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ON THE UNITARY GROUPS $U_5(q)$

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1. Introduction

The projective special unitary group $PSU_n(q^2)$ is isomorphic to the twisted Chevalley group ${}^2A_{n-1}(q)$ of type A_{n-1}, where q is a prime power. Use $U_n(q)$ to denote the group $PSU_n(q^2)$. The simple group ${}^2A_{n-1}(q)$ can be studied in terms of its (B,N)-pair structure, and so the group $U_n(q)$ can be done in this manner. The structure of the group $U_4(q)$ has been explicitly determined in [3].

The purpose of this paper is to study the structure of the group $U_5(q)$ by using the (B,N)-pair structure of the group ${}^2A_4(q)$. This paper is organized as follows. In section 2, we explicitly determine elements and a (B,N)-pair of the twisted Chevalley group ${}^2A_4(q)$. And an explicit isomorphism of ${}^2A_4(q)$ onto $U_5(q)$ is also given. In section 3, we determine maximal parabolic subgroups of $U_5(q)$ and study the structure of some subgroups which are contained in those parabolic subgroups. We also determine the elements of order p in $U_5(q)$ and study the structure of their centralizers in $U_5(q)$. In section 4, we determine involutions in $U_5(q)$ and the structure of their centralizers.

The notation and terminology in this paper are standard. They are taken from [4] for the general finite groups and from [2] for the finite simple groups of Lie type.

2. The group ${}^2A_4(q)$

Let F be a finite field with q^2 elements, where $q = p^m$ and p is a prime. Let σ be the automorphism of F of order 2 defined by $\sigma(\alpha) = \alpha^q$ for $\alpha \in F$. Denote $\sigma(\alpha)$ by $\bar{\alpha}$, and let $F_0 = \{\alpha \in F \mid \alpha = \bar{\alpha}\}$ be the subfield of F consisting of all elements left invariant by σ.

Let f be a non-degenerate Hermitian form on the five-dimensional vector space F^5 defined by

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$$f(u,v) = \alpha_1\bar{\beta}_5 - \alpha_2\bar{\beta}_4 + \alpha_3\bar{\beta}_3 - \alpha_4\bar{\beta}_2 + \alpha_5\bar{\beta}_1,$$

where $u = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $v = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$. Let J be the matrix associated with the form f with respect to the standard basis for F^5. For every 5×5 matrix A over F, let A^* be the matrix obtained from A by replacing each entry α_{ij} with $\overline{\alpha_{ij}}$ and then taking the transpose. Then the general unitary group $GU_5(q^2)$ is the set of all matrices A satisfying $AJA^* = J$. And we have

$$SU_5(q^2) = \{A \in GU_5(q^2) \mid \det A = 1\}, \quad U_5(q) = SU_5(q^2)/Z,$$

where $Z = \{\operatorname{diag}\{\lambda,\lambda,\lambda,\lambda,\lambda\} \mid \lambda \in F,\ \lambda^5 = 1,\ \lambda\bar{\lambda} = 1\}$.

The simple group $U_5(q)$ is isomorphic to the twisted Chevalley group ${}^2A_4(q)$, and this will be treated later in this section. Here we will give a more detailed description for ${}^2A_4(q)$ than that in [2].

Let Φ be the set of all roots of the simple Lie algebra $\mathcal{L}$ of type A_4 over the complex field. Relative to some fixed ordering $<$ on Φ, the system of fundamental roots may be denoted by $\Pi = \{a, b, c, d\}$ with $a < b < c < d$. The set of all positive roots is then $\Phi^+ = \{a,\ b,\ c,\ d,\ a+b,\ b+c,\ c+d,\ a+b+c,\ b+c+d,\ a+b+c+d\}$.

Let $\{h_s \mid s \in \Pi\} \cup \{e_r \mid r \in \Phi\}$ be a standard Chevalley basis for $\mathcal{L}$, and $N_{r,s}$ be the structure constant determined by the relation $[e_r, e_s] = N_{r,s}e_{r+s}$ for $r, s \in \Phi$. Then $N_{r,s}$ is either ± 1 or 0 according as $r+s$ is a root or not. The signs of the structure constants $N_{r,s}$ may be chosen arbitrarily for the pairs (a, b), (b, c), (c, d), $(a, b+c)$, $(b, c+d)$, $(a+b, c+d)$, and then the structure constants for all pairs are uniquely determined. We will set

$$(*) \quad N_{a,b} = -1, \quad N_{b,c} = N_{c,d} = N_{a,b+c} = N_{b,c+d} = N_{a+b,c+d} = 1.$$

Then it is easy to prove the following proposition.

(2.1) *We have*

$$N_{c,a+b} = N_{d,a+b+c} = 1, \quad N_{d,b+c} = N_{a,b+c+d} = -1.$$

Let $x_r(\alpha)$, n_r, $h_r(\lambda)$ and $h(\chi)$ be elements of the Chevalley group $A_4(q^2)$ as defined in [2], where $r \in \Phi^+$, $\alpha \in F$, $\lambda \in F^* = F - \{0\}$ and χ is an F-character. And let $\mathcal{B} = \{h_s \mid s \in \Pi\} \cup \{e_r \mid r \in \Phi\}$ be the basis of the extended Lie algebra $\mathcal{L}_F$ over the field F by $\mathcal{L}$. Then the elements n_r and $h_r(\lambda)$ act on $\mathcal{B}$ in the following manners:

$$(**) \qquad n_r.h_s = h_{w_r(s)}, \quad n_r.e_s = \eta_{r,s}e_{w_r(s)} \quad \text{for } s \in \Phi,$$

$$h_r(\lambda).h_s = h_s, \quad h_r(\lambda).e_s = \lambda^{A_{rs}} e_s \quad \text{for } s \in \Phi,$$

where w_r is the reflection defined by r and $A_{rs} = \dfrac{2(r, s)}{(r, r)}$.

From $(*)$, $(**)$ and (2.1), we have the following three propositions.

