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EXISTENCE OF EQUILIBRIUM IN NON–COMPACT SETS AND ITS APPLICATION

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1. Introduction

In the last twenty years, the classical Arrow-Debreu result [1] on the existence of Walrasian equilibria has been generalized in many directions. Mas-Colell [15] has first shown that the existence of equilibrium can be established without assuming preferences to be total or transitive. Next, by using a maximal element existence theorem, Gale and Mas-Colell [9] gave a proof of the existence of a competitive equilibrium without ordered preferences. By using Kakutani's fixed point theorem, Shafer and Sonnenschein [19] proved the powerful result on 'the Arrow-Debreu lemma for abstract economies' for the case where preferences may not be total or transitive but has open graph. In fact, they imposed stronger condition on the preference correspondences and weaker condition on the feasible correspondences.

On the other hand, Borglin and Keiding [3] proved a new existence theorem for a compact abstract economy with KF-majorized preference correspondences, which are weaker than Shafer-Sonnenschein's assumption. Following their ideas, till now there have been a number of generalizations of the existence of equilibria for compact abstract economies(e.g. Toussaint [21], Tulcea [22] and Yannelis-Prabhakar [23]), and recently Ding-Kim-Tan [7] extended Borglin-Keiding's result [3] to more general preferences and infinite number of agents.

As in [3,9,19,23], in most results on the existence of equilibria for abstract economies, the underlying spaces (commodity spaces or choice sets) are always compact and convex. However, in recent papers [7,11, 20] the underlying spaces are not always compact nor paracompact. Moreover it should be noted that we will encounter many kinds of preferences in various economic situations; so it is important that we

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shall consider several types of preferences and obtain some existence results for such correspondences in non-compact (or non-paracompact) settings.

In [4], Chang proved a generalization which combines the results of both of the above types (i.e. Shafer-Sonnenschein [19] and Yannelis-Prabhakar [23]), and his commodity spaces are locally convex and the agents are uncountable. In a recent paper [13], Kim-Lee-Park obtained two equilibrium existence theorems in non-compact sets by using Himmelberg's fixed point theorem and proved a generalization of Schafer-Sonnenschein's result under weak constraint correspondences and also obtain a quasi-variational inequality as an application.

The purpose of this paper is two-fold. First, we prove two general equilibrium existence theorems in non-compact abstract economies with general preference correspondences by using Himmelberg's fixed point theorem, which includes the previous equilibrium existence results due to Shafer-Sonnenschein [19], Borglin-Keiding [3], Yannelis-Prabhakar [23], Chang [4] and Kim-Lee-Park [13]. Second we shall prove a non-compact quasi-variational inequality by using an equilibrium existence theorem.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by $cl\,A$ the closure of A in X. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. If A is a non-empty subset of a topological vector space X and $S, T : A \to 2^X$ are correspondences, then $co\,T, cl\,T, T \cap S : A \to 2^X$ are correspondences defined by $(co\,T)(x) = co\,T(x), (cl\,T)(x) = cl\,T(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a correspondence. The correspondence T is said to be *open* or have *open graph* if the graph of $T (= \{(x, T(x)) \in X \times Y : x \in X\})$ is open in $X \times Y$. We may call $T(x)$ the *upper section* of T and $T^{-1}(y)$ the *lower section* of T. It is easy to check that if T has open graph, then the upper and lower sections of T are open ; however the converse is not true in general (see [21, p.104]). A correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X

such that $T(y) \subset V$ for each $y \in U$; and a correspondence $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, then there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.

Finally we recall the following general definitions of equilibrium theory in mathematical economics. Let I be a finite or an infinite set of agents. For each $i \in I$, let X_i be a non-empty set of actions. An *abstract economy* (or *generalized game*) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i) where X_i is a non-empty topological vector space (a choice set), $A_i, B_i : \Pi_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and $P_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\hat{x} \in X = \Pi_{i \in I} X_i$ such that for each $i \in I, \hat{x}_i \in cl\, B_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. When $A_i = B_i$ for each $i \in I$, our definitions of an abstract economy and an equilibrium coincide with the standard definition of Shafer-Sonnenschein [19] or in [3,21,23].

Now we shall need the following lemmas.

