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THE WIENER INTEGRAL OVER PATHS IN ABSTRACT WIENER SPACE

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1. Introduction

We summarize the properties of Wiener measure on the space of continuous functions y on $[a,b]$ which values in an abstract Wiener space B such that $y(a)=0$. Such properties are very briefly in the literature [6] with the exception of scaling properties. Since the scaling properties will be especially impotant to us, we consider them in some detail. The main results of this paper is in section 3 where we study the operator-valued function space integral of functionals on $C_0(B)$. Our results are related to works of Johnson and Lapidus (see section 2 and 3 of [4]) except that we work in more complicated Wiener space $C_0(B)$ instead of ordinary Wiener space. On the other hand, our results are restrict to real $\lambda>0$ whereas the results of [4] are not. We express the operator which are the values of the function space integrals as generalized Dyson series much as in [4]. And these series can be viewed as disentangling the operator's involved. For a discussion of disentangling, see [4].

2. Wiener measure on the space of paths in an arbitrary abstract wiener space

Let $(\mathbb{B},\beta(\mathbb{B}),m)$ be an abstract Wiener space. For $\lambda>0$, let m_λ be the Borel measure on $\mathbb{B}$ given by $m_\lambda(B)=m(\lambda^{-1}B)$ for Borel subsets B of $\mathbb{B}$. Let $C_0(\mathbb{B})$ denote the set of all continuous functions on $[a,b]$ into $\mathbb{B}$ which vanish at a. Then $C_0(\mathbb{B})$ is a real separable Banach space in the norm $\|y\|_{C_0(\mathbb{B})}\equiv\sup_{a\le t\le b}\|y(t)\|_{\mathbb{B}}$. And from [6], the minimal σ-algebra $\beta(C_0(\mathbb{B}))$ making the mapping $y\to y(t)$ measurable consists of the Borel subsets of $C_0(\mathbb{B})$. Further, Brownian motion in $\mathbb{B}$ induces

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a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \beta(C_0(\mathbb{B})))$ which is mean-zero Gaussian.

We will find a concrete form of $m_{\mathbb{B}}$. Let $\overrightarrow{t} = (t_1, t_2, \ldots, t_n)$ be given with $a = t_0 < t_1 < t_2 < \cdots < t_n \leq b$. Let $T_{\overrightarrow{t}} : \mathbb{B}^n \to \mathbb{B}^n$ be given by

(1.1)
$$T_{\overrightarrow{t}}(x_1, x_2, \ldots, x_n) = (\sqrt{t_1 - t_0}\, x_1, \sqrt{t_1 - t_0}\, x_1 + \sqrt{t_2 - t_1}\, x_2, \ldots, \sum_{j=1}^{n} \sqrt{t_j - t_{j-1}}\, x_j).$$

We define a set function $v_{\overrightarrow{t}}$ on $\beta(\mathbb{B}^n)$ given by

$$v_{\overrightarrow{t}}(B) = (\overset{n}{X} m)(T_{\overrightarrow{t}}^{-1}(B)). \tag{1.2}$$

Then $v_{\overrightarrow{t}}$ is a Borel measure. Let $f_{\overrightarrow{t}} : C_0(\mathbb{B}) \to \mathbb{B}^n$ be the function with

$$f_{\overrightarrow{t}}(y) = (y(t_1), y(t_2), \ldots, y(t_n)). \tag{1.3}$$

For Borel subsets $B_1, B_2, \ldots, B_n$ of $\mathbb{B}$, $f_{\overrightarrow{t}}^{-1}(\overset{n}{\underset{i=1}{X}} B_i)$ is called the I-set with respect to $B_1, B_2, \ldots, B_n$. Then the collection $\mathcal{I}$ of all I-sets is an algebra. We define a set function $m_{\mathbb{B}}$ on $\mathcal{I}$ by

$$m_{\mathbb{B}}(f_{\overrightarrow{t}}^{-1}(\overset{n}{\underset{i=1}{X}} B_i)) = v_{\overrightarrow{t}}(\overset{n}{\underset{i=1}{X}} B_i). \tag{1.4}$$

Then $m_{\mathbb{B}}$ is well-defined and countably additive on I. Using the Carathéodory process, we have a Borel measure $m_{\mathbb{B}}$ on $\beta(C_0(\mathbb{B}))$.

By the change of variable theorem, we have

LEMMA 1. (Wiener Integration Theorem). *Let* $\overrightarrow{t} = (t_1, t_2, \ldots, t_n)$ *be given with* $a = t_0 < t_1 < t_2 < \cdots < t_n \leq b$ *and let* $f : \mathbb{B}^n \to \mathbf{C}$ *be a Borel measurable function. Then*

(1.5)
$$\int_{C_0(\mathbb{B})} f(y(t_1), y(t_2), \cdots, y(t_n))\, dm_{\mathbb{B}}(y) \overset{*}{=} \int_{\mathbb{B}^n} f \circ T_{\overrightarrow{t}}(x_1, x_2, \ldots, x_n)\, d(\overset{n}{X} m)(x_1, x_2, \ldots, x_n)$$

where by $\stackrel{*}{=}$ we mean that if either side exists, both sides exists and they are equal.