(2.2) *If* $r, s, r+s \in \Phi^+$, *then* $\eta_{r,s} = N_{r,s}$ *and* $\eta_{r,r} = \eta_{r,-r} = -1$. *Futhermore, we have*

$$\begin{aligned}\eta_{a,c} &= \eta_{a,d} = \eta_{a,a+b} = \eta_{a,c+d} = \eta_{a,a+b+c+d} = \eta_{b,d} = \eta_{b,a+b+c}\\ &= \eta_{b,a+b+c+d} = \eta_{c,a} = \eta_{c,b+c} = \eta_{c,b+c+d} = \eta_{c,a+b+c+d}\\ &= \eta_{d,a} = \eta_{d,b} = \eta_{d,a+b} = \eta_{d,c+d} = \eta_{d,b+c+d} = 1,\\ \eta_{a,a+b+c} &= \eta_{b,a+b} = \eta_{b,b+c} = \eta_{b,b+c+d} = \eta_{c,c+d} = \eta_{c,a+b+c}\\ &= \eta_{d,a+b+c+d} = -1.\end{aligned}$$

(2.3) *For any* $r, s \in \Phi^+$, *we have*

$$x_r(\alpha)x_r(\beta) = x_r(\alpha+\beta), \qquad [x_r(\alpha), x_s(\beta)] = x_{r+s}(N_{r,s}\alpha\beta),$$
$$h_r(\lambda)h_r(\mu) = h_r(\lambda\mu), \qquad [h_r(\lambda), h_s(\mu)] = 1.$$

(2.4) *For any* $r, s \in \Phi^+$, *we have*

$$h_r(\lambda)^{-1}x_s(\alpha)h_r(\lambda) = x_s(\lambda^{-A_{rs}}\alpha),$$
$$n_r{}^{-1}x_s(\alpha)n_r = x_{w_r(s)}(\eta_{r,\,w_r(s)}\alpha), \quad n_r{}^{-1}h_s(\lambda)n_r = h(\chi),$$

where χ *is the* F-*character defined by* $\chi(t) = \lambda^{A_{s\,w_r(t)}}$.

Let ρ denote the nontrivial symmetry of the Dynkin diagram for $\mathcal{L}$ which is given by $\rho(a) = d$, $\rho(b) = c$, $\rho(c) = b$, $\rho(d) = a$. Then the sets of the partition of Φ^+ defined by Steinberg are

$S_1 = \{a, d\}$, $\quad S_2 = \{b, c, b+c\}$,

$S_3 = \{a+b+c, b+c+d\}$, $\quad S_4 = \{a+b, c+d, a+b+c+d\}$.

Define $w_1 = w_a \circ w_d$ and $w_2 = w_{b+c} = w_b \circ w_c \circ w_b$. Then

$w_1(a) = -a$, $\quad w_1(b) = a+b$, $\quad w_1(c) = c+d$, $\quad w_1(d) = -d$,

$w_2(a) = a+b+c$, $\quad w_2(b) = -c$, $\quad w_2(c) = -b$, $\quad w_2(d) = b+c+d$.

Thus, w_1 and w_2 are involutions satisfying $(w_1 \circ w_2)^4 = 1$, and the action of these involutions on the sets S_i are given by

$$w_1(S_2) = S_4, \qquad w_1(S_1) = -S_1, \qquad w_1(S_3) = S_3,$$
$$w_2(S_2) = -S_2, \qquad w_2(S_1) = S_3, \qquad w_2(S_4) = S_4.$$

Hence, the Weyl group W of the twisted Chevalley group ${}^2A_4(q)$ is generated by w_1 and w_2, and $W = \langle w_1, w_2\rangle$ is a dihedral group of order 8.

Now, we define the elements of the twisted Chevalley group ${}^2A_4(q)$ as follows:

$x_1(\alpha) = x_a(\alpha)x_d(\bar{\alpha}), \quad \alpha \in F\,;$
$x_2(\beta,\,\gamma) = x_b(\beta)x_c(\bar{\beta})x_{b+c}(\gamma), \quad \beta, \gamma \in F, \ \gamma + \bar{\gamma} = -\beta\bar{\beta}\,;$
$x_3(\delta) = x_{a+b+c}(\delta)x_{b+c+d}(\bar{\delta}), \quad \delta \in F\,;$
$x_4(\xi,\,\eta) = x_{a+b}(\xi)x_{c+d}(\bar{\xi})x_{a+b+c+d}(\eta), \quad \xi, \eta \in F, \ \eta + \bar{\eta} = -\xi\bar{\xi}\,;$
$h_1(\mu) = h_a(\mu)h_d(\bar{\mu}), \quad \mu \in F^* = F - \{0\};$
$h_2(\lambda) = h_b(\lambda)h_c(\bar{\lambda}), \quad \lambda \in F^*; \qquad n_1 = n_a n_d, \qquad n_2 = n_b n_c n_b.$

Futhermore, we set

$$U_1 = \{x_1(\alpha)\,|\,\alpha \in F\}, \quad U_2 = \{x_2(\beta,\,\gamma)\,|\,\beta, \gamma \in F, \ \gamma + \bar{\gamma} = -\beta\bar{\beta}\},$$
$$U_3 = \{x_3(\delta)\,|\,\delta \in F\}, \quad U_4 = \{x_4(\xi,\,\eta)\,|\,\xi, \eta \in F, \ \eta + \bar{\eta} = -\xi\bar{\xi}\},$$
$$U = U_1U_2U_3U_4.$$

Using $(*), (**)$ and (2.1) $\sim$ (2.4), we can easily prove the following properties (2.5) $\sim$ (2.12) of ${}^2A_4(q)$.

(2.5) *The multiplication in each* U_i *is given by*

$$x_1(\alpha)x_1(\alpha') = x_1(\alpha + \alpha'), \ x_2(\beta,\,\gamma)x_2(\beta',\,\gamma') = x_2(\beta + \beta',\,\gamma + \gamma' - \bar{\beta}\beta'),$$
$$x_3(\delta)x_3(\delta') = x_3(\delta + \delta'), \ x_4(\xi,\,\eta)x_4(\xi',\,\eta') = x_4(\xi + \xi',\,\eta + \eta' - \bar{\xi}\xi').$$

The subgroups U_1, U_2, U_3 *and* U_4 *are p-subgroups of* ${}^2A_4(q)$ *such that* $|U_1| = |U_3| = q^2$ *and* $|U_2| = |U_4| = q^3$. *Moreover, both* U_1 *and* U_3 *are elementary abelian p-subgroups of* ${}^2A_4(q)$.

(2.6) *The commutator relations between elements of the* U_i *are*

$$[x_1(\alpha)\,,\,x_2(\beta,\gamma)] = x_3(-\alpha\bar{\gamma})x_4(-\alpha\beta,\,\alpha\bar{\alpha}\bar{\gamma}),$$
$$[x_1(\alpha)\,,\,x_3(\delta)] = x_4(0,\,\bar{\alpha}\delta - \alpha\bar{\delta}), \quad [x_2(\beta,\gamma)\,,\,x_4(\xi,\eta)] = x_3(\bar{\beta}\xi),$$

and all other types of commutators between elements of the various U_i *are trivial.*

Every element of U *is uniquely expressed as a product*

$$x_1(\alpha)x_2(\beta,\gamma)x_3(\delta)x_4(\xi,\eta).$$

The subgroup U *is a Sylow p-subgroup of* ${}^2A_4(q)$ *of order* q^{10}, *and its center* $Z(U) = \{x_4(0,\eta)\,|\,\eta + \bar{\eta} = 0\}$ *is of order* q.

