J. Korean Math. Soc. **29** (1992), No. 2, pp. 391–400

# STRICTLY 2–CONVEX LINEAR 2–NORMED SPACES

R.W. FREESE, Y.J. CHO AND S.S. KIM

## I. Introduction

In [19] the concept of a linear 2-normed space was introduced as a natural 2-metric analog of that of a normed linear space. Linear 2-normed spaces were investigated in [19] as well as in numerous other articles by different authors from many points of view ([4], [6], [7], [9], [12], [13], [16], [18]–[21], [28], [29]).

Let $X$ be a linear space of dimension $> 1$ and let $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

($N_1$) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent,
($N_2$) $\|x, y\| = \|y, x\|$,
($N_3$) $\|\alpha x, y\| = |\alpha|\,\|x, y\|$, where $\alpha$ is real,
($N_4$) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The function $\|\cdot,\cdot\|$ is called a *2-norm* on $X$ and $(X, \|\cdot,\cdot\|)$ a *linear 2-normed space*. Some of the basic properties of the 2-norms are that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for every $x, y$ in $X$ and every real $\alpha$.

Note that in the definition of 2-norm if the condition ($N_4$) is replaced by the following condition:

($SN_4$) $\|x, y\| = \|x, y - x\|$,

then the function $\|\cdot,\cdot\|$ is called a *semi-2-norm* on $X$ and $(X, \|\cdot,\cdot\|)$ a *semi-2-normed space* ([16]).

Linear 2-normed spaces are special cases of a large class called 2-metric spaces. A 2-metric space is a set $X$ with a real-valued non-negative function $\sigma$ defined on $X \times X \times X$ which satisfies the following conditions:

($D_1$) For each pair of points $x, y$ in $X$ with $x \neq y$, there exists a point $z$ in $X$ such that $\sigma(x, y, z) \neq 0$,

Received December 8, 1991.

This paper was partially supported by Non-directed Research Fund. of Korea Research Fundation, 1989–1991.

($D_2$) $\sigma(x, y, z) = 0$, whenever at least two of the points $x, y, z$ are equal,

($D_3$) $\sigma(x, y, z) = \sigma(y, x, z) = \sigma(y, z, x)$,

($D_4$) $\sigma(x, y, z) \leq \sigma(x, y, w) + \sigma(x, z, w) + \sigma(y, z, w)$.

The function $\sigma$ is called a *2-metric* for the space $X$ and $(X, \sigma)$ a *2-metric space.* We remark that, in the definition of 2-metric, if the condition ($D_4$) is deleted, then the function $\sigma$ is called a *semi-2-metric.* If $(X, \|\cdot,\cdot\|)$ is a semi-2-normed space, then the function $\sigma(x, y, z) = \|x - z, y - z\|$ defines a semi-2-metric on $(X, \|\cdot,\cdot\|)$ ([16]). For more details on 2-metric spaces, refer to [1], [15]–[17], [18]–[20], [22] and [27]. Especially, if $(X, \|\cdot,\cdot\|)$ is a linear 2-normed space, then the function $\sigma(x, y, z) = \|x - z, y - z\|$ defines a 2-metric on $(X, \|\cdot,\cdot\|)$ ([19]). Every linear 2-normed space will be considered to be a 2-metric space with the 2-metric defined in this sense. Clearly, any finite example of a 2-metric space shows that not every 2-metric space is a linear 2-normed space. Conversely, in [17], some conditions for a 2-metric space to be a linear 2-normed space are given.

For non-zero vectors $x, y$ in $X$, let $V(x, y)$ denote the subspace of $X$ generated by $x$ and $y$. A linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *strictly convex* ([5]) if $\|x + y, z\| = \|x, z\| + \|y, z\|$ and $z \notin V(x, y)$ imply that $y = \alpha x$ for some $\alpha > 0$. Some characterizations of strictly convex linear 2-normed spaces are given in [2]–[5], [8], [11]–[16], [23]–[26], [28], and [31]. A linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *strictly 2-convex* ([8]) if $\|x, y\| = \|x, z\| = \|y, z\| = \|x + z, y + z\|/3 = 1$ implies that $z = x + y$. These spaces have been studied in [3], [8], [10]–[13], [15], [16], [23], [30] and [31]. It is easy to see that every strictly convex linear 2-normed space is always strictly 2-convex but the converse is not necessarily true ([8]) and that every linear 2-normed space of dimension 2 is strictly 2-convex.