LEMMA 1 [23]. *Let X, Y be topological vector spaces and $T : X \to 2^Y$ be a correspondence with open lower sections. Then the correspondence $co\, T$ has also open lower sections.*

LEMMA 2 [23]. *Let X, Y be two topological spaces and $T : X \to 2^Y$ has open lower sections. Then the correspondence T is lower semicontinuous.*

Finally, the following continuous selection theorem is essential in proving our main result:

LEMMA 3 [23]. *Let X be a non-empty paracompact Hausdorff topological space and Y be a Hausdorff topological vector space. Let $T : X \to 2^Y$ be a correspondence such that each $T(x)$ is non-empty convex and for each $y \in Y$, $T^{-1}(y)$ is open in X. Then T has a continuous selection, i.e. there exists a continuous map $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$.*

3. Equilibrium existence theorems and its application

First we prove the following non-compact equilibrium existence theorem which is a slight generalization of Theorem 1 in [13]:

THEOREM 1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy where I is a (possibly uncountable) set of agents such that for each $i \in I$,*

(1) X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space and D_i is a non-empty compact subset of X_i,

(2) for each $x \in X = \Pi_{i\in I} X_i$, $A_i(x)$ is non-empty and $A(x) \subset B_i(x)$,

(3) the correspondence $cl\, B_i : X \to 2^{D_i}$ is upper semicontinuous such that $B_i(x)$ is convex for each $x \in X$,

(4) for each $y \in D_i$, $(A_i \cap P_i)^{-1}(y)$ is open in X,

(5) for each $x \in X$, $x_i \notin co\, P_i(x)$,

(6) the set $W_i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is paracompact.

Then Γ has an equilibrium choice $\hat{x} \in X$, i.e. for each $i \in I$, $\hat{x}_i \in cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. By the assumption (4) and Lemma 1, the correspondence $co(A_i \cap P_i)$ has also open lower sections and the set $W_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\} = \cup_{y\in D_i}(A_i \cap P_i)^{-1}(y)$ is open in X. Then the correspondence $co(A_i \cap P_i) : W_i \to 2^{D_i}$ satisfies the whole assumptions of Lemma 3, and hence there exists a continuous selection f_i of $co(A_i \cap P_i)$ on W_i, i.e. $f_i(x) \in co(A_i \cap P_i)(x)$ for each $x \in W_i$. Define a correspondence $\phi_i : X \to 2^{D_i}$ by

$$\phi_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } \ x \in W_i, \\ cl\, B_i(x), & \text{if } \ x \notin W_i. \end{cases}$$

Then for each $x \in X$, $\phi_i(x)$ is a non-empty closed convex subset of D_i. To show ϕ_i is upper semicontinuous, we must show that the set $U := \{x \in X : \phi_i(x) \subset V\}$ is open in X for every open subset V of D_i. Since $f_i(x) \subset co\, A_i(x) \subset cl\, B_i(x)$ for each $x \in W_i$, we have

$$\begin{aligned} U &= \{x \in X : \phi_i(x) \subset V\} \\ &= \{x \in W_i : \phi_i(x) \subset V\} \cup \{x \in X \setminus W_i : \phi_i(x) \subset V\} \\ &= \{x \in W_i : f_i(x) \in V\} \cup \{x \in X \setminus W_i : cl\, B_i(x) \subset V\} \\ &= \{x \in W_i : f_i(x) \in V\} \cup \{x \in X : cl\, B_i(x) \subset V\}. \end{aligned}$$

It follows from the upper semicontinuity of $cl\,B_i$ that the set $\{x \in X : cl\,B_i(x) \subset V\}$ is open in X and by the continuity of f_i, the set $\{x \in W_i : f_i(x) \in V\}$ is open in W_i and hence is open in X ; thus U is open in X and ϕ_i is upper semicontinuous.

Finally we define $\Phi : X \to 2^D$, where $D = \Pi_{i\in I} D_i$, by

$$\Phi(x) := \Pi_{i\in I}\, \phi_i(x) \quad \text{for each } x \in X.$$

Then each $\Phi(x)$ is a non-empty closed convex subset of D and Φ is also upper semicontinuous by Lemma 3 in [8]. Therefore, by Himmelberg's fixed point theorem [10], there exists $\hat{x} \in X$ such that $\hat{x} \in \Phi(\hat{x})$. Therefore by the assumption (5), for each $i \in I$, we have $\hat{x}_i \in cl\,B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. This completes the proof.