(2.7) *Let* $H_1 = \{h_1(\mu)\ |\ \mu \in F^*\}$, $H_2 = \{h_2(\lambda) | \lambda \in F^*\}$, $H = H_1H_2$, *and let* $d = (5, q+1)$. *Then the following hold.*

(1) *The subgroups* H_1 *and* H_2 *are cyclic groups of order* $q^2 - 1$ *which are isomorphic to* F^*, *and we have*

$$h_1(\mu)h_1(\mu') = h_1(\mu\mu'), \qquad h_2(\lambda)h_2(\lambda') = h_2(\lambda\lambda').$$

(2) *The subgroup* H *is abelian of order* $\frac{1}{d}(q^2-1)^2$, *and every element of* H *can be expressed as a product* $h_1(\mu)\,h_2(\lambda)$. *In addition, we have* $h_1(\mu)h_2(\lambda) = 1 \Longleftrightarrow \mu^d = 1$ *and* $\lambda = \mu^2$.

If $d=1$, *then* $H=H_1\times H_2$. *Otherwise, if* $d=5$, *then* $H_1\cap H_2=\{\, h_1(\mu)\mid \mu\in F,\ \mu^5=1\,\}$ *and its order is* 5.

(2.8) *Let* $B=UH$. *Then* B *is the normalizer of* U *in* ${}^2A_4(q)$ *and the action of* $h=h_1(\mu)h_2(\lambda)$ *on* U *is given by*

$$h^{-1}x_1(\alpha)h=x_1(\mu^{-2}\lambda\alpha),\quad h^{-1}x_2(\beta,\gamma)h=x_2(\mu\lambda^{-2}\bar\lambda\beta,\ \mu\bar\mu\lambda^{-1}\bar\lambda^{-1}\gamma),$$
$$h^{-1}x_3(\delta)h=x_3(\mu^{-1}\bar\mu\bar\lambda^{-1}\delta),\ h^{-1}x_4(\xi,\eta)h=x_4(\mu^{-1}\lambda^{-1}\bar\lambda\xi,\ \mu^{-1}\bar\mu^{-1}\eta).$$

(2.9) *Let* $M=\langle n_1,n_2\rangle$, *and let* $N=HM$.

If q *is even, then* M *is a dihedral group of order* 8 *generated by two involutions* n_1 *and* n_2 *satisfying* $(n_1n_2)^4=1$ *and* $M\cap H=\{1\}$.

If q *is odd, then* M *is a non-abelian group of order 16 such that*

$$n_1^2=h_1(-1),\quad n_2^2=1,\quad (n_1n_2)^4=1,$$
$$Z(M)=\langle h_1(-1)\rangle\times\langle (n_1n_2)^2\rangle,\ M'=\langle [n_1,n_2]\rangle,\ M\cap H=\langle h_1(-1)\rangle.$$

The subgroup H *is normal in* N *and the action of* M *on* H *is given by*

$$n_1^{-1}h_1(\mu)n_1=h_1(\mu^{-1}),\qquad n_1^{-1}h_2(\lambda)n_1=h_1(\lambda)h_2(\lambda),$$
$$n_2h_1(\mu)n_2=h_1(\mu)h_2(\mu\bar\mu),\qquad n_2h_2(\lambda)n_2=h_2(\bar\lambda^{-1}).$$

In addition, there is an isomorphism of N/H *onto* W *which sends* n_1H *into* w_1 *and* n_2H *into* w_2, *and we have* $H=B\cap N$.

(2.10) *The elements* n_1 *and* n_2 *transform the elements of* U *in the following manners:*

$$n_1^{-1}x_1(\alpha)n_1=x_1(-\alpha^{-1})h_1(\alpha^{-1})n_1x_1(-\alpha^{-1}),\quad \alpha\in F^*;$$
$$n_2x_2(\beta,\gamma)n_2=x_2(\beta\gamma^{-1},\gamma^{-1})h_2(-\gamma^{-1})n_2x_2(\beta\bar\gamma^{-1},\gamma^{-1}),\quad \gamma\in F^*;$$
$$n_1^{-1}x_2(\beta,\gamma)n_1=x_4(\beta,\gamma),\quad n_1^{-1}x_3(\delta)n_1=x_3(\bar\delta),$$
$$n_1^{-1}x_4(\xi,\eta)n_1=x_2(-\xi,\eta),\quad n_2x_1(\alpha)n_2=x_3(\alpha),$$
$$n_2x_3(\delta)n_2=x_1(\delta),\quad n_2x_4(\xi,\eta)n_2=x_4(-\xi,\eta).$$

(2.11) *The subgroups* B *and* N *form a* (B,N)*-pair of* ${}^2A_4(q)$. *The group* ${}^2A_4(q)$ *is the disjoint union of eight double cosets of the form* BnB, *where* n *runs through the transversal*

$$\overline{N}=\{1,\ n_1,\ n_2,\ n_1n_2,\ n_2n_1,\ n_1n_2n_1,\ n_2n_1n_2,\ (n_1n_2)^2\}$$

of H *in* N. *In fact, we have* $BnB=BnU_n$, *where* U_n *is the subgroup of* U *given in the following table.*

n :	1	n_1	n_2	n_1n_2	n_2n_1	$n_1n_2n_1$	$n_2n_1n_2$	$(n_1n_2)^2$
U_n :	1	U_1	U_2	U_2U_3	U_1U_4	$U_1U_3U_4$	$U_2U_3U_4$	U

Moreover, U_n *is a complement of* $U\cap n^{-1}Un$ *in* U.

Each element of $^2A_4(q)$ *can be uniquely expressed as a product* bny, *where* $b \in B$, $n \in \overline{N}$ *and* $y \in U_n$.

(2.12) *There exists an isomorphism of* $^2A_4(q)$ *onto the unitary group* $U_5(q) = SU_5(q^2)/Z$ *which sends*

$$x_1(\alpha) \text{ into } \begin{pmatrix} 1 & \alpha & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{\alpha} \\ & & & & 1 \end{pmatrix} Z, \quad x_2(\beta,\gamma) \text{ into } \begin{pmatrix} 1 & & & & \\ & 1 & \beta & -\bar{\gamma} & \\ & & 1 & \bar{\beta} & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} Z,$$

$$x_3(\delta) \text{ into } \begin{pmatrix} 1 & & \delta & & \\ & 1 & & & \bar{\delta} \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} Z, \quad x_4(\xi,\eta) \text{ into } \begin{pmatrix} 1 & & -\xi & & \bar{\eta} \\ & 1 & & & \\ & & 1 & & \bar{\xi} \\ & & & 1 & \\ & & & & 1 \end{pmatrix} Z,$$

$$h_1(\mu)h_2(\lambda) \text{ into } \begin{pmatrix} \mu & & & & \\ & \mu^{-1}\lambda & & & \\ & & \lambda^{-1}\bar{\lambda} & & \\ & & & \bar{\mu}\bar{\lambda}^{-1} & \\ & & & & \bar{\mu}^{-1} \end{pmatrix} Z,$$

$$n_1 \text{ into } \begin{pmatrix} & 1 & & & \\ -1 & & & & \\ & & 1 & & \\ & & & & 1 \\ & & & -1 & \end{pmatrix} Z, \quad n_2 \text{ into } \begin{pmatrix} 1 & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix} Z,$$

where Z *is the center of* $SU_5(q^2)$.

