j : C \to X$ is the trivial group and hence $N_{\{1\}}(f; X, C) = N(f; X, C)$.

2. Basic properties.

In this section, we show that every result about the relative Nielsen number $N(f; X, A)$ applies to the relative mod K Nielsen number $N_K(f; X, A)$.

THEOREM 2.1. *Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra and A be path connected. If either X is simply connected or if X is connected and $f : (X, A) \to (X, A)$ is homotopic to the identity map $id : (X, A) \to (X, A)$, then*

$$N_K(f; X, A) = \begin{cases} 0 & \text{if } N_K(\overline{f}) = 0 \text{ and } N(f) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Under the above condition, $R(f) = 1 = R_K(\overline{f})$. If $N(f) = 0$ and $N_K(\overline{f}) = 0$, then $N_K(f; \overline{f}) = 0$ and hence $N_K(f; X, A) = 0$. Now consider the following three cases.

(i) $N(f) \neq 0$ and $N_K(\overline{f}) = 0$.

Then $f : X \to X$ has only one essential fixed point class and $N_K(f; \overline{f}) = 0$. Thus we have $N_K(f; X, A) = 1$.

(ii) $N(f) = 0$ and $N_K(\overline{f}) \neq 0$.

Then $\overline{f} : A \to A$ has only one essential mod K fixed point class and $N_K(f; \overline{f}) = 0$. Thus we have $N_K(f; X, A) = 1$.

(iii) $N(f) \neq 0$ and $N_K(\overline{f}) \neq 0$.

Then we have $N(f) = N_K(\overline{f}) = N_K(f; \overline{f}) = 1$ and hence we have $N_K(f; X, A) = 1$.

EXAMPLE 2.2. Let X be the unit disk and A be its boundary circle. Let $f : (X, A) \to (X, A)$ be the selfmap satisfying $f(re^{i\theta}) = re^{3\theta i}$ if $re^{i\theta} \in A$. Then we have $N(f) = 1$. Since X is simply connected, $N_K(f; X, A) = 1$ by Theorem 2.1. Furthermore we have $N(f; X, A) = 2$ and hence we have $N_K(f; X, A) < N(f; X, A)$.

THEOREM 2.3. (Homotopy invariance.) *Let (X, A) be a pair of compact polyhedra. If the maps $f_0, f_1 : (X, A) \to (X, A)$ are homotopic, then $N_K(f_0; X, A) = N_K(f_1; X, A)$.*

Proof. It suffices to show that $N_K(f_0; \bar{f}_0) = N_K(f_1; \bar{f}_1)$. Let $H = \{h_t, \bar{h}_t\} : (X \times I, A \times I) \to (X, A)$ be a homotopy from f_0 to f_1. Then there exists an index preserving bijection $\{h_t\} : FPC(f_0) \to FPC(f_1)$. We show that $\{h_t\}$ sends common mod K fixed point classes of f_0 and $\bar{f}_0$ to common mod K fixed point classes of f_1 and $\bar{f}_1$. Let $F_0 = p\text{Fix}(\tilde{f}_0)$ be a common mod K lifting class of f_0 and $\bar{f}_0$. Then there exists a mod K lifting class $[\tilde{\bar{f}}_{0,K}]$ of $\bar{f}_0$ such that $i_{FPC}[\tilde{\bar{f}}_{0,K}] = [\tilde{f}_0]$. Let $\{\bar{h}_t\}$ send $[\tilde{\bar{f}}_{0,K}]$ to $[\tilde{\bar{f}}_{1,K}]$. Then we have a commutative diagram by [5]

$$\begin{array}{ccc} [\tilde{\bar{f}}_{0,K}] & \stackrel{\{\bar{h}_t\}}{\longmapsto} & [\tilde{\bar{f}}_{1,K}] \\ {\scriptstyle i_{FPC}}\downarrow & & \downarrow{\scriptstyle i_{FPC}} \\ [\tilde{f}_0] & \stackrel{\{h_t\}}{\longmapsto} & [\tilde{f}_1]. \end{array}$$

Therefore $\{h_t\}$ sends $[\tilde{f}_0]$ to $[\tilde{f}_1] = i_{FPC}[\tilde{\bar{f}}_{1,K}]$. So we have the conclusion.