From Lemma 1, we can easily check that $m_{\mathbb{B}}$ is a Brownian motion in $\mathbb{B}$ which is a mean-zero Gaussian measure.

Given partition $\prod_n$ of $[a,b]; a = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = b$ with $\mu(\prod_n) = \max_{1\le p\le k(n)} \|t_p^n - t_{p-1}^n\| \to 0$ as $n \to \infty$ and y in $C_0(\mathbb{B})$, let

$$S_{\prod_n}(y) = \sum_{j=1}^{k(n)} \|y(t_j^n) - y(t_{j-1}^n)\|_{\mathbb{B}}^2. \tag{1.6}$$

By [3], $\int_{\mathbb{B}} \|x\|_{\mathbb{B}}^2 dm(x)$ and $\int_{\mathbb{B}} \|x\|_{\mathbb{B}}^4 dm(x)$ are finite. Let $\alpha = (b - a)\int_{\mathbb{B}} \|x\|_{\mathbb{B}}^2 dm(x)$ and $\beta = \int_{\mathbb{B}} \|x\|_{\mathbb{B}}^4 dm(x) - \{\int_{\mathbb{B}} \|x\|_{\mathbb{B}}^2 dm(x)\}^2$. Then, using Lemma 1,

$$\begin{aligned} &\int_{C_0(\mathbb{B})} S_{\prod_n}(y)dm_{\mathbb{B}}(y) \\ &= \int_{\mathbb{B}^{k(n)}} \sum_{j=1}^{k(n)} \| \sum_{i=1}^{j} \sqrt{t_i^n - t_{i-1}^n}\, x_i \\ &\qquad - \sum_{i=1}^{j-1} \sqrt{t_i^n - t_{i-1}^n}\, x_i \|_{\mathbb{B}}^2 d(\mathop{X}_{i=1}^{k(n)} m)(x_i) \\ &= \sum_{j=1}^{k(n)} (t_j^n - t_{j-1}^n) \int_{\mathbb{B}^{k(n)}} \|x_j\|_{\mathbb{B}}^2 d(\mathop{X}_{i=1}^{k(n)} m)(x_i) \\ &= \alpha. \end{aligned} \tag{1.7}$$

And

$$\begin{aligned} &\int_{C_0(\mathbb{B})} (S_{\prod_n}(y) - \alpha)^2 dm_{\mathbb{B}}(y) \\ &= \int_{C_0(\mathbb{B})} \{\sum_{j=1}^{k(n)} \|y(t_j) - y(t_{j-1})\|_{\mathbb{B}}^2\}^2 dm_{\mathbb{B}}(y) - \alpha^2 \\ &= \int_{\mathbb{B}^{k(n)}} \{\sum_{j=1}^{k(n)} (t_j^n - t_{j-1}^n)\|x_j\|_{\mathbb{B}}^2\}^2 d(\mathop{X}_{i=1}^{k(n)} m)(x_i) - \alpha^2 \end{aligned} \tag{1.8}$$

$$= \sum_{p=1}^{k(n)} \sum_{q=1,q\neq p}^{k(n)} (t_p^n - t_{p-1}^n)(t_q^n - t_{q-1}^n)$$

$$\int_{\mathbb{B}^{k(n)}} \|x_p\|_{\mathbb{B}}^2 \|x_q\|_{\mathbb{B}}^2 dm(\overset{k(n)}{\underset{i=1}{X}} m)(x_i)$$

$$+ \sum_{p=1}^{k(n)} (t_p^n - t_{p-1}^n)^2 \int_{\mathbb{B}^{k(n)}} \|x_p\|_{\mathbb{B}}^4 d(\overset{k(n)}{\underset{i=1}{X}} m)(x_i) - \alpha^2$$

$$= \{\sum_{p=1}^{k(n)} \sum_{q=1}^{k(n)} (t_p^n - t_{p-1}^n)(t_q^n - t_{q-1}^n\}$$

$$\{\int_{\mathbb{B}} \|x\|_{\mathbb{B}}^2 dm(x)\}^2 + \beta \sum_{p=1}^{k(n)} (t_p^n - t_{p-1}^n)^2 - \alpha^2$$

$$= \beta \sum_{p=1}^{k(n)} (t_p^n - t_{p-1}^n)^2.$$

Since $\mu(\prod_n) \to 0$ as $n \to \infty$, we can choose a subsequence $\langle \prod_{\sigma(n)} \rangle$ of $\langle \prod_n \rangle$ such that $\sum_{n=1}^{\infty} \sum_{p=1}^{k(\sigma(n))} (t_p^{\sigma(n)} - t_{p-1}^{\alpha(n)})^2$ is finite. Then by (1.8),

$$\text{(1.9)} \qquad \sum_{n=1}^{\infty} \int_{C_0(\mathbb{B})} \{S_{\prod_{\sigma(n)}}(y) - \alpha\}^2 dm_{\mathbb{B}(y)}$$
$$= \beta \sum_{n=1}^{\infty} \sum_{p=1}^{k(\sigma(n))} (t_p^{\sigma(n)} - t_{p-1}^{\sigma(n-1)})^2$$
$$< \infty.$$

Hence, by [7, p.173],

$$\text{(1.10)} \qquad \text{for} \quad m_{\mathbb{B}} - \text{a.e. } y, \quad \lim_{n\to\infty} S_{\prod_{\sigma(n)}}(y) = 0.$$