3. Parabolic subgroups and centralizers of elements of order p

From now on we identify $U_5(q)$ with the twisted Chevalley group $^2A_4(q)$, where $q = p^m$ and p is a prime. The letters denoting the elements or the specific subgroups of $U_5(q)$ which are introduced in section 2 will keep their meaning throughout this paper.

In this section we will discuss some properties of several subgroups contained in maximal parabolic subgroups of $U_5(q)$. We also explicitly determine the elements of $U_5(q)$ of order p and the centralizers of some elements of order p.

(3.1) *Let*

$$B_1 = B \cup Bn_1U_1, \quad B_2 = B \cup Bn_2U_2.$$

Then B_1 *and* B_2 *are maximal parabolic subgroups of* $U_5(q)$. *And every parabolic subgroups of* $U_5(q)$ *is conjugate to either* B, B_1, B_2 *or* $U_5(q)$.

Proof. This follows from the (B,N)-pair structure of $U_5(q)$.

(3.2) *Let*

$$B_1 = B \cup Bn_1U_1, \ P_1 = UH_1 \cup UH_1n_1U_1, \ P_2 = U_1H_1 \cup U_1H_1n_1U_1.$$

Then $B_1 \supset P_1 \supset P_2$ *and we have the following.*

(1) B_1 *has a maximal normal p-subgroup* $U_2U_3U_4$ *of order* q^8. *The subgroup* $U_1H \cup U_1Hn_1U_1$ *is a complement of* $U_2U_3U_4$ *in* B_1.

(2) P_1 *is a normal subgroup of* B_1 *such that* $B_1 = P_1H_2$ *and* $P_1 \cap H_2 = H_1 \cap H_2$. *Thus* $B_1/P_1 \cong H_2/H_1 \cap H_2$, *and so* B_1/P_1 *is cyclic of order* $\frac{1}{d}(q^2-1)$, *where* $d = (5, q+1)$.

(3) P_2 *is a complement of a normal subgroup* $U_2U_3U_4$ *in* P_1.

Proof. The assertions follow from (2.5) $\sim$ (2.10). For the proof of (2), notice that

$$h_1(\mu)^{-1}n_1h_1(\mu) = h_1(\mu^{-2})n_1, \quad h_2(\lambda)^{-1}n_1h_2(\lambda) = h_1(\lambda)n_1.$$

(3.3) *Let*

$$B_2 = B \cup Bn_2U_2, \ Q_1 = UH_2 \cup UH_2n_2U_2, \ Q_2 = U_2H_2 \cup U_2H_2n_2U_2.$$

Then $B_2 \supset Q_1 \supset Q_2$ *and we have the following.*

(1) B_2 *has a maximal normal p-subgroup* $U_1U_3U_4$ *of order* q^7. *The subgroup* $U_2H \cup U_2Hn_2U_2$ *is a complement of* $U_1U_3U_4$ *in* B_2.

(2) Q_1 *is a normal subgroup of* B_2 *such that* $B_2 = Q_1H_1$ *and* $Q_1 \cap H_1 = H_1 \cap H_2$. *Thus* $B_2/Q_1 \cong H_1/H_1 \cap H_2 \cong H_2/H_1 \cap H_2 \cong B_1/P_1$, *and so* B_2/Q_1 *is cyclic of order* $\frac{1}{d}(q^2-1)$, *where* $d = (5, q+1)$. *Moreover,* Q_1 *is isomorphic to a subgroup of* $SU_5(q^2)$.

(3) Q_2 *is a complement of a normal subgroup* $U_1U_3U_4$ *in* Q_1, *and*

$$Q_2 \cong SU_3(q^2).$$

Proof. The assertion (1) and the first part of (3) are obtained from (2.5), (2.6), (2.8) and (2.10). Since

$$h_1(\mu)^{-1}n_2h_1(\mu) = h_2(\mu\bar{\mu})n_2, \quad h_2(\lambda)^{-1}n_2h_2(\lambda) = h_2(\lambda^{-1}\bar{\lambda}^{-1})n_2,$$

the first assertion of (2) follows from (2.5) $\sim$ (2.10), and so the second one follows from (2.7) and (3.2).

By (2.12), there exists a monomorphism of Q_1 into $SU_5(q^2)$ which sends $x_1(\alpha)$, $x_2(\beta,\gamma)x_3(\delta)x_4(\xi,\eta)$, $h_2(\lambda)$, n_2 into

$$\begin{pmatrix} 1 & \alpha & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \bar{\alpha} \\ & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -\xi & \delta & \bar{\eta} & \\ & 1 & \beta & -\bar{\gamma} & \beta\bar{\xi}+\bar{\delta} \\ & & 1 & \bar{\beta} & \bar{\xi} \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & & & & \\ & \lambda & & & \\ & & \lambda^{-1}\bar{\lambda} & & \\ & & & \bar{\lambda}^{-1} & \\ & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix},$$

respectively. Thus Q_1 is isomorphic to a subgroup of $SU_5(q^2)$. By using this monomorphism, we can show that there exists an isomorphism of Q_2 onto $SU_3(q^2)$ which sends $x_2(\beta,\gamma)$, $h_2(\lambda)$, n_2 into

$$\begin{pmatrix} 1 & \beta & -\bar{\gamma} \\ & 1 & \bar{\beta} \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & & \\ & \lambda^{-1}\bar{\lambda} & \\ & & \bar{\lambda}^{-1} \end{pmatrix}, \quad \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

respectively. Hence, we have $Q_2 \cong SU_3(q^2)$.