In this paper, we give new characterizations of strictly 2-convex linear 2-normed spaces in terms of extreme points.

## II. Strict 2-convexity

A point $p$ of a 2-metric space $(X, \sigma)$ formed from a linear 2-normed space is called a *2-metric midpoint* of three non-collinear points $a, b, c$ of $X$ $(\sigma(a, b, c) \neq 0)$ if $\sigma(a, b, p) = \sigma(a, p, c) = \sigma(p, b, c) = \sigma(a, b, c)/3$.

For three non-collinear points $a, b, c$ in $X$, let

$$T(a,b,c) = \{x \in X \mid \sigma(a,b,c) = \sigma(a,b,x) + \sigma(a,x,c) + \sigma(x,b,c)\}.$$

$T(a,b,c)$ will be called the *triangle with vertices* $a, b$ and $c$. Furthermore, we will refer to $\sigma(a,b,c)$ as the area of $T(a,b,c)$. A point $p$ will be called a *center* of $T(a,b,c)$ if $p$ is a 2-metric midpoint of $a, b$ and $c$. If $a, b$ and $c$ are three non-collinear points in $X$, let $C(a,b,c)$ denote the *convex envelope* of $\{a,b,c\}$, that is, $C(a,b,c)$ is the smallest convex set containing $\{a,b,c\}$. In particular,

$$C(a,b,c) = \{\alpha a + \beta b + \gamma c \mid \alpha, \beta, \gamma \geq 0,\ \alpha + \beta + \gamma = 1\}.$$

The following theorem is proved in [8] and [10]:

THEOREM 2.1. *The following statements are equivalent:*

(1) $(X, \|\cdot,\cdot\|)$ *is strictly 2-convex.*
(2) $T(a,b,c)$ *has a unique center.*
(3) *If $a, b$ and $c$ are three non-collinear points in $X$, then* $T(a,b,c) = C(a,b,c)$.
(4) *If $a$ and $b$ are two points in $X$ with $\|a,b\| > 0$, then there exists a unique point $c$ in $X$ such that $0$ is a center of $T(a,b,c)$.*
(5) *If $\|a,b\| = \|b,c\| = \|c,a\| = 1$, $c \neq -(a+b)$, $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$, then $\sigma(\alpha a, \beta b, \gamma c) < 1/3$.*

A point $x$ of a semi-2-normed space $(X, \|\cdot,\cdot\|)$ is called an ***algebraic between point*** of three points $a, b, c$ in $X$ if there exist real numbers $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$ and $x = \alpha a + \beta b + \gamma c$. In the case $\alpha = \beta = \gamma = 1/3$, $x$ is said to be an ***algebraic midpoint*** of $a, b$ and $c$. A point $x \in X$ is called a ***$\sigma$-between point*** of three non-collinear points $a, b, c$ in $X$ if $\sigma(a,b,c) = \sigma(a,b,x) + \sigma(a,x,c) + \sigma(x,b,c)$. In the case $\sigma(a,b,x) = \sigma(a,x,c) = \sigma(x,b,c)$, $x$ is said to be a ***$\sigma$-midpoint*** of $a, b$ and $c$.

Thus every algebraic midpoint of three points is an algebraic between point of these points and every $\sigma$-midpoint of three non-collinear points is a $\sigma$-between point of these points.

In [16], the following theorems are proved:

THEOREM 2.2. *In a semi-2-normed space* $(X, \|\cdot,\cdot\|)$, *the following properties hold:*

(1) *An algebraic midpoint of three non-collinear points in* $X$ *is a* $\sigma$*-midpoint of these points.*
(2) *An algebraic between point of three non-collinear points in* $X$ *is a* $\sigma$*-between point of these points.*

THEOREM 2.3. *In a semi-2-normed space* $(X, \|\cdot,\cdot\|)$, *the following statements are equivalent:*

(1) *A* $\sigma$*-midpoint of three non-collinear points in* $X$ *is a algebraic midpoint of these points.*
(2) *For points* $a, b, c \in X$ *with* $\|a, b\| = \|b, c\| = \|c, a\| = \|a + c, b + c\|/3 = 1$, *it follows that* $c = a + b$.