For $\lambda \geq 0$, let

$$\text{(1.11)} \qquad \Omega_\lambda = \{y \text{ in } C_0(\mathbb{B}) \mid \lim_{n\to\infty} S_{\prod_{\sigma(n)}}(y) = \lambda^2 \alpha\}$$

and let

$$D^* = \{y \text{ in } C_0(\mathbb{B}) \mid \lim_{n\to\infty} S_{\Pi_{\sigma(n)}}(y) \text{ doesn't exist}\}. \tag{1.12}$$

Then for two positive reals $\lambda_1, \lambda_2, \lambda_1\Omega^*_{\lambda_2} = \Omega^*_{\lambda_1\lambda_2}, \Omega^*_\lambda$, $\lambda \geq 0$, and D^* are Borel subsets, $C_0(\mathbb{B})$ is the disjoint union of the sets Ω^*_λ, $\lambda \geq 0$, and D^* and $m_{\mathbb{B}}(\Omega^*_\lambda) = 0$ if and only if $\lambda \neq 1$.

For $\lambda > 0$, we define a Borel measure $m_{\mathbb{B},\lambda}$ on $\beta(C_0(\mathbb{B}))$ by $m_{\mathbb{B},\lambda}(B) = m_{\mathbb{B}}(\lambda^{-1}B)$. Then using Lemma 1, we easily check that for two positive reals p and q,

$$m_{\mathbb{B},p} * m_{\mathbb{B},q} = m_{\mathbb{B},\sqrt{p^2+q^2}} \quad \text{on} \quad \beta(C_0(\mathbb{B})). \tag{1.13}$$

3. Operator-valued function space integral on $C_0(\mathbb{B})$

In [2], Chung considered the Borel subsets Ω_λ, $\lambda > 0$ and D of an abstract Wiener space $\mathbb{B}$ which satisfies the following; for two positive reals λ_1 and λ_2, $\lambda_1\Omega_{\lambda_2} = \Omega_{\lambda_1\lambda_2}$ and $\mathbb{B}$ is the disjoint union of this family of sets. Also $m(\Omega_\lambda) = 0$ if and only if $\lambda \neq 1$. Let $(\mathbb{B}, \overline{\beta(\mathbb{B})}, \overline{m})$ be the completion of $(\mathbb{B}, \beta(\mathbb{B}), m)$.

DEFINITION 1. Let $L_{1\infty}$ be the class of all C-valued Borel measurable function ψ on $\mathbb{B}$ such that for each $\lambda > 0$, $\psi(\lambda(\cdot))$ is m-integrable and

$$\|\psi\|_{1\infty} \equiv \sup_{\lambda>0} \|\psi(\lambda(\cdot))\|_1 \equiv \sup_{\lambda>0} \int_{\mathbb{B}} |\psi(\lambda x)| dm(x) \tag{2.1}$$

is finite. For f and g in $L_{1\infty}$, we say that f is equivalent to g, denote $f \sim g$, if $\{\lambda_x \in \mathbb{B} \mid f(x) \neq g(x)\}$ is an m-null set for all $\lambda > 1$. Clearly, $\sim$ is an equivalent relation on $L_{1\infty}$. Hence, we obtain a quotient space $L_{1\infty}/\sim$ which we denote $L_{1\infty}$.

THEOREM 1. *$(L_{1\infty}, \|\cdot\|_{1\infty})$ is a Banach space.*

Proof. It is clear that $(L_{1\infty}, \|\cdot\|_{1\infty})$ is a semi-normed linear space. We suppose that $\|\psi\|_{1\infty} = 0$. Then for $\lambda > 0$,

$$\int_{\Omega_\lambda} |\psi(x)| dm_\lambda(x) = \int_{\mathbb{B}} |\psi(\lambda x)| dm(x) = 0. \tag{2.2}$$

Hence for $\lambda > 0$, there is a N_λ in $\beta(\Omega_\lambda)$ with $m_\lambda(N_\lambda) = 0$ such that $\psi(x) = 0$ on $\Omega_\lambda \backslash N_\lambda$. Put $N^* = (\bigcup_{\lambda>0} N_\lambda) \cup \Omega_0 \cup D$ and $N = \{x \in \mathbb{B} \,|\, \psi(x) \neq 0\}$. Then $N \subset N^*$ and since $\overline{m}(\lambda N^*) = 0$ for all $\lambda > 0$, N is a Borel scale-invariant null subset of $\mathbb{B}$, that is, $\psi \sim 0$. Therefore, $(L_{1\infty}, \|\cdot\|_{1\infty})$ is a normed linear space. We need to show that $(L_{1\infty}, \|\cdot\|_{1\infty})$ is complete. Suppose $\langle\psi_n\rangle$ is an absolutely summable sequence in $(L_{1\infty}, \|\cdot\|_{1\infty})$. Then

$$
\begin{aligned}
(2.3) \qquad & \int_{\mathbb{B}} \sum_{n=1}^{\infty} |\psi_n(x)| dm(x) \\
& \leq \sup_{\lambda>0} \sum_{n=1}^{\infty} \int_{\mathbb{B}} |\psi_n(\lambda x)| dm(x) \\
& \leq \sum_{n=1}^{\infty} \|\psi_n\|_{1\infty} \\
& < +\infty.
\end{aligned}
$$

Now, let

$$
(2.4) \qquad \psi(x) = \sum_{n=1}^{\infty} \psi_n(x).
$$

By (2.3), for $\lambda > 0$, the series as (2.4) is absolutely convergent for m_λ-a.e. x. Hence ψ is well-defined except for some Borel scale-invariant null subset and ψ is in $L_{1\infty}$. Therefore $\langle\psi_n\rangle$ is a summable sequence, as desired.