(3.4) *Let*

$$P = x_1(F_0)H_3 \cup x_1(F_0)H_3n_1x_1(F_0),$$
$$T = n_2Pn_2 = x_3(F_0)H_5 \cup x_3(F_0)H_5n_2n_1n_2x_3(F_0),$$

where $x_1(F_0) = \{x_1(\alpha) \mid \alpha \in F_0\}$, $x_3(F_0) = \{x_3(\delta) \mid \delta \in F_0\}$, $H_3 = \{h_1(\mu) \mid \mu \in F_0{}^*\}$ *and* $H_5 = n_2H_3n_2 = \{h_1(\mu)h_2(\mu^2) \mid \mu \in F_0{}^*\}$. *Then the following hold.*

(1) P *and* T *are subgroups of* $P_2 = U_1H_1 \cup U_1H_1n_1U_1$, *and we have*

$$P \cong T \cong SL_2(q^2) \cong SU_2(q^2).$$

(2) *Let* $E = PT$. *Then* E *is a subgroup of* $U_5(q)$, *and we have*

$$E = x_1(F_0)x_3(F_0)H_3H_5 \cup x_1(F_0)x_3(F_0)H_3H_5n_1x_1(F_0) \cup x_1(F_0)\cdot x_3(F_0)H_3H_5n_2n_1n_2x_3(F_0) \cup x_1(F_0)x_3(F_0)H_3H_5(n_1n_2)^2x_1(F_0)x_3(F_0).$$

In addition, $[P,T] = \{1\}$ *and* $Z(E) = Z(P) = Z(T) = P \cap T = H_3 \cap H_5 = \langle h_1(-1)\rangle$.

(3) *If q is even, then $E = P \times T$, $Z(E) = \{1\}$ and $|E| = q^2(q-1)^2(q+1)^2$. If q is odd, then E is the central product of P and T such that $Z(E) = \langle h_1(-1) \rangle$ and $|E| = \frac{1}{2}q^2(q-1)^2(q+1)^2$.*

Proof. Let

$$y(\alpha) = \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}, \quad k(\mu) = \begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix}, \quad m = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

where $\alpha \in F_0$ and $\mu \in F_0{}^*$. Then we can show that there exists an isomorphism of P onto $SU_2(q^2)$ which sends $x_1(\alpha)$, $h_1(\mu)$, n_1 into $y(\alpha)$, $k(\mu)$, m, respectively. The assertions (2) and (3) follow from (2.6) $\sim$ (2.10).

Compared to the subgroup P_2, $Q_2 = U_2H_2 \cup U_2H_2n_2U_2$ has a nice property as we can see in the next proposition. This proposition will be useful in proving (4.2).

For the remainder of this paper, F_1 denotes $\{\gamma \in F \mid \gamma + \bar{\gamma} = 0\}$.

(3.5) *Let*

$$Q = x_2(0, F_1)H_0 \cup x_2(0, F_1)h_2(F_1{}^*)n_2x_2(0, F_1),$$
$$R = U_4H_4 \cup U_4H_4n_1n_2n_1U_4,$$

where $x_2(0, F_1) = \{x_2(0, \gamma) \mid \gamma \in F_1\}$, $h_2(F_1^*) = \{h_2(\lambda) \mid \lambda \in F_1^*\}$, $H_0 = \{h_2(\lambda) \mid \lambda \in F_0{}^*\}$ and $H_4 = n_1{}^{-1}H_2n_1 = \{h_1(\mu)h_2(\mu) \mid \mu \in F^*\}$. Then the following hold.

(1) *The group Q is a subgroup of $Q_2 = U_2H_2 \cup U_2H_2n_2U_2$, and we have*

$$Q \cong SU_2(q^2) \cong SL_2(q^2), \quad \langle Q_2, n_1{}^{-1}Q_2n_1 \rangle = U_5(q).$$

(2) *R is a subgroup of $U_5(q)$ such that $R \cong Q_2 \cong SU_3(q^2)$.*

(3) *Let $D_2 = QR$. Then D_2 is a subgroup of $U_5(q)$, and we have*

$$D_2 = x_2(0,F_1)U_4H_0H_4 \cup x_2(0,F_1)U_4H_4h_2(F_1{}^*)n_2x_2(0,F_1) \cup x_2(0,F_1)\cdot U_4H_0H_4n_1n_2n_1U_4 \cup x_2(0,F_1)U_4H_4h_2(F_1{}^*)(n_1n_2)^2x_2(0,F_1)U_4.$$

Moreover, $[Q, R] = \{1\}$, $D_2 = Q \times R$ and $|D_2| = q^4(q^2-1)^2(q^3+1)$.

Proof. Choose any non-zero element ε in F_1. Then $\varepsilon F_0 = F_1$. Hence, every element γ' of F_1 is uniquely expressed as a product of the form $\gamma' = \varepsilon\gamma$ $(\gamma \in F_0)$. Set $n = h_2(\varepsilon)n_2$. And let

$$y_0(\gamma) = \begin{pmatrix} 1 & \gamma \\ & 1 \end{pmatrix}, \quad k_0(\lambda) = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \quad m_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

where $\gamma \in F_0$ and $\lambda \in F_0{}^*$. Then there exists an isomorphism of Q onto $SU_2(q^2)$ which sends $x_2(0, \varepsilon\gamma)$, $h_2(\lambda)$, n_1 into $y_0(\gamma)$, $k_0(\lambda)$, m_0, respectively. Thus, $Q \cong SU_2(q^2) \cong SL_2(q^2)$.

Observe that $n_1{}^{-1}Q_2n_1 = U_4H_4 \cup U_4H_4h_1(-1)n_1n_2n_1U_4$. Since $[U_2, U_4] = U_3$, $n_2U_3 = U_1n_2$, and since $H_2H_4 = H$, the subgroup $\langle\, Q_2,\ n_1{}^{-1}Q_2n_1 \,\rangle$ properly contains a maximal parabolic subgroup $B_2 = B \cup Bn_2U_2$. Hence, we have $\langle\, Q_2,\ n_1{}^{-1}Q_2n_1 \,\rangle = U_5(q)$ by (3.1).

Now, set $h(\mu) = h_1(\mu)h_2(\mu)$, $l = n_1n_2n_1$. And let

$$y(\xi,\eta) = \begin{pmatrix} 1 & \xi & -\bar{\eta} \\ & 1 & \bar{\xi} \\ & & 1 \end{pmatrix}, \quad k(\mu) = \begin{pmatrix} \mu & & \\ & \mu^{-1}\bar{\mu} & \\ & & \bar{\mu}^{-1} \end{pmatrix}, \quad m = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

where $\xi, \eta \in F, \eta + \bar{\eta} = -\xi\bar{\xi}$ and $\mu \in F^*$. Then there exists an isomorphism of R onto $SU_3(q^2)$ which sends $x_4(\xi,\eta)$, $h(\mu)$, l into $y(\xi,\eta)$, $k(\mu)$, m, respectively. Thus, $R \cong Q_2 \cong SU_3(q^2)$ by (3.3).

The assertion (3) follows from (2.6) $\sim$ (2.10).

