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ON THE DISCRETE GALERKIN METHODS FOR NONLINEAR INTEGRAL EQUATIONS

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1. Introduction

We consider the discrete Galerkin and iterated discrete Galerkin method for the numerical solution of nonlinear operator equation

$$x = Kx, \tag{1}$$

where K is a completely continuous operator defined on a Banach space X and x is the solution to be determined. Our main application is to let $X = C[0,1]$, the space of continuous functions equipped with the norm

$$\| x \| = \sup_{t\in[0,1]} | x(t) |, \qquad x \in C[0,1],$$

and (1) is the nonlinear integral equation

$$\begin{aligned} x(t) &= (Kx)(t) \\ &= \int_0^1 k(t,s,x(s))\ ds, \quad t \in [0,1]\ ,\ x \in C[0,1], \end{aligned} \tag{2}$$

with $k(t,s,u)$ sufficiently smooth on $G \equiv [0,1]\times[0,1]\times R$ so that $Kx \in C[0,1]$. Further, it is assumed that the solution x^* to be determined is geometrically isolated (see [13]) - that is to say, there is some ball $\Omega_\alpha \equiv \{x \in X : \|x - x^*\| \leq \alpha\}$, with $\alpha > 0$, that contains no solutions of (1) other than x^*.

Let S_n be a finite $n-$ dimensional approximating subspace of X and K_n be a completely continuous operator mapping on X. Then a solution of equation,

$$x_n = K_n x_n, \tag{3}$$

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may be considered an approximate equation of (1). When $K_n = P_n K$, where P_n is the unique orthogonal projection from a Hilbert space containing X into S_n, the equation (3) is called the Galerkin equation. An analysis of this method is given in Krasnoselskii et al. (1972). Assuming x_n exists, the iterated Galerkin approximation $\tilde{x}_n$ is given by

$$\tilde{x}_n = K x_n. \tag{4}$$

Applying the Galerkin method to the equation (2), discrete Galerkin and iterated discrete Galerkin method arises when numerical integration is used to calculate the required integrals. To see the form of integrals that arises, we consider integral form of the Galerkin method applied to the equation (2). To this end, let $\{u_{n_1}, u_{n_2}, \ldots, u_{n_n}\}$ be a basis of S_n. Assume

$$x_n(t) = \sum_{j=1}^{n} a_{n_j} u_{n_j}(t), \qquad t \in [0,1].$$

Then equation (3) is equivalent to solving for $\{a_{n_j}\}$ in the nonlinear system

$$\begin{aligned} &\sum_{i=1}^{n} a_{n_i} < u_{n_i}, u_{n_j} > \\ &= < \int_0^1 k(t,s,\sum_{i=1}^{n} a_{n_i} u_{n_i}(s))\,ds, u_{n_j} >, \\ &\qquad j = 1, \ldots, n, \end{aligned} \tag{5}$$

where $< .,. >$ denotes the usual $L^2[0,1]$ innerproduct, that is

$$< u, v > = \int_0^1 u(s)v(s)\,ds, \qquad u, v \in L^2[0,1].$$

Assumig x_n exists, iterated Galerkin approximation $\tilde{x}_n$ is given by

$$\begin{aligned} \tilde{x}_n(t) &= (K x_n)(t) \\ &= \int_0^1 k(t,s,x_n(s))\; ds, \qquad t \in [0,1]. \end{aligned} \tag{6}$$

To solve (5) and (6), we need the numerical integration scheme. Let I_n be a sequence of quadrature formulae;

$$(7) \qquad I_n x = \sum_{i=1}^{m_n} w_{i,m_n} x(t_{i,m_n}), \qquad x \in C[0,1],$$

with all $t_{i,m_n} \in [a,b]$ and all $w_{i,m_n} > 0$. Here m_n is the number of abscissas. We assume that the numerical integration scheme $I_n x$ converges to Ix as $n \to \infty$ for all $x \in C[a,b]$. Here Ix denote the usual integration,

$$Ix = \int_0^1 x(t)\ dt.$$

With the assumptions on the quadrature formulae, we may suppose that there exists a constant $w > 0$ such that , for details see [6] ,