REMARK. Since $\|\chi_{\Omega_{\lambda_1}} - \chi_{\Omega_{\lambda_2}}\|_{1\infty} = 1$ for distinct two positive reals λ_1 and λ_2, $L_{1\infty}$ is not separable.

REMARK. In [1], Cameron and Storvick introduced the space $W(C_1[\alpha,\beta])$ which is the class of Borel measurable functionals defined on the concrete Wiener space $C_1[\alpha,\beta]$ such that $\psi(\gamma y + x)$ is Wiener integrable in y over $C_1[\alpha,\beta]$ for each positive γ and each x in $C_1[\alpha,\beta]$. They considered an operator-valued function space integral acting on this space. However, $W(C_1,[\alpha,\beta])$ is not Banach space whereas the space $L_{1\infty}$ which we are considering is a Banach space.

DEFINITION 2. For $\lambda > 0$, we define an operator C_λ on $L_{1\infty}$ given by

$$(2.5) \qquad (C_\lambda\psi)(x) = \int_{\mathbb{B}} \psi(\lambda^{-\frac{1}{2}}x_1 + x)dm(x_1) \quad \text{for} \quad \psi \text{ in } L_{1\infty}.$$

LEMMA 2. *For $\lambda > 0$, C_λ is a bounded linear operator from $L_{1\infty}$ into itself. Moreover $\|C_\lambda\| \leq 1$.*

Proof. Clearly C_λ is linear and $C_\lambda\psi$ is Borel measurable for ψ in $L_{1\infty}$ and for $\lambda > 0$. And

$$(2.6) \qquad \begin{aligned} &\|C_\lambda\psi\|_{1\infty} \\ &\leq \sup_{\mu>0} \int_{\mathbb{B}}\int_{\mathbb{B}} |\psi(\lambda^{\frac{-1}{2}}x_1 + \mu x)|dm(x_1)dm(x) \\ &= \sup_{\mu>0} \int_{\mathbb{B}} |\psi(\sqrt{1/\lambda + \mu^2}\, z)\,|\,dm(z) \\ &\leq \|\psi\|_{1\infty}, \quad \text{as desired.} \end{aligned}$$

REMARK. Let $\psi(x) = \chi_{\Omega_{\sqrt{2}}}(x)$ on $\mathbb{B}$. Let $\lambda > 0$ be given and let x be in $\mathbb{B}$. Then

$$(2.7) \qquad \begin{aligned} \int_{\mathbb{B}} (C_\lambda\psi)(x)dm(x) &= \int_{\mathbb{B}} \chi_{\Omega_{\sqrt{2}}}(\sqrt{1/\lambda + 1}\, z)dm(z) \\ &= \begin{cases} 1 & \text{if } \lambda = 1 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Since $(C_\lambda\psi)(x) \leq 1$ m-a.e. x, if $\lambda = 1$ then $(C_\lambda\psi)(x) = 1$ m-a.e. x. And if $\lambda \neq 1$, then $(C_\lambda\psi)(x) = 0$ m-a.e. x. Hence $\lim_{\lambda\to 1} C_\lambda\psi \neq C_1\psi$ as in the L_1-norm sense. Thus $\lim_{\lambda\to 1} C_\lambda\psi \neq C_1\psi$ as in the $L_{1\infty}$-norm sense, that is, C_λ is not strongly continuous.

REMARK. Let λ and μ be positive reals, let ψ be in $L_{1\infty}$ and let x be in $\mathbb{B}$. Then

$$(2.8) \qquad [(C_\lambda \circ C_\mu)\psi](x)$$

$$= \int_{\mathbb{B}}\{\int_{\mathbb{B}}\psi(\mu^{\frac{-1}{2}}x_1+\lambda^{-\frac{1}{2}}x_2+x)dm(x_1)\}dm(x_2)$$
$$= \int_{\mathbb{B}}\psi(\sqrt{1/\mu+1/\lambda}\,z+x)dm(x)$$
$$= [C_{\lambda\mu/(\lambda+\mu)}\psi](x).$$

Let $C_\lambda = C^*_{1/\lambda}$ for $\lambda > 0$. Then C^*_λ has the semi-group property with respect to λ.

DEFINITION 3. For a bounded Borel measurable functional θ_1 on $\mathbb{B}$, we define the multiplication operator M_{θ_1} by

$$(2.9) \qquad (M_{\theta_1}\psi)(x) = \theta_1(x)\psi(x) \quad \text{for} \quad \psi \text{ in } L_{1\infty}.$$

Let $\theta : [a,b]\times\mathbb{B} \to \mathbf{C}$ be a bounded Borel measurable function. Let $\theta(s)$ denote the operator $M_{\theta(s,\cdot)}$ of multiplication by $\theta(s,\cdot)$, acting in $L_{1\infty}$.