THEOREM 2.4. (Commutativity.) *Let (X, A) and (Y, B) be pairs of compact polyhedra. Let $i : A \to X$ and $j : B \to Y$ be the inclusions, $i_\pi : \pi_1(A) \to \pi_1(X)$, and $j_\pi : \pi_1(B) \to \pi_1(Y)$ be the induced homomorphisms. Let K be the kernel of i_π and K' be the kernel of j_π. If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ are maps, then $N_K(g \circ f; \bar{g} \circ \bar{f}) = N_{K'}(f \circ g; \bar{f} \circ \bar{g})$ and $N_K(g \circ f; X, A) = N_{K'}(f \circ g; Y, B)$.*

Proof. In fact, $\bar{f}_\pi(K) \subset K'$ and $\bar{g}_\pi(K') \subset K$. Now let F be an essential common mod K fixed point class of $g \circ f$ and $\bar{g} \circ \bar{f}$. Then $f(F)$ is an essential fixed point class of $f \circ g$ and contains an essential mod K' fixed point class of $\bar{f} \circ \bar{g}$. Since $FPC(g \circ f) \rightleftarrows FPC(f \circ g)$ is a pair of homeomorphism and they respect fixed point classes, $N_K(g \circ f; \bar{g} \circ \bar{f}) = N_{K'}(f \circ g; \bar{f} \circ \bar{g})$. Therefore we have $N_K(g \circ f; X, A) = N_{K'}(f \circ g; Y, B)$ by [5].

Two maps of pairs of spaces $f : (X, A) \to (X, A)$ and $g : (Y, B) \to (Y, B)$ are said to be *maps of the same homotopy type* if there exists a homotopy equivalence $h : (X, A) \to (Y, B)$ so that the maps of spaces $h \circ f$, $g \circ h : (X, A) \to (Y, B)$ are homotopic.

THEOREM 2.5. (Homotopy type invariance.) *Let (X, A) and (Y, B) be two pairs of compact polyhedra. Let $i : A \to X$ and $j : B \to Y$ be the inclusions, $i_\pi : \pi_1(A) \to \pi_1(X)$, and $j_\pi : \pi_1(B) \to \pi_1(Y)$ be the induced homomorphisms. Let K be the kernel of i_π and K' be the kernel of j_π. If $f : (X, A) \to (X, A)$ and $g : (Y, B) \to (Y, B)$ are maps of the same homotopy type, then $N_K(f; X, A) = N_{K'}(g; Y, B)$.*

Proof. Using Theorem 2.3 and Theorem 2.4.

3. Applications.

In this section, we give a product formula relating Nielsen numbers of the fibre maps. There have been several improvements of the formula since Brown published a product formula in 1967. Let $q : E \to B$ be a fibration in which E, B and all fibres are compact connected ANR's and let $f : E \to E$ be a fibre preserving map inducing selfmaps f_B on B and f_b on the fibre F_b over some fixed point $b \in B$. Then, for some $b \in \Phi(f_B) = \{b \in B : f_B(b) = b\}$, a fibre preserving map $f : E \to E$ can induce the pair map $f : (E, F_b) \to (E, F_b)$ which has the restriction $f_b : F_b \to F_b$ of the pair map f to the fibre F_b. Let K be the kernel of the induced homomorphism $i_\pi : \pi_1(F_b) \to \pi_1(E)$ by the inclusion map $i : F_b \to E$. For a pair map $f : (E, F_b) \to (E, F_b)$ inducing $f_b : F_b \to F_b$, we have the relative Nielsen number $N(f; E, F_b)$ and the relative mod K Nielsen number $N_K(f; E, F_b)$. By Theorem 1.7, we knew that $N_K(f; E, F_b) \leq N(f; E, F_b)$. In the following theorem, we show that $N(f; f_b) = N_K(f; f_b)$ for the fibration.

THEOREM 3.1. *Let $f : (E, F_b) \to (E, F_b)$ be a pair map inducing $f_b : F_b \to F_b$. Then we have $N_K(f; f_b) = N(f; f_b)$.*

Proof. Since $K \subset J(F_b)$ which is a Jiang subgroup by [5], we have the conclusion by Lemma 1.4.

Jiang ([5]) and You ([10]) introduced the product formula $N(f) = N(f_B) \cdot N(f_b)$ of the Nielsen number of a fibre map. They found the conditions that the product formula $N(f) = N(f_B) \cdot N(f_b)$ holds. Now we consider new product formula of Nielsen numbers

$$\text{(P1)} \qquad \begin{aligned} N(f) &= N(f_B) \cdot N_K(f; f_b) \\ &= N(f_B) \cdot N(f; f_b). \end{aligned}$$

It does not always hold and we discuss the conditions which imply (P1).