REMARKS. (i) It should be noted that if X_i is metrizable for each $i \in I$ and I is countable, then the set $X = \Pi_{i\in I} X_i$ is also metrizable and hence X is perfectly normal. Then by the assumption (4), the set $W_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\} = \cup_{y\in D_i}(A_i \cap P_i)^{-1}(y)$ is open in X. Hence W_i is an F_σ-set and so W_i is paracompact. Thus the assumption (6) of Theorem 1 is automatically satisfied. In this case, Theorem 1 is a non-compact generalization of Theorem 3.1 of Chang [4] using hypothesis (5′). In fact, Theorem 1 generalizes Theorem 3.1 [4] in three aspects; i.e. the commodity space X_i need not be a compact subset of a finite dimensional space and the set I of agents need not be countable.

(ii) Recently, Tian [20] proved a non-compact generalization of Theorem 6.1 of Yannelis-Prabhakar [23] using some compact convex assumptions (vii) 1–3 ; however these are very complicated and we may only assume the range of the correspondence $cl\,B_i$ is contained in a non-empty compact subset D_i of X_i for each $i \in I$ and the paracompact assumption (6).

(iii) Theorem 1 further generalizes Theorem 1 in [13] in two aspects, i.e. the choice set X_i need not be metrizable and the set of agent I need not be countable.

In Theorem 1, the condition (4) is weaker than the corresponding open lower section assumptions in [6, Theorem 4]. In fact, we can give a simple example of a non-compact 1-person game which Theorem 6.1 in [23], Theorem in [19] or Theorem 4 in [6] can not be applied to this setting :

EXAMPLE. Let $X = [0, \infty)$ be the non-compact choice set and the correspondences $A = B, P : X \to 2^X$ be defined as follows :

$$A(x) := [0,2), \qquad \text{for each } x \in X,$$

$$P(x) := \begin{cases} (x, x + \dfrac{2}{1+x}), & \text{for each } x \in [0,2), \\ \{x + \dfrac{2}{1+x}\}, & \text{for each } x \in [2,\infty). \end{cases}$$

Then all hypotheses of Theorem 1 are satisfied; in fact the set $W_i = [0,2)$ is open, and hence paracompact. And the image of $cl\,B$ is contained in a compact set $[0,2]$. Therefore by Theorem 1, we can obtain an equilibrium point $2 \in X$ such that $2 \in cl\,A(2)$ and $A(2) \cap P(2) = \emptyset$.

When the commodity space X_i is compact convex and $A_i = B_i$ for each $i \in I$, we can obtain a generalization of Theorem 6.1 in [23] as a corollary:

COROLLARY. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy where I is a (possibly uncountable) set of agents such that for each $i \in I$,*

(1) X_i is a non-empty compact convex subset of a locally convex Hausdorff topological vector space,

(2) for each $x \in X = \Pi_{i \in I} X_i$, $A_i(x)$ is non-empty convex subset of X_i,

(3) the correspondence $cl\,A_i : X \to 2^{X_i}$ is upper semicontinuous,

(4) for each $y \in D_i$, $(A_i \cap P_i)^{-1}(y)$ is open in X,

(5) for each $x \in X, x_i \notin co\,P_i(x)$,

(6) the set $\{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is paracompact.

Then Γ has an equilibrium.

We now use the following notation as in [22]. Let X be a Hausdorff topological space, E a Hausdorff topological vector space and $Y \subset E$. For a correspondence $T : X \to 2^E$ and every set $U \subset E$, we define the correspondence $T_U : X \to 2^Y$ by

$$T_U(x) := (T(x) + U) \cap Y \qquad \text{for each } x \in X.$$

The following is very useful in relaxing the assumption of having open graph property.

