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ALMOST REGULAR OPERATORS

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1. Introduction

In 1988, Robin Harte suggested formulating a working concept of "almost regular" which is a weakened version of "regular" ([4] Preface vii). In this paper we make the formal definition of almost regular operators and then give the properties of almost regular operators.

2. Preliminaries

Throughout this paper, suppose X, Y and Z are non-zero normed spaces, write $BL(X,Y)$ for the set of all bounded linear operators from X to Y; I for the identity operator; $X^\dagger$ for the dual space of X. If $T \in BL(X,Y)$ write $T^\dagger$ for the adjoint of T in $BL(Y^\dagger, X^\dagger)$.

Recall [4] that if $k > 0$ and if $\|x\| \le k\|Tx\|$ for each $x \in X$ then we call $T \in BL(X,Y)$ *bounded below*, if $y \in \{Tx : \|x\| \le k\|y\|\}$ for each $y \in Y$ then we call T *open*, if $y \in \mathrm{cl}\{Tx : \|x\| \le k\|y\|\}$ *almost open*, and if $\mathrm{cl}\, T(X) = Y$ *dense*. The operator $T \in BL(X,Y)$ will be called *relatively open* [respectively, *relatively almost open*] if its truncation $T^\vee : X \longrightarrow T(X)$ is open [respectively, almost open](cf.[5],[6]). Thus bounded below is just relatively open one-one and almost open is relatively almost open dense. The mapping $\mathrm{core}(T) : X/T^{-1}(0) \longrightarrow \mathrm{cl}\, T(X)$ defined by setting

$$\mathrm{core}(T)(x + T^{-1}(0)) = Tx \in \mathrm{cl}\, T(X) \quad \text{for each } x \in X$$

is always one-one and dense; when it happens to be invertible the operator T is called *proper* ([4] Definition 3.2.7). We also recall ([4] Definition 3.7.1) that $T \in BL(X,Y)$ is said to be *almost left invertible* if there is (U_n) in $BL(Y,X)$ for which

$$\|I - U_nT\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \quad \text{with } \sup_n \|U_n\| < \infty$$

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and is said to be *almost right invertible* if there is (V_n) in $BL(Y,X)$ for which

$$\|I - TV_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \quad \text{with } \sup_n \|V_n\| < \infty.$$

If $T \in BL(X,Y)$ and if W is another normed space then we shall write

$$L_T = BL(W,T) : V \longmapsto TV \quad \text{from } BL(W,X) \text{ to } BL(W,Y)$$

and

$$R_T = BL(T,W) : U \longmapsto UT \quad \text{from } BL(Y,W) \text{ to } BL(X,W)$$

for the *left* and *right compositions* associated with T. Almost openness and bounded belowness can be tested by composition operators ([4] Theorem 5.6.3):

(2.1) $$T \text{ almost open} \Longleftrightarrow R_T \text{ bounded below}$$

and

(2.2) $$T \text{ bounded below} \Longleftrightarrow L_T \text{ bounded below.}$$

In each case forward implication is elementary, while we use the Hahn-Banach theorem for the reverse.

3. Almost regular operators.

We begin with a common generalization of almost left and almost right invertibility:

DEFINITION 3.1. $T \in BL(X,Y)$ is called *almost regular* if there is a sequence (T_n) in $BL(Y,X)$ for which

(3.1.1) $$\|T - TT_nT\| \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad \text{with } \sup_n \|T_n\| < \infty.$$

Evidently, almost left (respectively almost right) invertible operators are almost regular. If $T \in BL(X,Y)$ is almost regular then so are all composition operators $L_T = BL(W,T)$ and $R_T = BL(T,W)$. Indeed, if (T_n) satisfies (3.1.1) then

$$\|L_T - L_T L_{T_n} L_T\| \longrightarrow 0 \text{ and } \|R_T - R_T R_{T_n} R_T\| \longrightarrow 0.$$

In a sense, almost regularity lies somewhere between almost invertibility and relatively almost openess:

THEOREM 3.2. *If* $T \in BL(X,Y)$ *then*

(3.2.1) $\qquad T$ *almost regular* $\Longrightarrow T$ *relatively almost open.*

Proof. Suppose that (T_n) in $BL(Y,X)$ satisfies

$$\|T - TT_nT\| \longrightarrow 0 \text{ with } \sup_n\|T_n\| = k < \infty.$$

For each $y \in T(X)$, write $y = T(x)$ for a suitable x; then we have, with $x_n = T_n(y)$,

$$\|y - T(x_n)\| = \|T(x) - TT_nT(x)\| \leq \|T - TT_nT\|\|x\| \longrightarrow 0$$

and

$$\|x_n\| \leq \|T_n\|\|y\| \leq k\|y\|,$$

which says that T is relatively almost open.

In general, almost regularity does not imply almost left and right invertibility. We however have:

THEOREM 3.3. *If* $T \in BL(X,Y)$ *then*

(3.3.1) $\;T$ *almost regular and dense* $\Longrightarrow T$ *almost right invertible*

and

(3.3.2)

T *almost regular and bounded below* $\Longrightarrow T$ *almost left invertible.*

Proof. If (T_n) in $BL(Y,X)$ satisfies $\|T - TT_nT\| \longrightarrow 0$ with $\sup_n\|T_n\| < \infty$ and if T is dense then by (3.2.1) T is relatively almost open and dense, and hence T is almost open, which by (2.1) makes $R_T = BL(T,Y)$ bounded below. Therefore it follows that

$$\|R_T(I - TT_n)\| = \|(I - TT_n)T\| = \|T - TT_nT\| \longrightarrow 0$$
$$\Longrightarrow \|I - TT_n\| \longrightarrow 0,$$

giving (3.3.1). If instead T is bounded below, then by (2.2) $L_T = BL(X,T)$ is bounded below; thus it follows that

$$\|L_T(I - T_nT)\| = \|T(I - T_nT)\| = \|T - TT_nT\| \longrightarrow 0$$
$$\Longrightarrow \|I - T_nT\| \longrightarrow 0,$$

giving (3.3.2).

We now meet a condition that is actually weaker than properness:

DEFINITION 3.4. $T \in BL(X,Y)$ is said to be *almost right* [respectively, *left*] *proper* if core(T) is almost right [respectively, left] invertible. If T is almost right and almost left proper, it is called *almost proper.*

For example, the zero operator is almost proper. Evidently,

(3.4.1) $\qquad T$ almost right proper $\Longrightarrow T$ relatively almost open.

and

$$T \text{ almost invertible} \Longleftrightarrow T \text{ one-one, dense and almost proper.}$$

In particular, if X and Y are complete and $T \in BL(X,Y)$ then

$$T \text{ almost proper} \Longleftrightarrow T \text{ proper.}$$

One way round, almost regularity can be tested by almost right properness:

THEOREM 3.5. *If* $T \in BL(X,Y)$ *then*

(3.5.1) $\qquad$ *T almost regular $\Longrightarrow T$ almost right proper.*

Proof. Suppose that (T_n) in $BL(Y,X)$ satisfies $\|T - TT_nT\| \longrightarrow 0$ with $\sup_n \|T_n\| < \infty$. Define, for each $n \in N$, $S_n : \mathrm{cl}\, T(X) \longrightarrow X/T^{-1}(0)$ by setting

$$S_n(y) = T_n(y) + T^{-1}(0) \quad \text{for each } \; y \in \mathrm{cl}\, T(X).$$

Now observe that

$$S_n \in BL(\mathrm{cl}\, T(X),\ X/T^{-1}(0)), \ \ \sup_n \|S_n\| < \infty$$

and

$$\|\mathrm{core}(T) - \mathrm{core}(T) S_n \mathrm{core}(T)\| = \|T - TT_nT\| \longrightarrow 0,$$

which says that core(T) is almost regular. Since core(T) is always dense, it follows from (3.3.1) that core(T) is almost right invertible; therefore T is almost right proper.