We will now explicitly determine elements of order p in $U_5(q)$. Since U is a Sylow p-subgroup of $U_5(q)$, it suffices to find all elements of order p in U.

(3.6) *The following is the list of all elements of order p which are contained in the Sylow p-subgroup $U = U_1U_2U_3U_4$.*

(1) *the case when $p = 2$.*

(i) *every element of the form* $x_1(\alpha)x_3(\delta)x_4(\xi,\eta)$, *where* $\alpha \in F_0^*$, $\delta, \xi, \eta \in F$, $\alpha\bar{\delta} + \bar{\alpha}\delta + \xi\bar{\xi} = 0$ *and* $\eta + \bar{\eta} + \xi\bar{\xi} = 0$.

(ii) *every element of the form* $x_2(0,\gamma)x_3(\delta)x_4(0,\eta)$, *where* $\gamma \in F_0{}^*$ $\delta \in F$ *and* $\eta \in F_0$.

(iii) *every non-identity element of the form* $x_3(\delta)x_4(0,\eta)$, *where* $\delta \in F$ *and* $\eta \in F_0$.

(2) *the case when $p = 3$.*

(i) *every element of the form* $x_1(\alpha)x_3(\delta)x_4(\xi,\eta)$, *where* $\alpha \in F^*$, $\delta, \xi, \eta \in F$ *and* $\eta + \bar{\eta} + \xi\bar{\xi} = 0$.

(ii) *every non-identity element of the form* $x_2(0,\gamma)x_3(\delta)x_4(\xi,\eta)$, *where* $\gamma \in F_0$ $\delta, \xi, \eta \in F$ *and* $\eta + \bar{\eta} + \xi\bar{\xi} = 0$.

(3) *the case when* $p \geq 5$.
every non-identity element in U.

Proof. The proof of (3) is almost identical to that of the corresponding result in [3], and an easy calculation yields (1) and (2).

We now determine the centralizers of some elements of order p. The element $x_3(1)$ is clearly a non-central element in U of order p. Let ε be a fixed non-zero element such that $\varepsilon + \bar{\varepsilon} = 0$. Then $x_4(0, \varepsilon)$ is a central element in U of order p by (2.5) and (2.6). Two elements $x_3(1)$ and $x_4(0, \varepsilon)$ are not conjugate in $U_5(q)$. Hence, there are at least two conjugacy classes of elements of order p. In particular, if q is even, that is, if $p = 2$, then there are exactly two conjugacy classes of involuitons in $U_5(q)$ (*cf.* [6], p.255).

By (2.5) $\sim$ (2.10) and (3.3) $\sim$ (3.4), it is easy to prove the following two propositions.

(3.7) *Let* C_3 *be the centralizer of* $x_3(1)$ *in* $U_5(q)$ *and* $d = (5, q + 1)$. *Then we have the following.*

(1) C_3 *is a subgroup of* $B_1 = B \cup Bn_1U_1$, *and we have*

$$C_3 = x_1(F_0)U_2U_3U_4H_6 \cup x_1(F_0)U_2U_3U_4H_6n_1x_1(F_0),$$

$$| C_3 | = \frac{1}{d}q^9(q+1)(q^2-1),$$

where $H_6 = \{h_1(\mu)h_2(\mu\bar{\mu}^{-1}) \mid \mu \in F^*\}$.

(2) *Let* D_3 *be the normal closure of the Sylow* p*-subgroup* $x_1(F_0) \cdot U_2U_3U_4$ *in* C_3. *Then* C_3/D_3 *is cyclic of order* $\frac{1}{d}(q+1)$. *In fact,*

$$D_3 = x_1(F_0)U_2U_3U_4H_3 \cup x_1(F_0)U_2U_3U_4H_3n_1x_1(F_0),$$

where $H_3 = \{h_1(\mu) \mid \mu \in {F_0}^*\}$.

(3) *Let*

$$E_3 = U_2U_3U_4, \qquad P_3 = x_1(F_0)H_6 \cup x_1(F_0)H_6n_1x_1(F_0).$$

Then E_3 *is a maximal normal p-subgroup of* C_3 *and* P_3 *is a complement of* E_3 *in* C_3. *Moreover,* E_3 *is a maximal normal* p*-subgroup of* D_3 *and* $P = x_1(F_0)H_3 \cup x_1(F_0)H_3n_1x_1(F_0)$ *is a complement of* E_3 *in* D_3. *Thus,*

$$D_3/E_3 \cong P \cong SU_2(q^2).$$

(3.8) *Let* C_4 *be the centralizer of* $x_4(0, \varepsilon)$ *in* $U_5(q)$ *where* ε *is a fixed non-zero element in* F *such that* $\varepsilon + \bar{\varepsilon} = 0$, *and let* $d = (5, q+1)$. *Then we have the following.*

(1) *C_4 is a subgroup of $B_2 = B \cup Bn_2U_2$, and we have*

$$C_4 = UH_7H_2 \cup UH_7H_2n_2U_2, \qquad |C_4| = \frac{1}{d}q^{10}(q+1)(q^2-1)(q^3+1),$$

where $H_7 = \{h_1(\mu) \mid \mu \in F^*,\ \mu\bar{\mu} = 1\}$.

(2) *Let D_4 be the normal closure of the Sylow p-subgroup U in C_4. Then C_4/D_4 is cyclic of order $\frac{1}{d}(q+1)$. In fact,*

$$D_4 = UH_2 \cup UH_2n_2U_2.$$

(3) *Let*

$$E_4 = U_1U_3U_4, \qquad Q_4 = U_2H_7H_2 \cup U_2H_7H_2n_2U_2.$$

Then E_4 is a maximal normal p-subgroup of C_4 and Q_4 is a complement of E_4 in C_4. Moreover, E_4 is a maximal normal p-subgroup of D_4 and $Q_2 = U_2H_2 \cup U_2H_2n_2U_2$ is a complement of E_4 in D_4. Thus,

$$D_4/E_4 \cong Q_2 \cong SU_3(q^2).$$

4. Involutions and their centralizers

An important insight into the structure of a finite simple group has been obtained by studying that of centralizers for involutions.

In this section we will explicitly determine involutions of $U_5(q)$ and the centralizers of involutions in $U_5(q)$.