$$(8) \qquad \sum_{i=1}^{m_n} w_{i,m_n} \leq w, \qquad \forall\, n \geq 1.$$

Ordinarily, the weights and the abscissas will be written simply as w_k and t_k. Applying (7) to (5) and (6), we obtain the discrete Galerkin scheme for the integral equation (2);

$$z_n(t) = \sum_{i=1}^{n} b_{n_i} u_{n_i}(t),$$

$$(9) \qquad \sum_{i=1}^{n} b_{n_i} < u_{n_i}, u_{n_j} >_n$$
$$= < \sum_{k=1}^{m_n} k(t, s_k, \sum_{i=1}^{n} b_{n_i} u_{n_i}(s_k)) w_k, u_{n_j} >_n,$$

$$(10) \qquad \tilde{z}_n(t) = \sum_{i=1}^{m_n} k(t, s_i, z_n(s_i)) w_i,$$

where $< .,. >_n$ is the discrete inner product (see [2]) defined by

$$(11) \qquad < f,g >_n = \sum_{k=1}^{m_n} w_k f(t_k) g(t_k), \qquad f,g \in C[0,1].$$

The principal task in this paper is to show that the approximation $\tilde{z}_n$ converges , under suitable conditions , to an exact solution x^* of (1), and to analysis the rate of this convergence. In fact we will show that , under mild conditions , $\tilde{z}_n$ has a higer order of convergence than the discrete Galerkin approximation z_n converging to x^*. This phenomenion is known as super-convergence , and has been studied in K. E. Atkinson and A. Bogomolny (1987), K. E. Atkinson and F. A. Potra (1989), Chatelin and Lebber (1984) and S. Joe (1987) for the linear integral equations. A more recent contribution to nonlinear integral equation is K. E. Atkinson and F. A. Potra (1987) and S. Kumar and I. H. Sloan (1987).

In section 2, we give some necessary background material including estimates of the order of convergence for the Galerkin scheme, and the analysis of discrete Galerkin scheme is given in Section 3. Section 4 contains some numerical examples.

2. Background material

In this section, we reformulate the theory [11],[12],[13] of the approximate solutions. And then, we apply the results to the Galerkin and iterated Galerkin equations. Also, in section 3 , we apply this results to the discrete Galerkin scheme.

The fundamental theorem about approximate equation (3) was given by Krasnoselskii et al. (see [12], lemma 19.1). The following version of this theorem is sufficient for our purposes.

LEMMA 1. *Let x^* be a solution of (1). Suppose that the operator K and K_n, $n \geq 1$, are completely continuous on X and Frechet differentiable on some neighborhood V of x^*. Further, we assume that $K_n x^* \to K x^*$ and $K_n'(x^*) \to K'(x^*)$ on $B(X)$, the space of bounded linear operators on X, as $n \to \infty$. Suppose that $\{K_n'(x^*) : n \geq 1\}$ is a family of collectively compact operators on X, and 1 is not an eigenvalue of $K'(x^*)$. If a family of mappings*

$$K_n' \; : \; x \in X \longrightarrow K_n'(x) \in B(X) \, , \qquad n \geq 1,$$

is equicontinuous on some neighborhood $U \subseteq V$ of x^* ; i.e., given $\epsilon > 0$ there is $\delta > 0$ such that $\|x - y\| < \delta$ implies

$$\|K_n'(x) - K_n'(y)\| < \epsilon, \qquad x, y \in U.$$

Then for sufficiently large n, equation $K_n x = x$ has a unique solution x_n in some neighborhood $\Omega \subseteq U$ of x^* and the solution satisfies the estimate

$$\|x_n - x^*\| \le c\,\|x^* - K_n x^*\|, \qquad c > 0. \tag{12}$$

Proof. Since K is completely continuous, $K'(x^*)$ is compact in X (see Krasnoselkii and Zabreiko, 1984, p.77). Then a direct application of theorem 1.6 in [1] shows that for sufficiently large n, $(1 - K_n'(x^*))^{-1}$ exists and uniformly bounded, say $\|(1 - K_n'(x^*))^{-1}\| \le M$. To simplify the notation, we will suppose that the above argument holds for all n ≥ 1. Now fix some q $(0 < q < 1)$. From the equicontinuity of the mapping $x \to K_n'(x)$, we can take $\delta_0 > 0$ so small that the ball $\Omega = \{x : \|x - x^*\| < \delta_0\}$ is contained in U and