LEMMA 4 [4,22]. *Let X be a Hausdorff topological space, E a Hausdorff topological vector space and $Y \subset E$. If the correspondence $T : X \to 2^E$ is lower semicontinuous and $U \subset E$ is non-empty open, then the correspondence T_U has open graph in $X \times Y$.*

We now prove the following

LEMMA 5. *Let X be a Hausdorff topological space, E a Hausdorff topological vector space and $Y \subset E$. If the correspondence $T : X \to 2^E$ is upper semicontinuous and U is a non-empty open balanced neighborhood of 0 in E, then the correspondence $T_U : X \to 2^Y$ is also upper semicontinuous.*

Proof. For any $x \in X$, let V be an arbitrary open subset in Y containing $T_U(x)$. Then there exists an open subset W of E such that $T(x)+U \subset W$ and $T_U(x) = (T(x)+U)\cap Y \subset V = W\cap Y$ since $T(x)+U$ is open in E. Since U is balanced, $T(x) \subset W - U = W + U$. Since T is upper semicontinuous and $W + U$ is an open set containing $T(x)$, there exists an open neighborhood N of x in X such that $T(y) \subset W + U$ for each $y \in N$. Then for each $y \in N$, $T(y) - U = T(y) + U \subset W$, so that $T_U(y) = (T(y) + U) \cap Y \subset W \cap Y = V$. Therefore T_U is upper semicontinuous at x. This completes the proof.

Recall that a topological space is called *perfectly normal* if it is normal and each open subset is an F_σ-set, and note that if A has open lower sections, then A is lower semicontinuous; however the converse is not true in general (e.g. see [23, Remark 4.1]).

Using Lemmas 4 and 5, we can prove the following generalization of Theorem 1 by relaxing the open lower section assumption on A_i :

THEOREM 2. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy where I is a (possibly uncountable) set of agents such that for each $i \in I$,*

(1) X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space and D_i is a non-empty compact subset of X_i,

(2) for each $x \in X = \Pi_{i\in I}X_i$, $A_i(x)$ is non-empty and $A(x) \subset B_i(x)$,

(3) the correspondence $B_i : X \to 2^{D_i}$ is upper semicontinuous,

(4) there exists a convex open neighborhood V_1 of 0 in E such that for each $x \in X$, $(B_i(x) + V_1) \cap D_i$ is convex,

(5) *for each* $y \in D_i$, $P_i^{-1}(y)$ *is open in* X,
(6) *the correspondence* $A_i : X \to 2^{D_i}$ *is lower semicontinuous*,
(7) *for each* $x \in X$, $x_i \notin co\, P_i(x)$.

If X *is perfectly normal, then* Γ *has an equilibrium choice* $\hat{x} \in X$, *i.e. for each* $i \in I$,
$\hat{x}_i \in cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. Since V_1 is a convex open neighborhood of 0, there exists a balanced convex open neighborhood W of 0 such that $W \subset V_1$. Let $\mathcal{U}$ be a local basis of balanced convex open neighborhood of 0 in E and let $U \in \mathcal{U}$. From now on, we simply denote a balanced convex open neighborhood $U \cap W$ by U for each $U \in \mathcal{U}$ without any confusion. Then by the assumption (6) and Lemma 4, the correspondence $(A_U)_i : X \to 2^{D_i}$, defined by $(A_U)_i(x) = (A_i(x) + U) \cap D_i$ for each $x \in X$, has open graph in $X \times D_i$, and by the assumption (3) and Lemma 5, the correspondence $(B_U)_i : X \to 2^{D_i}$, defined by $(B_U)_i(x) = (B_i(x) + U) \cap D_i$ for each $x \in X$, is upper semicontinuous. Note that $cl\,(B_U)_i$ is also upper semicontinuous. Since $(A_U)_i$ has open graph, $(A_U)_i$ has open lower sections, i.e. $(A_U)_i^{-1}(y)$ is open in X for each $y \in D_i$. Hence the set $\{x \in X : ((A_U)_i \cap P_i)(x) \neq \emptyset\} = \cup_{y \in D_i}((A_U)_i \cap P_i)^{-1}(y)$ is open in X ; thus it is paracompact since X is perfectly normal. And by the assumption (4), $(B_U)_i(x)$ is non-empty convex for each $x \in X$. Therefore the associated abstract economy $\Gamma_U = (X_i, (A_U)_i, (B_U)_i, P_i)_{i \in I}$ satisfies the whole assumptions of Theorem 1 and hence there exists an equilibrium $x_u \in X$ such that $(x_u)_i \in cl\,(B_U)_i(x_u)$ and $(A_U)_i(x_u) \cap P_i(x_u) = \emptyset$ for each $i \in I$.