(4.1) *Assume that q is odd, and let C_1 be the centralizer of the involution $h_1(-1)$ in $U_5(q)$ and $d = (5, q+1)$. Then the following hold.*

(1) *We have*

$$\begin{aligned} C_1 = {} & U_1x_2(0,F_1)U_3x_4(0,F_1)H \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)Hn_1U_1 \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)Hn_2x_2(0,F_1) \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)Hn_1n_2x_2(o,F_1)U_3 \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)Hn_2n_1U_1x_4(0,F_1) \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)Hn_1n_2n_1U_1U_3x_4(0,F_1) \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)Hn_2n_1n_2x_2(0,F_1)U_3x_4(0,F_1) \\ & \cup U_1x_2(0,F_1)U_3x_4(0,F_1)H(n_1n_2)^2U_1x_2(0,F_1)U_3x_4(0,F_1), \end{aligned}$$

$$|C_1| = \frac{1}{d}q^6(q^2-1)^2(q+1)(q^2+1)(q^3+1),$$

where $x_4(0,F_1) = \{x_4(0,\eta) \mid \eta \in F_1\}$.

(2) *Let D_1 be the normal closure of the Sylow p-subgroup $S_1 =$*

$U_1x_2(0,F_1)U_3x_4(0,F_1)$ *in* C_1. *Then* $D_1 \cong SU_4(q^2)$ *and* C_1/D_1 *is cyclic of order* $\frac{1}{d}(q+1)$. *In fact,*

$$\begin{aligned}D_1 &= U_1x_2(0,F_1)U_3x_4(0,F_1)H_1H_0\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1H_0n_1U_1\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1h_2(F_1{}^*)n_2x_2(0,F_1)\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1h_2(F_1^*)n_1n_2x_2(0,F_1)U_3\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1h_2(F_1{}^*)n_2n_1U_1x_4(0,F_1)\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1h_2(F_1{}^*)n_1n_2n_1U_1U_3x_4(0,F_1)\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1H_0n_2n_1n_2x_2(0,F_1)U_3x_4(0,F_1)\\ &\cup U_1x_2(0,F_1)U_3x_4(0,F_1)H_1H_0(n_1n_2)^2U_1x_2(0,F_1)U_3x_4(0,F_1),\end{aligned}$$

where $H_0 = \{h_2(\lambda) \mid \lambda \in F_0^*\}$.

Proof. By (2.8) and (2.9), it is easy to prove that the centralizer of $h_1(-1)$ in U is $U_1x_2(0,F_1)U_3x_4(0,F_1)$ and N is centralized by $h_1(-1)$. From these facts and (2.11), we obtain the result (1).

Since D_1 is a normal subgroup of C_1 containing S_1, we have

$$D_1 \supset U_1(n_1{}^{-1}U_1n_1)U_1 \supset H_1n_1,$$
$$D_1 \supset x_2(0,F_1)\big(n_2x_2(0,F_1)n_2\big)x_2(0,F_1) \supset h_2(F_1{}^*)n_2$$

by (2.10). On the other hand, by (2.7) $\sim$ (2.8), $\langle H_1n_1,\ h_2(F_1^*)n_2\rangle = H_1H_0 \cup H_1H_0Ln_1 \cup H_1h_2(F_1^*)n_2 \cup H_1h_2(F_1^*)n_1n_2 \cup H_1h_2(F_1^*)n_2n_1 \cup H_1h_2(F_1^*)n_1n_2n_1 \cup H_1H_0n_2n_1n_2 \cup H_1H_0(n_1n_2)^2$. Thus, the last assertion of (2) holds by (2.5) $\sim$ (2.11). Moreover, this gives $C_1 = D_1H$, $D_1 \cap H = H_1H_0$. Hence, $C_1/D_1 \cong H/H_1H_0$, and so C_1/D_1 is cyclic of order $\frac{1}{d}(q+1)$.

Let ε be a fixed non-zero element in F_1. As noted in the proof of (3.5), every element γ' of F_1 is uniquely expressed as a product of the form $\gamma' = \varepsilon\gamma$ $(\gamma \in F_0)$. Set $l_1 = n_1$, $l_2 = h_2(\varepsilon)n_2$. And let

$$y_1(\alpha) = \begin{pmatrix}1 & \alpha & & \\ & 1 & & \\ & & 1 & \bar\alpha \\ & & & 1\end{pmatrix},\quad y_2(\gamma)y_3(\delta)y_4(\eta) = \begin{pmatrix}1 & & \delta & -\eta \\ & 1 & \gamma & -\bar\delta \\ & & 1 & \\ & & & 1\end{pmatrix},$$

$$k_1(\mu) = \begin{pmatrix}\mu & & & \\ & \mu^{-1} & & \\ & & \bar\mu & \\ & & & \bar\mu^{-1}\end{pmatrix},\qquad k_2(\lambda) = \begin{pmatrix}1 & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & 1\end{pmatrix},$$

$$m_1 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \qquad m_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix},$$

where $\alpha, \delta \in F$, $\mu \in F^*$, $\gamma, \eta \in F_0$ and $\lambda \in F_0^*$. Then we can show that there is an isomorphism of D_1 onto $SU_4(q^2)$ which sends $x_1(\alpha)$, $x_2(0, \varepsilon\gamma)x_3(\varepsilon\delta)x_4(0, \varepsilon\eta)$, $h_1(\mu)$, $h_2(\lambda)$, l_1, l_2 into $y_1(\alpha)$, $y_2(\gamma)y_3(\delta)y_4(\eta)$, $k_1(\mu)$, $k_2(\lambda)$, m_1, m_2, respectively (*cf.* [3]). Thus, the assertion (2) holds.

(4.2) *Assume that q is odd, and let C_2 be the centralizer of the involution $h_2(-1)$ in $U_5(q)$ and $d = (5, q+1)$. Then the following hold.*

(1) *We have*

$$C_2 = x_2(0,F_1)U_4H \cup x_2(0,F_1)U_4Hn_2x_2(0,F_1) \cup x_2(0,F_1)U_4Hn_1n_2n_1U_4 \cup x_2(0,F_1)U_4H(n_1n_2)^2x_2(0,F_1)U_4,$$

$$| C_2 | = \frac{1}{d}q^4(q^2-1)^2(q+1)(q^3+1).$$

(2) *Let D_2 be the normal closure of the Sylow p-subgroup $S_2 = x_2(0,F_1)U_4$ in C_2. Then $D_2 = Q \times R$, $Q \cong SU_2(q^2)$, $R \cong SU_3(q^3)$ and C_2/D_2 is cyclic of order $\frac{1}{d}(q+1)$. In fact,*

$$D_2 = x_2(0,F_1)U_4H_4H_0 \cup x_2(0,F_1)U_4H_4h_2(F_1^*)n_2x_2(0,F_1) \cup x_2(0,F_1)\cdot U_4H_4H_0n_1n_2n_1U_4 \cup x_2(0,F_1)U_4H_4h_2(F_1^*)(n_1n_2)^2x_2(0,F_1)U_4,$$

$$Q = x_2(0,F_1)H_0 \cup x_2(0,F_1)h_2(F_1^*)n_2x_2(0,F_1),$$

$$R = U_4H_4 \cup U_4H_4n_1n_2n_1U_4,$$

where $H_0 = \{h_2(\lambda) \mid \lambda \in F_0^*\}$ and $H_4 = \{h_1(\lambda)h_2(\lambda) \mid \lambda \in F^*\}$.