$$\|K_n'(x) - K_n'(x^*)\| \le \frac{q}{M}$$

whenever $x \in \Omega$. So the condition (19.3) in [11] is satisfied. Now $K_n x^* \to x^*$ implies that for sufficiently large n,

$$\|K_n x^* - x^*\| \le \frac{\delta_0(1-q)}{M}.$$

So the condition (19.4) in [12] is satisfied. Estimate (12) is that of (19.5) in [12].

To use lemma 1, we need to write the Galerkin and iterated Galerkin equations (3),(4) in the operator form. To this end, we firstly define P_n , the (unique) $L^2[a, b]$ orthogonal projection onto the space S_n. For all $\phi \in S_n$, the operator P_n satisfies

$$< P_n g, \phi > = < g, \phi >, \qquad \forall\, g \in L^\infty[a, b].$$

As mentioned in the introduction, the Galerkin approximation z_n will belong to a finite dimensional space $S_n \subseteq C[a,b]$, which is of finite element character. For our analysis, a detailed description of S_n is not required. However, we shall assume that the subspaces S_n and the node points $\{ t_k \}$ are such that

A1. $\lim_{n\to\infty} \| x - P_n x \| = 0 \qquad \textit{for all } x \in C[a,b].$

Usually some form of quasi-uniformity on the partition is required. For more general conditions , see Gusmann (1980). Using the orthogonal projection P_n, equation (3) may be written as

$$x_n = P_n K x_n, \qquad x_n \in S_n, \tag{13}$$

and iterated Galerkin approximation $\tilde{x}_n = K x_n$ satisfies the equation

$$x = K P_n x, \qquad x \in C[0,1]. \tag{14}$$

We remark that if K is Frechet differentiable on X then KP_n , $n \geq 1$ is Frechet differentiable on X and the Frechet derivative of KP_n at $x \in X$ is given by:

$$(KP_n)'(x) = K'(P_n x)P_n, \qquad x \in X\ ,\ n \geq 1.$$

Frechet differentiability of the integral operator K usually depends on the differentiability of the kernel $k(t,s,u)$. We now state the following results concerning the properties of $k(t,s,u)$.

PROPOSITION 1. *Suppose that the kernel $k(t,s,u)$ is continuous and has a continuous partial derivative $\frac{\partial k(t,s,u)}{\partial u}$ for all $(t,s,u) \in [0,1] \times [0,1] \times R$. Then*

(i) *K is a completely continuous operator from $C[0,1]$ into itself.*
(ii) *K is Frechet differentiable on $C[0,1]$; its Frechet derivative at $x \in C[0,1]$ is the linear integral operator given by*

$$(K'x)(y)(t) = \int_0^1 \frac{\partial k(t,s,x(s))}{\partial u} y(s)\,ds, \qquad y \in C[0,1].$$

(iii) *mapping $x \to K'(x)$ is continuous from $C[0,1]$ into $B(C[0,1])$.*

Proof. Part (i),(ii) are well-known (see [13], p. 83). To prove (iii), note that $\frac{\partial k(t,s,u)}{\partial u}$ is uniformly continuous on any compact subset of $[0,1] \times [0,1] \times R$. So given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{\partial k}{\partial u}(t,s,a) - \frac{\partial k}{\partial u}(t,s,b) \right| < \epsilon$$

whenever $|a-b| < \delta$. Now let $x, y, u \in C[0,1]$ and $\|x-y\| < \delta$, $\|u\| \leq 1$. Then for any $t \in [0,1]$

$$|(K'x)(u)(t) - (K'y)(u)(t)| \leq \epsilon$$

This proves our theorem.

From A1, P_n is uniformly bounded on $C[0,1]$. Hence from the argument in [13] (p.74), P_nK and KP_n are completely continuous.