THEOREM 2.4. *In a semi-2-normed space* $(X, \|\cdot,\cdot\|)$, *the following statements are equivalent:*

(1) *A* $\sigma$*-between point of three non-collinear points in* $X$ *is an algebraic between point of these points.*
(2) *For points* $a, b, c \in X$ *with* $\|a-c, b-c\| \neq 0$ *and* $d = \delta a+(1-\delta)b$, $\delta \in (0,1)$, *the validity of* $\sigma(a, c, d) = \sigma(a, c, e) + \sigma(a, e, d)$ *and* $\sigma(b, c, d) = \sigma(b, c, e) + \sigma(b, e, d)$ *implies the existence of a real number* $\lambda \in [0,1]$ *with* $e = \lambda c + (1 - \lambda)d$.
(3) *For arbitrary points* $a, b, c$ *in* $X$ *with* $\|a + c, b + c\| = \|a, b\| + \|b, c\| + \|c, a\|$ *and* $\|a, b\| \cdot \|b, c\| \cdot \|c, a\| \neq 0$, *there exist real numbers* $\alpha, \beta > 0$ *such that* $c = \alpha a + \beta b$.

REMARK. (1) A linear 2-normed space $(X, \|\cdot,\cdot\|)$ in which the properties from Theorems 2.3 and 2.4 hold is said to be strictly 2-convex.

(2) By means of the corollary to Theorem 8 ([8]), it follows that every semi-2-normed linear space of dimension 2 is a strictly 2-convex linear 2-normed space.

## III. Extreme points

In this section, we give some geometric characterizations of strictly 2-convex linear 2-normed spaces in terms of extreme points.

In a 2-metric space $(X, \sigma)$, a point $p$ is said to be a *2-metric extreme point* of a set $M$ provided for arbitrary points $x, y, z$ in $M$ with

$\sigma(x,y,z) = \sigma(x,y,p) + \sigma(x,z,p) + \sigma(y,z,p)$, at least one of the terms on the right-hand side is zero. A point $p$ is said to be a *2-metric ultra-extreme point* of a set $M$ provided for arbitrary points $x, y, z$ in $M$ with $\sigma(x,y,z) = \sigma(x,y,p) + \sigma(x,z,p) + \sigma(y,z,p)$, at least two of the terms on the right-hand side are equal to zero. In a linear 2-normed space $(X, \|\cdot,\cdot\|)$, a point $p$ is said to be an *algebraic extreme point* of a set $M$ provided for arbitrary points $x, y, z$ in $M$ with $p = \alpha x + \beta y + \gamma z$, $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \in [0,1]$, at least one of $\alpha, \beta, \gamma$ is zero. A point $p$ is said to be an *algebraic ultra-extreme point* of a set $M$ provided for arbitrary points $x, y, z$ in $M$ with $p = \alpha x + \beta y + \gamma z$, $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \in [0,1]$, at least two of $\alpha, \beta, \gamma$ are equal to zero

REMARK. (1) A 2-metric (resp. algebraic) ultra-extreme point is a 2-metric (resp. algebraic) extreme point but not conversely.

(2) In the euclidean plane a 2-metric extreme point is an algebraic extreme point and conversely.

We are now ready to present our main theorems:

THEOREM 3.1. *A 2-metric extreme point of a set $M$ in a 2-metric space $(X, \sigma)$ is an algebraic extreme point of $M$.*

*Proof.* Suppose that there exists a point $p$ in $M$ such that $p$ is a 2-metric extreme point of $M$ but not an algebraic extreme point of $M$. This means that there exist three points $x, y, z$ in $M$ such that $p = \alpha x + \beta y + \gamma z$, $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \in [0,1]$, with each of $\alpha, \beta, \gamma$ are non-zero. Therefore $z - p$ and $y - p$ are linearly independent and hence we have $\sigma(p, y, z) = \|z - p, y - p\| \neq 0$. Similarly we have $\sigma(p, x, z) \neq 0$ and $\sigma(p, x, y) \neq 0$, which is contrary to $p$ being a 2-metric extreme point. This completes the proof.