REMARK. If θ_1 is bounded by K in the above Definition, M_{θ_1} is bounded linear operator from $L_{1\infty}$ into itself and $\|M_{\theta_1}\| \le K$.

NOTATION. Let $C(\mathbb{B})$ be the space of all $\mathbb{B}$-valued continuous functions on $[a,b]$. z in $C(\mathbb{B})$ has a unique decomposition $z = y + x$ where x is in $\mathbb{B}$ and y is in $C_0(\mathbb{B})$.

DEFINITION 4. Let $F : C(\mathbb{B}) \to \mathbf{C}$ be a function, let $\lambda > 0$ be given, let ψ be in $L_{1\infty}$ and let x be in $\mathbb{B}$. We consider the expression

$$(2.10) \quad [K_\lambda(F)\psi](x) = \int_{C_0(\mathbb{B})} F(\lambda^{\frac{-1}{2}}y+x)\psi(\lambda^{-\frac{1}{2}}y(b)+x)dm_{\mathbb{B}}(y).$$

If $K_\lambda(F)$ exists and $K_\lambda(F)$ is a bounded linear operator from $L_{1\infty}$ into itself for all $\lambda > 0$, we say that the operator-valued function space integral $K_\lambda(F)$ exists for all $\lambda > 0$.

The rest of this section, we adopt the following notations and assumptions. Let $\theta : [a,b]\times\mathbb{B} \to \mathbf{C}$ be a bounded Borel measurable function and let η be a $\mathbf{C}$-valued Borel measure on (a,b). $\eta = \mu + \nu$ will be the decomposition of η into its continuous μ and discrete parts ν. And let

$$(2.11) \qquad F_n(y) = (\int_{(a,b)}\theta(s,y(s))d\eta(s))^n \quad \text{for} \quad y \text{ in } C(\mathbb{B}).$$

Let δ_{τ_p} be the Dirac measure with total mass one concentrated at τ_p.

THEOREM 2. (finite supported case). *Let* $\eta = \mu+\nu=\mu+\sum_{p=1}^{h} w_p\delta_{\tau_p}$ *where* $a < \tau_1 < \tau_2 < \cdots < \tau_h < b$ *and the* w_p*'s* $(p = 1,2,\dots,h)$ *are in* $\mathbf{C}$. *Then the operator-valued function space integral* $K_\lambda(F_n)$ *exists for all* $\lambda > 0$ *and for* $\lambda > 0$, ψ *in* $L_{1\infty}$ *and* x *in* $\mathbb{B}$ *except for some Borel scale-invariant null subset,*

(2.12)

$$[K_\lambda(F_n)\psi](x) = \sum_{q_0+q_1+\cdots+q_h=n} n!\frac{w_1^{q_1}w_2^{q_2}\cdots w_h^{q_h}}{q_1!\,q_2!\cdots q_h!}$$

$$\times \sum_{j_1+j_2+\cdots+j_{h+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_2,j_{h+1}}} [L_0\circ L_1\circ\cdots\circ L_h\psi](x)$$

$$d(\mathop{X}_{p=0}^{h}\mathop{X}_{i=1}^{j_{p+1}}\mu)(s_{p,i})$$

where for $k = 1,2,\cdots,h$, $L_k = C_{\alpha_{k+1,1}}\circ\theta(s_{k,1})\circ C_{\alpha_{k+1,2}}\circ\theta(s_{k,2})\circ\cdots\circ C_{\alpha_{k+1,j_{k+1}}}\circ\theta(s_{k,j_{k+1}})\circ\{\theta(\tau_k)\}^{q_{k+1}}$, $\Delta_{q_0;j_1,j_2,\dots,j_{h+1}} =\{(s_{0,1},s_{0,2},\cdots, s_{0,j_1},s_{1,1},s_{1,2},\cdots,s_{h,j_{h+1}})\,|\,a= s_{0,0} < s_{0,1} < \cdots < s_{0,j_1} < \tau_1 < s_{1,1} < s_{1,2} < \cdots < s_{h-1,j_h} < \tau_h < s_{h,1} <\cdots < s_{h,j_{h+1}} < b = \tau_{h+1}\}$ *and for* $p = 1,2,\cdots,h+1$ *and for* $i = 1,2,\cdots,j_{p+1}$, $\alpha_{p,i} = \lambda/(s_{p-1,i}-s_{p-1,i-1})$ *and* $\alpha_{p+1,j_{p+1}+1} = \lambda/(s_{p,0} - s_{p-1,j_{p+1}}) = \lambda/(\tau_p - s_{p-1,j_{p+1}})$.