THEOREM 3.2. (P1) *holds if one of the following conditions is satisfied:*

(i) $N(f_B) \leq 1$
(ii) $N(f_b) \leq 1$ *for any* $b \in \overline{F}_i$, $i = 1, \ldots, N(f_B)$.

Proof. According to [10, Theorem 4.1], $N(f_B) = 0$ implies $N(f) = 0$. If $N(f_B) = 1$, then $N(f) = N(f; f_b)$. If $N(f_b) = 0$, then $N(f; f_b) = 0$ and $N(f) = 0$ by [10, Theorem 4.1]. Finally $N(f_b) = 1$ implies $N(f; f_b) = 1$ and $N(f) = N(f_B)$ for any $b \in \overline{F}_i$, $i = 1, \ldots, N(f_B)$. Thus (P1) holds.

COROLLARY 3.3. *Suppose that* B *or* F_b *is simply connected for any* $b \in \overline{F}_i$, $i = 1, \ldots, N(f_B)$. *Then* (P1) *holds.*

THEOREM 3.4. *Let* $f : (E, F_b) \to (E, F_b)$ *be a pair map inducing* $f_b : F_b \to F_b$ *and let* $\overline{F}_1, \ldots, \overline{F}_n$ *be essential fixed point classes of* f_B, *where* $n = N(f_B)$. *Let* $F_{i_1}, F_{i_2}, \ldots, F_{i_{c_i}}$ *be essential fixed point classes of* f *such that* $q(F_{i_j}) \subset \overline{F}_i$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq c_i$. *If* $c_i = c$ *is constant for all* i, *then* (P1) *holds.*

Proof. $N(f; f_b) = c$. Since $N(f) = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} c = c \cdot n = N(f; f_b) \cdot N(f_B)$, (P1) holds.

Now we consider another point of view of (P1). It is due to P. R. Heath ([4]). Consider a group G (not necessarily abelian) and a homomorphism $\phi : G \to G$.

DEFINITION 3.5. ([4]) The Reidemeister operation of ϕ on G is the left action of G on itself given by $(g_1, g_2) \mapsto g_1 + g_2 - \phi(g_1)$. Let $(1-\phi) : G \to G$ denote the function defined by $(1-\phi)(g) = g - \phi(g)$; then by a slight abuse, we write the set of orbits of the operation as $\mathrm{Coker}(1-\phi)$ with elements $[g]$ for $g \in G$. We observe that if $j : G \to \mathrm{Coker}(1-\phi)$

has $j(g) = [g]$, then $j(g_1) = j(g_2)$ iff there exists a $g \in G$ such that $g_1 = g + g_2 - \phi(g)$.

We know that since $(1-\phi)$ need not be a homomorphism, Coker$(1-\phi)$ need not be the quotient of G by a subgroup. The order $\#(\text{Coker}(1-\phi))$ of the orbit set is called the *Reidemeister number* of ϕ and is written by $R(\phi)$.

Let $f : (E, F_b) \to (E, F_b)$ be a pair map inducing $f_b : F_b \to F_b$. From now on, we choose $x \in \Phi(f)$ in E and $b = q(x)$ in B as base points. Heath ([4]) introduced the following definitions.

DEFINITION 3.6. Consider the homomorphism $f^x_{b_\pi/K} : \pi_1(F_b, x)/K \to \pi_1(F_b, x)/K$, where $f^x_{b_\pi/K}$ is given by $f^x_{b_\pi/K}(K + \langle\alpha\rangle) = K + \langle f_b(\alpha)\rangle$ for $\langle\alpha\rangle \in \pi_1(F_b, x)$. The K-Reidemeister number of f_b, written $R_K(f_b)$, is the Reidemeister number of $f^x_{b_\pi/K}$ i.e., the order of the orbit set Coker$(1 - f^x_{b_\pi/K})$. $R_K(f_b)$ is well–defined. Similarly we can define the orbit sets Coker$(1 - f^x_\pi)$ and Coker$(1 - f^{q(x)}_{B_\pi})$ and we have their orders $R(f)$ and $R(f_B)$, respectively, where $f^x_\pi : \pi_1(E, x) \to \pi_1(E, x)$ and $f^{q(x)}_{B_\pi} : \pi_1(B, q(x)) \to \pi_1(B, q(x))$ are the induced homomorphisms.