Since $(x_u)_i \in cl\,(B_U)_i(x_u) \subset D_i$ for each $i \in I$ and $U \in \mathcal{U}$, x_u is in a compact set $D = \Pi_{i \in I} D_i$ for each $U \in \mathcal{U}$. Therefore we can find a convergent subnet $\{x_v : V \in \mathcal{V}\}$, where $\mathcal{V}$ is a cofinal subset of $\mathcal{U}$, of $\{x_u : U \in \mathcal{U}\}$ such that $\hat{x} = \lim_{\mathcal{V}} x_v$; then $\hat{x}_i = \lim_{\mathcal{V}} (x_v)_i$ for each $i \in I$. Since $(A_V)_i$ has open graph and, by the assumption (5), P_i is lower semicontinuous, the correspondence $(A_V)_i \cap P_i$ is also lower semicontinuous (e.g. see [21, Remark 2.6(a)]), so that the set $\{x \in X : ((A_V)_i \cap P_i)(x) \neq \emptyset\}$ is closed in X and hence $(A_V)_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$ and $V \in \mathcal{V}$. Since $A_i(x) \subset (A_V)_i(x)$ for each $x \in X$ and $i \in I$, we have $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$.

It remains to show that $\hat{x}_i \in cl\, B_i(\hat{x})$ for each $i \in I$. For every

$V \in \mathcal{V}$ and $i \in I$,

$$(x_v)_i \in cl\,(B_V)_i(x_v) = cl\,[(B_i(x_v) + V) \cap D_i] \\ \subset (cl\,(B_i(x_v) + cl\,V) \cap D_i,$$

whence $(x_v)_i = (t_v)_i + (s_v)_i$ where $(t_v)_i \in cl\,B_i(x_v)$ and $(s_v)_i \in cl\,V$. For each $i \in I$, since $cl\,B_i$ is upper semicontinuous and $\mathcal{V}$ is cofinal, the family $\{(t_v)_i\}_{V\in\mathcal{V}}$ has a limit $t_i \in D_i$ with respect to $\mathcal{V}$, and hence $\{(s_v)_i\}_{V\in\mathcal{V}}$ has a limit $s_i \in cl\,V$ with respect to $\mathcal{V}$. Since $\mathcal{U}$ is a basis of balanced convex open neighborhood of 0 and $\mathcal{V}$ is a cofinal subset of $\mathcal{U}$, we can deduce $s_i = 0$ and hence $\hat{x}_i = t_i$ for each $i \in I$. Since $(t_v)_i \in cl\,B_i(x_v)$ for each $i \in I$ and $V \in \mathcal{V}$, and $cl\,B_i$ is upper semicontinuous, we have $\hat{x}_i \in cl\,B_i(\hat{x})$ for each $i \in I$. Therefore $\hat{x}$ is an equilibrium for the abstract economy Γ. This completes the proof.

REMARKS. (i) When $X_i = D_i$ and $B_i(x)$ is convex for each $i \in I$, the assumption (4) is automatically satisfied.

(ii) In Theorem 2.5 [21], Toussaint generally assumed that the commodity space X_i lies in an arbitrary topological vector space (but our choice set lies in locally convex space); however he needed the stronger assumption that each $A_i^{-1}(y)$ is open in X. As he remarked in [21], as in many economic applications, we have in mind this point makes no difference. In this case, our Theorem 2 further generalizes Toussaint's Theorem 2.5 with weak constraint and preference correspondences in non-compact abstract economy.

Let E be a locally convex Hausdorff topological vector space, E^* be its dual space and denote the dual pairing on $E^* \times E$ by $< w, x >$ for each $w \in E^*, x \in E$. Recall that for a topological vector space E, the *strong topology* on its dual space E^* is the topology on E^* generated by the family $\{U(B;\epsilon) : B$ is a non-empty bounded subset of E and $\epsilon > 0\}$ as a basis for the neighborhood system at 0, where

$$U(B;\epsilon) = \{f \in E^* : \sup_{x\in B} | < f, x > | < \epsilon\}.$$

Finally we prove the following quasi-variational inequality in non-compact sets as an application of equilibrium existence Theorem 2:

THEOREM 3. *Let X be a non-empty perfectly normal convex subset of a locally convex Hausdorff topological vector space E, D be a non-empty compact subset of X and E^* be its dual space. Let $T : X \to 2^{E^*}$ be upper semicontinuous from the relative topology of X to the strong topology of E^* such that each $T(x)$ is a non-empty (strongly) compact convex subset of E^*. Let $A : X \to 2^D$ be continuous such that each $A(x)$ is non-empty closed convex. If there exists a non-empty convex open neighborhood V of 0 in E such that $(A(x)+V) \cap D$ is convex for each $x \in X$, then there exists a point $\hat{x} \in X$ satisfying the following conditions:*