Moreover,

$$\|K_\lambda(F_n)\| \le (\sup|\theta|\,\|\eta\|)^n. \tag{2.13}$$

Proof. Let $\lambda > 0$ be given, let ψ be in $L_{1\infty}$. Then for x in $\mathbb{B}$ except for some Borel scale-invariant null subset,

(2.14)

$$|F_n(\lambda^{\frac{-1}{2}}y + x)\psi(\lambda^{\frac{-1}{2}}y(b) + x)| \le (\sup|\theta|\,\|\eta\|)^n\psi(\lambda^{\frac{-1}{2}}y(b) + x) \quad \text{for } y \text{ in } C_0(\mathbb{B}).$$

Also $|\psi(\lambda^{\frac{-1}{2}}y(b)+x)|$ is integrable in y over $C_0(\mathbb{B})$ and $\int_{C_0(\mathbb{B})}|\psi(\lambda^{-\frac{1}{2}}y(b) +x)|dm_{\mathbb{B}}(y) = [C_{\lambda/(b-a)}(|\psi|)](x)$. Hence for x in $\mathbb{B}$ except for some Borel scale-invariant null subset,

(2.15)

$$[K_\lambda(F_n)\psi](x)$$

$$\overset{(1)}{=} \int_{C_0(\mathbb{B})} \{ \int_{(a,b)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + x) d\mu(s) + \sum_{p=1}^{h} w_p \theta(\tau_p, \lambda^{\frac{-1}{2}} y(\tau_p) + x) \}^n$$

$$\psi(\lambda^{\frac{-1}{2}} y(b) + x) dm_{\mathbb{B}}(y)$$

$$\overset{(2)}{=} \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_0! q_1! \cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h}$$

$$\int_{C_0(\mathbb{B})} (\int_{(a,b)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + x) d\mu(s))^{q_0}$$

$$(\prod_{p=1}^{h} \theta(\tau_p, \lambda^{-\frac{1}{2}} y(\tau_p) + x)^{q_p}) \psi(\lambda^{\frac{-1}{2}} y(b) + x) dm_{\mathbb{B}}(y)$$

$$\overset{(3)}{=} \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_1! q_2! \cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h}$$

$$\int_{C(\mathbb{B})} (\int_{\Delta_{q_0}} \prod_{i=1}^{q_0} \theta(s_i, \lambda^{-\frac{1}{2}} y(s_i) + x) d(\overset{q_0}{X} \mu)(s_i))$$

$$(\prod_{p=1}^{h} \theta(\tau_p, \lambda^{-\frac{1}{2}} y(\tau_p) + x)^{q_p}) \psi(\lambda^{\frac{-1}{2}} y(b) + x) dm_{\mathbb{B}}(y)$$

$$\overset{(4)}{=} \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_1! q_2! \cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h} \sum_{j_1+j_2+\cdots+j_{h+1}=q_0}$$

$$\{ \int_{\Delta_{q_0}; j_1, j_2, \ldots, j_{h+1}} (\int_{C_0(\mathbb{B})} (\prod_{i=1}^{q_0} \theta(s_i, \lambda^{-\frac{1}{2}} y(s_i) + x))$$

$$(\prod_{p=1}^{h} \theta(\tau_p, \lambda^{\frac{-1}{2}} y(\tau_p) + x)^{q_p} \psi(\lambda^{-\frac{1}{2}} y(b) + x) dm_{\mathbb{B}}(y)) d(\overset{q_0}{X} \mu)(s_i) \}$$

$$\overset{(5)}{=} \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_1! q_2! \cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h} \sum_{j_1+j_2+\cdots+j_{h+1}=q_0}$$

$$\int_{\Delta_{q_0}; j_1, j_2, \ldots, j_{h+1}} \int_{C_0(\mathbb{B})} \{ \prod_{p=1}^{h+1} \prod_{i=1}^{j_p} \theta(s_{p-1,i}, \lambda^{\frac{-1}{2}} y(s_{p-1,i}) + x) \}$$

$$\times \{\prod_{p=1}^{h} \theta(s_{p,0}, \lambda^{\frac{-1}{2}} y(s_{p,0}) + x)^{q_p}\}$$

$$\psi(\lambda^{\frac{-1}{2}} y(s_{h+1,0}) + x) dm_{\mathbb{B}}(y) d(\mathop{X}_{p=0}^{h} \mathop{X}_{i=1}^{j_{p+1}} \mu)(s_{p,i})$$

$$\stackrel{(6)}{=} \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_1! q_2! \cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h} \sum_{j_1+j_2+\cdots+j_{h+1}=q_0}$$

$$\int_{\Delta_{q_0;j_1,j_2,\cdots,j_{h+1}}} \int_{\mathbb{B}^{q_0+h+1}} (\prod_{p=1}^{h+1} \prod_{i=1}^{j_{p+1}} \theta(s_{p-1,i}, \beta_{p-1,i} + x))$$

$$\times (\prod_{p=1}^{h} \theta(s_{p,0}, \beta_{p,0} + x)^{q_p}) \psi(\beta_{h,j_{h+1}+1} + x)$$

$$d(\mathop{X}_{p=1}^{h+1} \mathop{X}_{i=1}^{j_{p+1}+1} m)(x_{p,i}) d(\mathop{X}_{p=0}^{h} \mathop{X}_{i=1}^{j_{p+1}} \mu)(s_{p,i})$$

$$\stackrel{(7)}{=} \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_1! q_2! \cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h} \sum_{j_1+j_2+\cdots+j_{h+1}=q_0}$$

$$\int_{\Delta_{q_0;j_1,j_2,\cdots,j_{h+1}}} (L_1 \circ L_2 \circ \cdots \circ L_h \psi)(x) d(\mathop{X}_{p=1}^{h+1} \mathop{X}_{i=1}^{j_{p+1}} \mu)(s_{p,i}).$$