DEFINITION 3.7. Let $x' \in F$ in $FPC(f)$. Define $\rho : FPC(f) \to$ Coker$(1 - f^x_\pi)$ by $\rho(F) = [\langle c - f(c)\rangle]$, where $c : x \mapsto x'$ is a path. Then the relation ρ is an injective function. Similarly we can define an injective function $\rho_K : FPC_K(f_b) \to$ Coker$(1 - f^x_{b_\pi/K})$ which is given by $\rho_K(\overline{F}) = [K + \langle c - f_b(c)\rangle]$, where $c : x \mapsto x'$ is a path for $x' \in \overline{F}$ in $FPC_K(f_b)$.

REMARK 3.8. For any path $u : x \mapsto x' \in \Phi(f)$, there exists a bijection $u^f_* :$ Coker$(1 - f^x_\pi) \to$ Coker$(1 - f^{x'}_\pi)$ given by $u^f_*([\langle\alpha\rangle]) = [\langle -u + \alpha + f(u)\rangle]$. (See [4].) Similarly there exists a bijection $\overline{u}^{f_B}_* :$ Coker$(1 - f^b_{B_\pi}) \to$ Coker$(1 - f^{b'}_{B_\pi})$ given by $\overline{u}^{f_B}_*([\langle\alpha\rangle]) = [\langle -\overline{u} + \alpha + f_B(\overline{u})\rangle]$, where $\overline{u} : b \mapsto b' \in \Phi(f_B)$ is a path.

From the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f_B} & B \end{array}$$

we write q_π^x for the induced function $\mathrm{Coker}(1-f_\pi^x) \to \mathrm{Coker}(1-f_{B_\pi}^{q(x)})$ which is given by $q_\pi^x[\langle\alpha\rangle] = [\langle q(\alpha)\rangle]$.

REMARK 3.9. If $x' \in \Phi(f)$, then for paths $u : x \mapsto x'$ and $q \circ u = \overline{u} : b \mapsto b'$, there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Coker}(1-f_\pi^x) & \xrightarrow{q_\pi^x} & \mathrm{Coker}(1-f_{B_\pi}^b) \\ {\scriptstyle u_*^f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{u}_*^{f_B}} \\ \mathrm{Coker}(1-f_\pi^{x'}) & \xrightarrow{q_\pi^{x'}} & \mathrm{Coker}(1-f_{B_\pi}^{b'}). \end{array}$$

The diagram is useful in that it induces a bijection (u_*^f) : $(q_\pi^x)^{-1}(q_\pi^x[\langle u - f(u)\rangle]) \cong \mathrm{Ker}\, q_\pi^{x'}$. (See [4].)

DEFINITION 3.10. ([4]) Define the index of element $[\langle\alpha\rangle]$ in $\mathrm{Coker}(1-f_\pi^x)$ as follows;

$$\mathrm{index}([\langle\alpha\rangle]) = \begin{cases} \mathrm{index}(f,F) & \text{if } \rho(F) = [\langle\alpha\rangle],\ F \in FPC(f), \\ 0 & \text{otherwise.} \end{cases}$$

Let $E(f) = \{[\langle\alpha\rangle] \in \mathrm{Coker}(1-f_\pi^x) : \mathrm{index}([\langle\alpha\rangle]) \neq 0\}$.

Since $N(f)$ is finite, $\#E(f)$ is finite.

LEMMA 3.11. ([4]) *The functions ρ and u_*^f are index – preserving.*

In [3], since we have an exact sequence

$$0 \to \pi_1(F_b, x)/K \xrightarrow{i_\pi^x} \pi_1(E,x) \xrightarrow{q_\pi^x} \pi_1(B, q(x)) \to 0,$$

there exists an exact sequence

$$\begin{aligned} 0 \to \mathrm{Fix}\, f_{b_\pi/K}^x \xrightarrow{i_\pi^x} \mathrm{Fix}\, f_\pi^x \xrightarrow{q_\pi^x} \mathrm{Fix}\, f_{B_\pi}^b \xrightarrow{\delta} \mathrm{Coker}(1-f_{b_\pi/K}^x) \\ \xrightarrow{i_\pi^x} \mathrm{Coker}(1-f_\pi^x) \xrightarrow{q_\pi^x} \mathrm{Coker}(1-f_{B_\pi}^b) \to 0 \end{aligned}$$

in which δ is given by $\delta(\langle\alpha\rangle) = [\langle\lambda - f(\lambda)\rangle]$, where $q_\pi^x(\langle\lambda\rangle) = \langle\alpha\rangle$.