(1) $\hat{x} \in A(\hat{x})$,

(2) *there exists a point $\hat{w} \in T(\hat{x})$ such that*

$$Re < \hat{w}, \hat{x} - x > \leq 0 \quad \text{for all } x \in A(\hat{x}).$$

Proof. In order to apply Theorem 2, we first define a correspondence $P : X \to 2^D$ by

$$P(x) := \{y \in D : \inf_{w \in T(x)} Re < w, x - y > > 0\} \quad \text{for each } x \in X.$$

Then for each $x \in X, x \notin co\,P(x)$. In fact, suppose that $x_0 \in co\,P(x_0)$ for some $x_0 \in X$. Then for some $n \in N$, there exist $y_1, ..., y_n \in P(x_0)$ and $t_1, ..., t_n \in [0,1]$ with $\Sigma_{i=1}^n t_i = 1$ such that $x_0 = \Sigma_{i=1}^n t_i\, y_i$. Then for each $i = 1, ..., n$, $\inf_{w \in T(x_0)} Re < w, x_0 - y_i > > 0$, so that

$$\begin{aligned} 0 &= \inf_{w \in T(x_0)} Re < w, x_0 - x_0 > \\ &= \inf_{w \in T(x_0)} Re < w, x_0 - \Sigma_{i=1}^n t_i\, y_i > \\ &\geq \Sigma_{i=1}^n t_i \cdot \inf_{w \in T(x_0)} Re < w, x_0 - y_i > > 0, \end{aligned}$$

which is a contradiction. And by Theorem 2.5.1 in [2] (or Lemma 2 in [12]), the correspondence $\phi : x \to \inf_{w \in T(x)} Re < w, x - y >$ is lower semicontinuous, so that each $P^{-1}(y) = \{x \in X : \phi(x) > 0\}$ is open, i.e. P has open lower sections. Therefore all hypotheses of Theorem 2 are satisfied, so that there exists $\hat{x} \in A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Therefore for all $x \in A(\hat{x})$, we have

$$(*) \qquad \inf_{w \in T(\hat{x})} Re < w, \hat{x} - x > \leq 0.$$

To prove the second assertion, we now define $f : A(\hat{x}) \times T(\hat{x}) \to R$ by

$$f(x, w) := Re < w, \hat{x} - x > \quad \text{for each } (x, w) \in A(\hat{x}) \times T(\hat{x}).$$

Note that for each fixed $x \in A(\hat{x})$, $x \to Re < w, \hat{x} - x >$ is continuous affine, and for each $w \in T(\hat{x})$, $x \to Re < w, \hat{x} - x >$ is affine. Thus, by Kneser's minimax theorem [14], we have

$$\inf_{w \in T(\hat{x})} \sup_{x \in A(\hat{x})} f(x, w) = \sup_{x \in A(\hat{x})} \inf_{w \in T(\hat{x})} f(x, w).$$

Thus

$$\inf_{w \in T(\hat{x})} \sup_{x \in A(\hat{x})} Re < w, \hat{x} - x > \leq 0 \quad \text{by } (*).$$

Since $T(\hat{x})$ is (strongly) compact, there exists $\hat{w} \in T(\hat{x})$ such that

$$\sup_{x \in A(\hat{x})} Re < \hat{w}, \hat{x} - x > = \inf_{w \in T(\hat{x})} \sup_{x \in A(\hat{x})} Re < w, \hat{x} - x > .$$

Therefore $Re < \hat{w}, \hat{x} - x > \leq 0$ for all $x \in A(\hat{x})$. This completes the proof.

REMARKS. (i) If $X = D$ is compact convex, then $(A(x) + V) \cap D$ is clearly convex for any convex set V in E. In this case, Theorem 3 is a non-compact version of the quasi-variational inequality in [13, Theorem 4].

(ii) In [4], Chang asserts that the graph of P is open without any proof. However, this is not a trivial thing; but in our proof we shall apply Theorem 2, and hence it suffices to show that the correspondence P has open lower sections.

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