Step (1) results from writing η as $\mu + \sum_{p=1}^{h} w_p \delta_{\tau_p}$ and carrying out the integral with respect to $\sum_{p=1}^{h} w_p \delta_{\tau_p}$. By the multinomial expansion theorem, we obtain Step (2). Let $\Delta_{q_0} = \{(s_1, s_2, \ldots, s_{q_0}) \mid a = s_0 < s_1 < s_2 < \cdots < s_{q_0} < b\}$. Since the integrand is invariant under permutations of s-variables and the integral over the $n!$ simplexes are equal, we have Step (3). Step (4) follows from the Fubini theorem which is justified above (2.14). After the relabeling $s_{j_1+j_2+\cdots+j_k+i} = s_{k,i}$, $\tau_k = s_{k,0}$ and $\tau_{h+1} = s_{h,j_{h+1}+1}$, we have Step (5). Letting $\beta_{k,i} = \sum_{u=1}^{k} \sum_{v=1}^{i} \alpha_{u,v}^{-\frac{1}{2}} x_{u,v} + \sum_{v=1}^{i} (\alpha_{k+1,v}^{\frac{-1}{2}} x_{k+1,v})$, by Lemma 1, we obtain Step (6). From the notation of C_λ, $\theta(\cdot)$ and L_k, we have Step (7), as desired.

And from (2.14),

(2.16)

$$\|K_\lambda(F_n)\psi\|_{1\infty}$$

$$\leq (\sup|\theta|\,\|\eta\|)^n \sup_{\mu>0} \int_{\mathbb{B}} \int_{C_0(\mathbb{B})} |\psi(\lambda^{\frac{-1}{2}} y(b) + \mu x)| dm_{\mathbb{B}}(y) dm(x)$$
$$= (\sup|\theta|\,\|\eta\|)^n \sup_{\mu>0} \int_{\mathbb{B}} |\psi(\sqrt{\frac{b-a}{\lambda} + \mu^2}\, x)| dm(x)$$
$$\leq (\sup|\theta|\,\|\eta\|)^n \|\psi\|_{1\infty}.$$

Thus, we obtain (2.13).

COROLLARY 1. ($\eta = \nu$ purely finite discrete case). *Let $\eta = \nu = \sum_{p=1}^{h} w_p \delta_\tau$ where $a < \tau_1 < \tau_2 < \cdots < \tau_h < b$. Then the operator-valued function space integral $K_\lambda(F_n)$ exists for $\lambda > 0$ and for $\lambda > 0$, ψ in $L_{1\infty}$ and x in $\mathbb{B}$ except for a Borel scale-invariant null subset,*

(2.17)
$$[K_\lambda(F_n)\psi](x) = \sum_{q_0+q_1+\cdots+q_h=n} \frac{n!}{q_1!q_2!\cdots q_h!} w_1^{q_1} w_2^{q_2} \cdots w_h^{q_h} [C_{\alpha_1} \circ \{\theta(\tau_1)\}^{q_1} \circ C_{\alpha_2} \circ \cdots \circ C_{\alpha_h} \circ \{\theta(\tau_h)\}^{q_h} \circ C_{\alpha_{h+1}} \psi](x)$$

where $\alpha_i = \lambda/(\tau_i - \tau_{i-1})$ $(i = 1, 2, \ldots, h+1)$. Moreover,

$$\|K_\lambda(F_n)\| \leq (\sup|\theta|\,\|\eta\|)^n. \tag{2.18}$$

COROLLARY 2. ($\eta = \mu$ purely continuous case). *Let $\eta = \mu$. Then the operator-valued function space integral $K_\lambda(F_n)$ exists for $\lambda > 0$ and for $\lambda > 0$, ψ in $L_{1\infty}$ and x in $\mathbb{B}$ except for a Borel scale-invariant null subset,*

(2.19)
$$[K_\lambda(F_n)\psi](x) = n! \int_{\Delta_n} [C_{\alpha_1} \circ \theta(s_1) \circ \cdots \circ \theta(s_n) \circ C_{\alpha_{n+1}} \psi] d(\overset{n}{X}\mu)(s_i)$$

where $\Delta_n = \{(s_1, s_2, \ldots, s_n) \,|\, a = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = b\}$ and $\alpha_i = \lambda/(s_i - s_{i-1})$ $(i = 1, 2, \ldots, n+1)$. Moreover,

$$\|K_\lambda(F_n)\| \leq (\sup|\theta|\,\|\eta\|)^n. \tag{2.20}$$

From the "$\aleph_0$-nomial formular" [see 4, p.41], we obtain the following Corollary.