By a similar argument used in Theorem 3.1, we have the following theorem:

THEOREM 3.2. *A 2-metric ultra-extreme point of a set $M$ in a 2-metric space $(X, \sigma)$ is an algebraic ultra-extreme point of $M$.*

THEOREM 3.3. *An algebraic ultra-extreme point of a set $M$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ that is a 2-metric extreme point of $M$ is a 2-metric ultra-extreme point of $M$.*

*Proof.* Suppose that there exists a point $p$ in $M$ such that $p$ is an algebraic ultra-extreme point of $M$ and a 2-metric extreme point but not a 2-metric ultra-extreme point. This means that there exist three distinct points $x, y, z$ in $M$ such that $\sigma(p,x,y) = 0$ but $\sigma(x,y,z) = \sigma(x,y,p) + \sigma(x,z,p) + \sigma(y,z,p)$. That is, $\sigma(x,y,z) = \sigma(x,z,p) + \sigma(y,z,p)$. Therefore, since $p$ is not an algebraic ultra-extreme point of $M$, then we have $p \neq x$, $p \neq y$, $p \neq z$ and similarly $x, y, z$ are distinct. From $\sigma(p,x,y) = 0$, we know that $p - x$ and $y - x$ are linearly dependent, that is, $p = \lambda x + (1 - \lambda)y$. Also from $\sigma(x,p,z) = 0$ we know that $p - x$ and $z - x$ are linearly dependent, that is, $p = \lambda^* x + (1 - \lambda^*)z$. Since $z - x$ and $y - x$ are linearly independent, then we have $p = x$, which is a contradiction. The proof is similar in the event that $\sigma(p,x,z) = 0$ or $\sigma(p,y,z) = 0$. This completes the proof.

Note that in the above theorem, if the hypothesis that a point is a 2-metric extreme point is deleted, the theorem is false as is shown by the following example:

EXAMPLE 3.4. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space that is not strictly 2-convex. Then by Theorem 2.1, there exists a triple $p, q, r$ in $X$ with a non-unique center. That is, $\sigma(p,q,r) = \sigma(p,q,m_1) + \sigma(p,r,m_1) + \sigma(q,r,m_1)$ and $\sigma(p,q,r) = \sigma(p,q,m_2) + \sigma(p,r,m_2) + \sigma(q,r,m_2)$ where it may be assumed without loss of generality that $m_1$ is an element of the algebraic triangle $T(p,q,r)$ with $m_1 \neq m_2$. Therefore, since $X$ is a linear space, it follows that $m_2$ is an algebraic ultra-extreme point of $M = T(p,q,r) \cup T(p,q,m_2) \cup T(p,r,m_2) \cup T(q,r,m_2)$. However, since none of the values $\sigma(p,q,m_2)$, $\sigma(p,r,m_2)$ and $\sigma(q,r,m_2)$ are zero, $m_2$ is not a 2-metric ultra-extreme point.

This leads naturally to the following theorem:

THEOREM 3.5. *In a strictly 2-convex linear 2-normed space $(X, \|\cdot,\cdot\|)$, a point of a set $M$ is an algebraic ultra-extreme point of $M$ if and only if it is a 2-metric ultra-extreme point of $M$.*

*Proof.* Since every 2-metric ultra-extreme point is an algebraic ultra-extreme point, it suffices to show that every algebraic ultra-extreme point $p$ of $M$ is a 2-metric ultra-extreme point. By Theorems 2.3 and 2.4, we know that in a strictly 2-convex linear 2-normed space, a point is an algebraic between point if and only if it is a 2-metric between point. Therefore, $p$ is a 2-metric ultra-extreme point of $M$. This completes the proof.