LEMMA 3.12. ([4]) *If* $[\langle\alpha\rangle] = i^x_\pi([K+\langle\alpha\rangle]) = i^x_\pi([K+\langle\theta\rangle]) = [\langle\theta\rangle]$, *then* $index([K+\langle\alpha\rangle]) = index([K+\langle\theta\rangle])$ *for* $[K+\langle\alpha\rangle], [K+\langle\theta\rangle] \in Coker(1-f^x_{b_\pi/K})$.

LEMMA 3.13. ([10]) *If* $[K+\langle\mu\rangle] \in Coker(1-f^x_{b_\pi/K})$, $[\langle\mu\rangle] = i_\pi([K+\langle\mu\rangle]) \in Coker(1-f^x_\pi)$ *and* $[\langle\overline{\mu}\rangle] = q^x_\pi([\langle\mu\rangle]) \in Coker(1-f^b_{B_\pi})$, *then* $index([\langle\mu\rangle]) \neq 0$ *iff* $index([K+\langle\mu\rangle]) \neq 0$ *and* $index([\langle\overline{\mu}\rangle]) \neq 0$.

THEOREM 3.14. *If the diagram (D)*

$$\begin{array}{ccc} FPC_K(f_b) & \xrightarrow{i_{FPC}} & FPC(f) \\ \rho_K \big\downarrow & & \big\downarrow \rho \\ Coker(1-f^x_{b_\pi/K}) & \xrightarrow{i^x_\pi} & Coker(1-f^x_\pi) \end{array}$$

commutes, then for each $[\langle\mu\rangle] \in Coker(1-f^x_\pi)$, $[\langle\mu\rangle] \in Ker\, q^x_\pi \cap E(f)$ *iff* $[\langle\mu\rangle]$ *corresponds to an essential common mod* K *fixed point class of* f *and* f_b.

Proof. Let $[\langle\mu\rangle] \in \operatorname{Ker} q^x_\pi \cap E(f)$. Since $\operatorname{Ker} q^x_\pi = \operatorname{Im} i^x_\pi$ by the exactness, there exists $[K+\langle\mu_i\rangle] \in \operatorname{Coker}(1-f^x_{b_\pi/K})$ such that $i^x_\pi([K+\langle\mu_i\rangle]) = [\langle\mu\rangle]$ for all $1 \leq i \leq n$. By Lemma 3.12 and Lemma 3.13, $\operatorname{index}([K+\langle\mu_i\rangle] \neq 0$ for all i. Since ρ and ρ_K are index-preserving and injective functions, there exists only one $\emptyset \neq \overline{F}_{\mu_i} \in FPC(f_b)$ for each i and only one $\emptyset \neq F \in FPC(f)$ such that $\rho_K(\overline{F}_{\mu_i}) = [K+\langle\mu_i\rangle]$ for each i and $\rho(F) = [\langle\mu\rangle]$. Also $\operatorname{index}(f_b, \overline{F}_{\mu_i}) \neq 0$ for each i and $\operatorname{index}(f, F) \neq 0$. By the commutative diagram, all $\overline{F}_{\mu_i}$ are contained in F. Hence $[\langle\mu\rangle]$ corresponds to F which is an essential common mod K fixed point class of f and f_b.

Conversely, suppose $[\langle\mu\rangle]$ corresponds to an essential common mod K fixed point class F of f and f_b. Then there exists an essential mod K fixed point class $\overline{F}_\mu$ of f_b such that $\overline{F}_\mu \subset F$. Denote $\rho_K(\overline{F}_\mu) = [K+\langle\sigma\rangle] \in \operatorname{Coker}(1-f^x_{b_\pi/K})$. By the commutative diagram, $i^x_\pi([K+\langle\sigma\rangle]) = [\langle\mu\rangle]$. Since F is essential, $[\langle\mu\rangle] \in E(f)$. Thus $[\langle\mu\rangle] \in E(f) \cap \operatorname{Im} i^x_\pi = E(f) \cap \operatorname{Ker} q^x_\pi$.

Our main theorem is

THEOREM 3.15. *If the diagram (D) commutes and* $\#(\mathrm{Ker}\, q^x_\pi \cap E(f))$ *is independent of* x *in any essential fixed point class of* $f : E \to E$, *then (P1) holds.*

Proof. Without loss of generality we assume $N(f) \neq 0$. Since $u^f_*(E(f)) = E(f)$, (See Lemma 3.11.) we see that for each $[\langle\lambda\rangle] \in E(f)$,

$$\#((q^x_\pi)^{-1} q^x_\pi([\langle\lambda\rangle]) \cap E(f)) = \#(\mathrm{Ker}\, q^{x'}_\pi \cap E(f)),$$

where $q^{x'}_\pi : \mathrm{Coker}(1 - f^{x'}_\pi) \to \mathrm{Coker}(1 - f^{b'}_{B_\pi})$ is the function for some $x' \in \Phi(f)$ in the class represented by $[\langle\lambda\rangle]$ and $q(x') = b'$.