We are ready for:

THEOREM 3.6. *Let $T \in BL(X,Y)$. If T is almost right proper and if $T^{-1}(0)$ and $\mathrm{cl}\, T(X)$ are both complemented, then T is almost regular.*

Proof. Suppose that both $T^{-1}(0)$ and $\mathrm{cl}\, T(X)$ are complemented; thus we may choose continuous projections $P = P^2 \in BL(X,X)$ and $Q = Q^2 \in BL(Y,Y)$ with

$$T^{-1}(0) = P^{-1}(0) \quad \text{and} \quad Q(Y) = \mathrm{cl}\, T(X).$$

Further, if T is almost right proper then the mapping

$$T^{\wedge} : P(X) \longrightarrow Q(Y)$$

induced by T is almost right invertible. Thus there is $(T_n^{\wedge})$ in $BL(Q(Y), P(X))$ for which

$$\|I - T^{\wedge}T_n^{\wedge}\| \longrightarrow 0 \quad \text{with } \sup_n \|T_n^{\wedge}\| < \infty.$$

Define $T_n : Y \longrightarrow X$ by setting

$$T_n(y) = T_n^{\wedge}Q(y) \in X \quad \text{for each } y \in Y.$$

Evidently, T_n is well defined, linear and bounded. We thus have

$$\begin{aligned}\|T - TT_nT\| &= \sup_{\|x\|=1} \|T(x) - TT_nT(x)\| \\ &= \sup_{\|x\|=1} \|T(x) - TT_n^{\wedge}Q(T(x))\| \\ &= \sup \|y - T^{\wedge}T_n^{\wedge}(y)\| \quad \text{with } y = T(x) \text{ and } \|x\| = 1 \\ &\leq \|T\| \|I - T^{\wedge}T_n^{\wedge}\| \longrightarrow 0,\end{aligned}$$

which says that T is almost regular.

REMARK. We recall, by [2], [3], [4] or [7] that $T \in BL(X,Y)$ is called *regular* if there is $T' \in BL(Y,X)$ for which

$$T = TT'T. \tag{3.6.1}$$

It is well known that T is regular if and only if T is proper and both $T^{-1}(0)$ and $\mathrm{cl}\, T(X)$ are complemented. We thus have

$$T \text{ regular} \Longrightarrow T \text{ almost regular} \tag{3.6.2}$$

and

$$(3.6.3) \qquad T \text{ invertible} \Longleftrightarrow T \text{ regular, one-one and dense.}$$

In general, the reverse of (3.6.2) does not hold. For example, in l_∞, let c_{00} be the subspace of all sequences with only finitely many non-zero terms. We now define $T : c_{00} \longrightarrow c_{00}$ by setting

$$T(\xi_1, \xi_2, \xi_3, \cdots) = (0, \xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \cdots).$$

Then T is a quasinilpotent operator, so that $I - T$ is almost invertible and hence one-one, dense and almost regular; indeed,

$$\|I - (I + T + \cdots + T^n)(I - T)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

We notice that $I - T$ is not onto because $(1, 0, 0, 0, \cdots) \notin \mathrm{Ran}(I - T)$; thus $I - T$ is not invertible and hence, by (3.6.3), not regular.

In the setting of Banach spaces, the implication

$$T \text{ almost regular} \Longrightarrow T \text{ regular}$$

remains an open problem: we have not yet found a counterexample.

In the context of Hilbert space we however have:

COROLLARY 3.7. *If X and Y are Hilbert spaces and $T \in BL(X, Y)$ then*

$$(3.7.1) \qquad T \textit{ almost regular} \Longleftrightarrow T \textit{ regular.}$$

Proof. If X and Y are Hilbert spaces then we have (cf.[5])

$$(3.7.2) \qquad T \text{ relatively almost open} \Longleftrightarrow T \text{ regular.}$$

Therefore (3.7.2) together with (3.2.1) gives (3.7.1).

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