Theorem 3.5 gives directly the following:

COROLLARY 3.6. *In a strictly 2-convex linear 2-normed space* $(X, \|\cdot,\cdot\|)$, *a point* $x$ *is an algebraic extreme point of a set* $M$ *if and only if it is a 2-metric extreme point of* $M$.

Furthermore, since the equivalence of algebraic and 2-metric interior points is equivalent to strict convexity and strict 2-convexity ([8], [16]), the equivalence of algebraic and 2-metric extreme points implies strict 2-convexity and thus we have shown the following:

THEOREM 3.7. *In a linear 2-normed space* $(X, \|\cdot,\cdot\|)$, *the set of algebraic extreme points of a set* $M$ *is identical to the set of 2-metric extreme points if and only if* $(X, \|\cdot,\cdot\|)$ *is strictly 2-convex.*

Note that the circle in the euclidean plane illustrates the fact that there exists a set $M$ in a strictly convex linear 2-normed space $(X, \|\cdot,\cdot\|)$ with an infinite number of both 2-metric and algebraic ultra-extreme points. Also the set of all points $(x, y)$ in the euclidean plane such that $1 \leq x \leq 2$ is a closed set $M$ which has no ultra-extreme points.

THEOREM 3.8. *In a strictly 2-convex linear 2-normed space* $(X, \|\cdot,\cdot\|)$, *the set of algebraic ultra-extreme points of a set* $M$ *is identical to the set of 2-metric ultra-extreme points of* $M$.

*Proof.* By Theorem 3.2, since every 2-metric ultra-extreme point is an algebraic ultra-extreme point, it suffices to show that every algebraic ultra-extreme point of a set $M$ is a 2-metric extreme point. Furthermore, by Theorem 3.3, if $p$ is an algebraic ultra-extreme point and a 2-metric extreme point of a set $M$, then $p$ is a 2-metric ultra-extreme point. Therefore, suppose that $p$ is an algebraic extreme point but neither a 2-metric ultra-extreme point nor a 2-metric extreme point. Then there exist three points $x, y, z$ in $M$ such that $\sigma(x, y, z) = \sigma(x, y, p) + \sigma(x, z, p) + \sigma(y, z, p)$ but $p$ is not in $T(x, y, z)$. Thus $p$ is a 2-metric between point of $x, y, z$ but not an algebraic between point of $x, y, z$. Therefore, by Theorems 2.2 and 2.3, $(X, \|\cdot,\cdot\|)$ is not strictly 2-convex, which is a contradiction. This completes the proof.

THEOREM 3.9. *In a linear 2-normed space $(X, \|\cdot,\cdot\|)$, if the set of algebraic ultra-extreme points of a set $M$ is identical to the set of 2-metric ultra-extreme points of $M$, then a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is strictly 2-convex.*

*Proof.* Suppose that $(X, \|\cdot,\cdot\|)$ is not strictly 2-convex. Then, by Theorem 2.1, there exist a point $p$ and a triple $x, y, z$ in $X$ such that $p$ is a 2-metric center of $x, y, z$ but $p$ is not the algebraic center of $x, y, z$. Then the set $M = T(x,y,p) \cup T(x,z,p) \cup T(y,z,p) \cup T(x,y,z)$ has $p$ as an algebraic ultra-extreme point but not a 2-metric extreme point. This contradiction proves this theorem. This completes the proof.

By Theorems 3.8 and 3.9, we have the following:

COROLLARY 3.10. *In a linear 2-normed space $(X, \|\cdot,\cdot\|)$, the set of algebraic ultra-extreme points of a set $M$ is identical to the set of 2-metric ultra-extreme points of $M$ if and only if the set of algebraic extreme points of a set $M$ is identical to the set of 2-metric extreme points of $M$.*