Using [4, Observation 1.10],

$$\#E(f) = \#((q^x_\pi)^{-1} q^x_\pi([\langle\lambda\rangle]) \cap E(f)) \cdot \# q^x_\pi(E(f)).$$

As $q^x_\pi(E(f)) = E(f_B)$ and the independence of x in any essential fixed point class of f,

$$\begin{aligned} \#E(f)/\#E(f_B) &= \#((q^x_\pi)^{-1} q^x_\pi([\langle\lambda\rangle]) \cap E(f)) \\ &= \#(\mathrm{Ker}\, q^x_\pi \cap E(f)) \\ &= N_K(f; f_b) \quad \text{(Theorem 3.14)}. \end{aligned}$$

Thus

$$N(f)/N(f_B) = N_K(f; f_b).$$

So $N(f) = N(f_B) \cdot N_K(f; f_b)$.

COROLLARY 3.16. *If* E *is simply connected, then (P1) holds.*

Proof. Since E is simply connected, the diagram (D) commutes.

If $N(f) = 0$, then $N_K(f; f_b) = 0$ implies (P1).

If $N(f) \neq 0$, then $\#(\mathrm{Ker}\, q^x_\pi \cap E(f))$ is independent of x in an essential fixed point class of f. Then we can obtain the result by Theorem 3.15.

EXAMPLE 3.17. Let $S^1 \to S^3 \xrightarrow{q} S^2$ be the Hopf fibration. Then if $f_B : S^2 \to S^2$ is a map of degree d, there is no obstruction to lifting f_B to a fibre map $f : S^3 \to S^3$. Suppose $d \neq -1$ so that f_B has a fixed point $b \in S^2$. It is easy to see that f_b also has degree d. Since S^3 and S^2 are simply connected, (P1) holds by Corollary 3.3 or Corollary 3.16.

But if we take $|d| \geq 3$, then the product formula $N(f) = N(f_B) \cdot N(f_b)$ does not hold. (See [5].)

We say that $f_{B_\pi} : \pi_1(B) \to \pi_1(B)$ is *nilpotent* if for some positive integer n, $f^n_{B_\pi} : \pi_1(B) \to \pi_1(B)$ is the trivial homomorphism.

THEOREM 3.18. *If* $f_{B_\pi} : \pi_1(B) \to \pi_1(B)$ *is nilpotent and* $N(f_B) \neq 0$, *then* $N_K(f_b) = N_K(f; f_b) = N(f)$.

Proof. Since $N(f_B) \neq 0$, $N(f_B) = 1$ by nilpotentness. Then

$$\begin{aligned} N(f) &= N_K(f_b) \cdot N(f_B) \quad ([4, \text{Corollary } 4.17]) \\ &= N_K(f; f_b) \cdot N(f_B) \quad (\text{Theorem } 3.2 \text{ (i)}). \end{aligned}$$

So Theorem 3.18 holds.

References

1. R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Co., Glenview Ill., 1979.
2. E. Fadell, *Nielsen numbers as a homotopy type invariant*, Pacific. J. Math., **63** (1976), 381–388.
3. P. R. Heath and K. H. Kams, *On exact orbit sequences*, Illinois. J. Math., **26** (1982), 149–154.
4. P. R. Heath, *Product formulae for Nielsen numbers of fibre maps*, Pacific. J. Math., **117** (1985), 267–289.
5. B. J. Jiang, *Lectures on Nielsen fixed point theory*, Contemp. Math., **14** (1983).
6. T. H. Kiang, *The theory of fixed point classes*, (in Chinese), Scientific Press, Peking, 1979.
7. H. Schirmer, *A relative Nielsen number*, Pacific. J. Math., **122** (1986), 459–473.
8. ———, *On the location of fixed points on pairs of spaces*, Topology Appl. **30** (1988), 253–266.
9. E. Spanier, *Algebraic Topology*, McGraw Hill Book Co., New York, 1986.
10. C. Y. You, *Fixed point classes of a fiber map*, Pacific. J. Math., **100** (1982), 217–241.

Korea University
Seoul 136–701, Korea