COROLLARY 3. (η an arbitrary Borel measure). *Let* $\eta = \mu + \sum_{p=1}^{\infty} w_p \delta_{\tau}$. *Then the operator-valued function space integral* $K_\lambda(F_n)$ *exists for* $\lambda > 0$ *and for* $\lambda > 0$, ψ *in* $L_{1\infty}$ *and* x *in* $\mathbb{B}$ *except for some Borel scale-invariant null subset,*

$$(2.21)\quad \begin{aligned}&[K_\lambda(F_n)\psi](x)\\ &= \sum_{h=1}^{\infty} \sum_{q_0+q_1+\cdots+q_h=n, q_h\neq 0} \frac{n!}{q_1!q_2!\cdots q_h!} w_1^{q_1} w_2^{q_2}\cdots w_h^{q_h}\\ &\quad \sum_{j_1+j_2+\cdots+j_{h+1}=q_0} \int_{\Delta^{\sigma}_{q_0;j_1,j_2,\ldots,j_{h+1}}} L_0^{\sigma}\circ L_1^{\sigma}\circ\cdots\circ L_h^{\sigma}\\ &\quad d\Big(\mathop{X}_{p=0}^{h}\mathop{X}_{i=1}^{j_{p+1}}\mu\Big)(s_{p,i})\end{aligned}$$

where for each h, σ *in the permutation of* $\{1,2,\ldots,h\}$ *such that* $\tau_{\sigma_1} < \tau_{\sigma_2} < \cdots < \tau_{\sigma(h)}$ *and* $\Delta^{\sigma}_{q_0;j_1,j_2,\ldots,j_{h+1}}$, $L_{1\infty}$ *and* $\alpha_{p,i}$ *are defined as in Theorem 2 except with* τ_p *replaced by* $\tau_{\sigma(p)}$. *Moreover,*

$$(2.22)\qquad \|K_\lambda(F_n)\| \leq (\sup|\theta|\,\|\eta\|)^n.$$

THEOREM 3. *Let*

$$(2.23)\qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an analytic function with radius of convergence strictly greater than $\sup|\theta|\,\|\eta\|$ *and let*

$$(2.24)\qquad F(y) = f\Big(\int_{(a,b)} \theta(s, y(s))d\eta(s)\Big) \quad \text{for} \quad y \text{ in } C(\mathbb{B}).$$

Then the operator-valued function space integral $K_\lambda(F)$ *exists for* $\lambda > 0$ *and for* $\lambda > 0$,

$$(2.25)\qquad K_\lambda(F) = \sum_{n=0}^{\infty} a_n K_\lambda(F_n)$$

where the series converges in the operator-norm topology. Moreover,

$$\|K_\lambda(F)\| \leq \sum_{n=0}^{\infty} |a_n|(\sup|\theta|\,\|\eta\|)^n. \tag{2.26}$$

Proof. Let $\lambda > 0$ be given, let ψ be in $L_{1\infty}$. Then for x in $\mathbb{B}$ except for some Borel scale-invariant null subset, $|\sum_{n=0}^{\infty} a_n F_n(\lambda^{\frac{-1}{2}} y + x)\psi(\lambda^{-\frac{1}{2}} y(b) + x)| \leq \sum_{n=0}^{\infty} |a_n(\sup|\theta|\,\|\eta\|)^n |\psi(\lambda^{-\frac{1}{2}} y(b) + x)|$. Also $|\psi(\lambda^{\frac{-1}{2}} y(b) + x)|$ is integrable in y. Hence, by the dominated convergence theorem, for x in $\mathbb{B}$ except for some Borel scale-invariant null subset,

$$[K_\lambda(F)\psi](x) = \sum_{n=0}^{\infty} a_n[K_\lambda(F_n)\psi](x). \tag{2.27}$$

Hence the operator-valued function space integral $K_\lambda(F)$ exists for $\lambda > 0$. And since

$$\begin{aligned} &\|K_\lambda(F) - \sum_{n=0}^{m} a_n K_\lambda(F_n)\psi\|_{1\infty} \\ &\leq \sum_{n=m+1}^{\infty} |a_n|\,\|K_\lambda(F_n)\psi\|_{1\infty} \\ &\leq (\sum_{n=m+1}^{\infty} |a_n|(\sup|\theta|\,\|\eta\|)^n)\|\psi\|_{1\infty}, \end{aligned} \tag{2.28}$$

we have

(2.29)

$$\|K_\lambda(F) - \sum_{n=0}^{m} a_n K_\lambda(F_n)\| \leq \sum_{n=m+1}^{\infty} |a_n|(\sup|\theta|\,\|\eta\|)^n \to 0 \quad \text{as} \quad m \to +\infty.$$

Thus we have (2.25). Moreover, from (2.22) and (2.25),

$$\begin{aligned} \|K_\lambda(F)\| &\leq \sum_{n=0}^{\infty} |a_n|\,\|K_\lambda(F_n)\| \\ &\leq \sum_{n=0}^{\infty} |a_n|(\sup|\theta|\,\|\eta\|)^n, \quad \text{as desired.} \end{aligned} \tag{2.30}$$

Corollary in the above Theorem 3 if $f(z) = \exp(z)$ then

$$K_\lambda(F) = \sum_{n=0}^{\infty} \frac{1}{n!} K_\lambda(F_n) \tag{2.31}$$

where the series converges in the operator-norm topology and

$$\|K_\lambda(F)\| \leq \exp(\sup|\theta|\, \|\eta\|). \tag{2.32}$$

REMARK. In the above Corollary, if we let η be Lebesque measure and let $\mathbb{B}$ be the concrete Wiener space $C_1[\alpha, \beta]$, the equation (2.31) and the equation (7,0) in Theorem 8 of [1, p.254] are essentially the same.

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