## References

1. E.Z. Andalafte and R. W. Freese, *Existence of 2-segments in 2-metric spaces*, Fund. Math. LX(1967), 201–208.
2. Y.J. Cho, K.S. Ha and W.S. Kim, *Strictly convex linear 2-normed spaces*, Math. Japon. **26** (1981), 495–498.
3. Y.J. Cho, B.H. Park and K.S. Park, *Strictly 2-convex linear 2-normed spaces*, Math. Japon. **27** (1982), 609–612.
4. C. Diminnie, S. Gähler and A. White, *2-inner product spaces*, Demonstratio Math. **6** (1973), 525–536.
5. ———, *Strictly convex linear 2-normed spaces*, Math. Nachr. **59** (1974), 319–324.
6. ———, *Remarks on generalizations of 2-inner products*, Math. Nachr. **74** (1976), 363–372.
7. ———, *2-inner product spaces* II, Demonstratio Math. **10** (1977), 169–188.
8. ———, *Remarks on strictly and strictly 2-convex 2-normed spaces*, Math. Nachr. **88** (1979), 363–372.
9. ———, *Non-expansive mappings in linear 2-normed spaces*, Math. Japon. **21** (1976), 197–200.
10. C. Diminnie and A. White, *Some geometric remarks concerning strictly 2-convex 2-normed spaces*, Math. Seminar Notes, Kobe Univ., **6** (1978), 245–253.
11. ———, *A characterization of strictly convex 2-normed spaces*, J. Korean Math. Soc. **11** (1984), 53–59.

12. ______, *2-norms generated by seminorms on the space of bivectors*, Math. Nachr. **106** (1982), 341–346.
13. R. Ehret, *Linear 2-normed spaces*, Doctoral Diss., Saint Louis Univ., 1969.
14. I. Franic, *Two results in 2-normed spaces*, Glasnik Mat. **17** (1982), 271–275.
15. R.W. Freese, and E.Z. Andalafte, *A characterization of 2-betweenness in 2-metric spaces*, Canad. J. Math. **18** (1966), 936–968.
16. R.W. Freese and S. Gähler, *Remarks on semi-2-normed spaces*, Math. Nachr. **105** (1982), 151–161.
17. R.W. Freese and Y.J. Cho, *A characterization of linear 2-normed spaces*, preprint.
18. S. Gähler, *2-metrische Räume und ihr topologische Struktur*, Math. Nachr. **26** (1963–64), 115–148.
19. ______, *Linear 2-normierte Räume*, Math. Nachr. **28** (1965), 1–45.
20. ______, *Zur geometric 2-metrische Räume*, Revue Roumaine de Math. Pures iet Appliquees XL(1966), 664–669.
21. ______, *Über 2-Banach Räume*, Math. Nachr. **42** (1969), 335–347.
22. K.E. Grant, *Axioms for n-metric structures*, Canad. J. Math. **25** (1973), 26–30.
23. K.S. Ha, Y.J. Cho and A. White, *Strictly convex and strictly 2-convex 2-normed spaces*, Math. Japon. **33** (1988), 375–384.
24. K.S. Ha, Y.J. Cho, S.S. Kim and M.S. Khan, *Strictly convex linear 2-normed spaces*, Math. Nachr. **146** (1990), 7–16.
25. Y. Ho and A. White, *A note on p-semi-inner product spaces*, Glasnik Mat. **42** (1989), 365–370.
26. K. Iseki, *On nonexpansive mappings in strictly convex linear 2-normed spaces*, Math. Seminar Notes, Kobe Univ., **3** (1975), 125–129.
27. G. Murphy, *Lines in a planar space*, Proc. Amer. Math. Soc. **19** (1968), 1106–1108.
28. M. Newton, *Uniform and strict convexity in linear 2-normed spaces*, Doctoral Diss., Saint Louis Univ., 1979.
29. A. White, *2-Banach spaces*, Math. Nachr. **42** (1969), 44–60.
30. ______, *A new characterization of strict convexity*, Math. Seminar Notes, Kobe Univ., **10** (1982), 619–620.
31. A. White and Y.J. Cho, *Linear mappings on linear 2-normed spaces*, Bull. Korean Math. Soc. **21** (1984), 1–6.

Department of Mathematics
Saint Louis University
St. Louis, MO 63103, U.S.A.

Department of Mathematics
Gyeongsang National University
Jinju 660–701, Korea

Department of Mathematics
Dongeui University
Pusan 614–714, Korea