Proof. By (2.8), the centralizer of $h_2(-1)$ in U is $x_2(0,F_1)U_4$ and the centralizer of this involution in N is $H\langle n_2, n_1n_2n_1\rangle$. From these facts and (2.11), we obtain the result (1).

Note that

$$(n_1n_2n_1)^{-1}x_2(0,\gamma)(n_1n_2n_1) = x_2(0,\gamma), \quad (n_1n_2n_1)^{-1}x_4(\xi,\eta)(n_1n_2n_1) = x_4(-\xi\eta^{-1},\eta^{-1})h_2(-1)h_1(\eta^{-1})h_2(\eta^{-1})n_1n_2n_1x_4(-\xi\bar{\eta}^{-1},\eta^{-1}),$$

where $\gamma \in F_1$, $\xi, \eta \in F$ and $\eta + \bar{\eta} = -\xi\bar{\xi}$. Hence, it follows from (2.10) that

$$D_2 \supset x_2(0,F_2)\big(n_2x_2(0,F_1)n_2\big)x_2(0,F_1) \supset h_2(F_1{}^*)n_2,$$

$$D_2 \supset U_4\big((n_1n_2n_1)^{-1}U_4(n_1n_2n_1)\big)U_4 \supset h_2(-1)H_4n_1n_2n_1.$$

On the other hand, by (2.7) and (2.10),

$$\langle h_2(F_1^*)n_2,\ h_2(-1)H_4n_1n_2n_1 \rangle$$
$$= H_4H_0 \cup H_4h_2(F_1^*)n_2 \cup H_4H_0n_1n_2n_1 \cup H_4h_2(F_1^*)(n_1n_2)^2.$$

Thus, we have the last assertion of (2). Moreover, this gives $C_2 = D_2H$, $D_2 \cap H = H_4H_0$. Hence $C_2/D_2 \cong H/H_4H_0$, and so C_2/D_2 is cyclic of order $\frac{1}{d}(q+1)$. On the other hand, $D_2 = Q \times R$, $Q \cong SU_2(q^2)$, $R \cong SU_3(q^3)$ by (3.5). Thus the assertion (2) holds.

(4.3) *Assume that q is odd. Then there are exactly two conjugacy classes of involutions in $U_5(q)$ with representatives $h_1(-1)$ and $h_2(-1)$. Thus, the centralizer of any involution in $U_5(q)$ is conjugate in $U_5(q)$ to either C_1 or C_2.*

Proof. Let $d = (5, q+1)$ and let 2^m be the highest power of 2 dividing $q-1$, that is, $2^m \parallel q-1$. Then it is easy to show that

$$2^{2m+5} \parallel |N|, \quad 2^{2m+5} \parallel |U_5(q)|.$$

Hence, any Sylow 2-subgroup of N is a Sylow 2-subgroup of $U_5(q)$, and so every involution of $U_5(q)$ is conjugate in $U_5(q)$ to an involution in N.

On the other hand, we have the following list of all involutions in N by (2.7) and (2.9).

(I) $h_1(-1)$;
$h_1(-\lambda^{-1}\bar\lambda)h_2(\lambda)n_2, \quad \lambda \in F^*,\ (\lambda^{-1}\bar\lambda)^d = 1$;
$h_1(\lambda^2\bar\lambda^{-1})h_2(\lambda)n_1n_2n_1, \quad \lambda \in F^*,\ (\lambda^{-1}\bar\lambda)^d = 1$.

(II) $h_2(-1)$;
$h_1(\mu)h_2(-\mu^2), \quad \mu \in F^*,\ \mu^d = -1$;
$h_1(\mu)h_2(\lambda)n_1, \quad \mu, \lambda \in F^*,\ \lambda^d = -1$;
$h_1(\lambda^{-1}\bar\lambda)h_2(\lambda)n_2, \quad \lambda \in F^*,\ (\lambda^{-1}\bar\lambda)^d = 1$;
$h_1(-\lambda^2\bar\lambda^{-1})h_2(\lambda)n_1n_2n_1, \quad \lambda \in F^*,\ (\lambda^{-1}\bar\lambda)^d = 1$;
$h_1(\mu)h_2(\mu\bar\mu\bar\lambda^{-1})n_2n_1n_2, \quad \mu, \lambda \in F^*,\ \lambda^d = -1$;
$h_1(\mu)h_2(\mu^2\xi_0{}^{-1})(n_1n_2)^2, \quad \mu \in F^*,\ (\mu^{-1}\bar\mu)^d = 1$
and ξ_0 is a primitive $(q-1)$-th root of unity in F^*.

Now we can show that involutions in (I) are conjugate in $U_5(q)$ to $h_1(-1)$ and involutions in (II) are conjugate to $h_2(-1)$.

(4.4) *Assume that q is even. Then there are exactly two conjugacy classes of involutions in $U_5(q)$ with representatives $x_3(1)$ and*

$x_4(0,1)$. *Thus the centralizer of any involution in* $U_5(q)$ *is conjugate in* $U_5(q)$ *to either* C_3 *or* C_4 *in (3.7) and (3.8).*

Proof. Since any Sylow 2-subgroup of U is a Sylow 2-subgroup of $U_5(q)$, every involution of $U_5(q)$ is conjugate in $U_5(q)$ to an involution in U.

On the other hand, the following is the list of all involutions in U.

(I) $x_1(\alpha)x_3(\delta)x_4(\xi,\eta)$, $\alpha \in F_0^*$, $\delta,\xi,\eta \in F$, $\alpha\bar{\delta}+\bar{\alpha}\delta+\xi\bar{\xi}=0$, $\eta+\bar{\eta}+\xi\bar{\xi}=0$;
$x_2(0,\gamma)x_3(\delta)x_4(0,\eta)$, $\gamma \in {F_0}^*$, $\delta \in F$, $\eta \in F_0$, $\delta\bar{\delta} \neq \gamma\eta$;
$x_3(\delta)x_4(0,\eta)$, $\delta \in F^*$, $\eta \in F_0$.

(II) $x_2(0,\gamma)x_3(\delta)x_4(0,\eta)$, $\gamma \in {F_0}^*$, $\delta \in F$, $\eta \in F_0$, $\delta\bar{\delta}=\gamma\eta$;
$x_4(0,\eta)$, $\eta \in {F_0}^*$.

Now we can show that involutions in (I) are conjugate in $U_5(q)$ to $x_3(1)$ and involutions in (II) are conjugate to $x_4(0,1)$.

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