PROPOSITION 2. *Let x^* be a solution of (1). Suppose A1 holds. Also suppose that the kernel $k(t,s,u)$ is continuous and has a continuous partial derivative $\frac{\partial k(t,s,u)}{\partial u}$ for all $(t,s,u) \in [0,1] \times [0,1] \times R$. Then*

- (i) *$x \to K'(P_nx)P_n$, $n \geq 1$, is a family of equicontinuous mappings from $C[0,1]$ into $B(C[0,1))$.*
- (ii) *$\{K'(P_nx^*)P_n : n \geq 1\}$ form a pointwise convergent and collectively compact family of operators on $C[0,1]$.*

Proof. Part (i) follows from proposition 1 and the uniform boundedness of P_n. To prove (ii), we note that

$$\| K'(P_nx^*) - K'(x^*) \| \longrightarrow 0 \quad as\, n \to \infty.$$

So, we may assume that there is a constant $\alpha > 0$ such that

$$\| K'(P_nx^*) \| \leq \alpha < \infty.$$

Now for $u \in C[0,1]$,

$$\|K'(P_nx^*)P_nu - K'(x^*)u\| \leq \alpha \|P_nu - u\| + \|K'(P_nx^*) - K'(x^*)\| \|u\|.$$

Proving that $K'(P_nx^*)P_nu \to K'(x^*)u$ as $n \to \infty$, for all $u \in C[0,1]$. Now for $u \in \Omega_1$, the closed unit ball in $C[0,1]$,

$$\| K'(P_nx^*)P_nu \| \le \|K'(P_nx^*)\| \, \|P_n\| \, \|u\|.$$

Now let $p = \sup_{n\ge 1} \|P_n\|$. Then given $\epsilon > 0$ there exists $\delta > 0$ such that

$$| \frac{\partial k}{\partial u}(t,s,P_nx^*(s)) - \frac{\partial k}{\partial u}(t',s,P_nx^*(s)) | < \frac{\epsilon}{p},$$

if $t, t' \in [0,1]$, $|t-t'| < \delta$. Hence for $u \in \Omega_1$,

$$| K'(P_nx^*)P_nu(t) - K'(P_nx^*)P_nu(t') | \le \epsilon$$

when $|t-t'| < \delta$. This complete the proof, by Arzela-Ascoli theorem.

Now we will establish the order of convergence of Galerkin and iterated Galerkin method for solving (1).

THEOREM 1. *Suppose that the hypothesis of proposition 2 holds. If 1 is not an eigenvalue of* $K'(x^*)$. *Then*

(i) *Equation (13) has a unique solution* x_n *in some neighborhood of* x^* *and the solution satisfies the estimate*

$$\| x_n - x^* \| \le \alpha \| x^* - P_nx^* \|, \qquad \alpha > 0. \tag{15}$$

(ii) *Equation (14) has a unique solution* $\tilde{x}_n$ *in some neighborhood of* x^* . *Further, if* $\frac{\partial^2 k}{\partial u^2}(t,s,u)$ *exists and continuous on* $[0,1] \times [0,1] \times R$ *then* $\tilde{x}_n$ *satisfies the estimate*

(16)

$$\|\tilde{x}_n - x^*\| \le \sup_{t\in[0,1]} | < (1-P_n)k_t \, , \, (1-P_n)x^* > | + \beta \| x^* - P_nx^* \|^2,$$

where $\beta > 0$ and $k_t(s) = \frac{\partial k}{\partial u}(t,s,x^*(s))$.

Proof. Part (i) is well known (see [13] , p. 326). Existence and uniqueness of the approximation $\tilde{x}_n$ follows from lemma 1. So we prove (16). By Taylor's formula we have

$$\begin{aligned} k(t,s,(P_nx^*)(s)) &= k(t,s,x^*(s)) + \frac{\partial k}{\partial u}(t,s,x^*(s))(x^*(s)-P_nx^*(s)) \\ &+ \frac{1}{2}\frac{\partial^2 k}{\partial u^2}(t,s,x^*(s)+\theta P_nx^*(s))(x^*(s)-P_nx^*(s))^2, \\ &(0<\theta<1). \end{aligned}$$

From the uniform continuity of $\frac{\partial k^2}{\partial u^2}$ on any compact subset of $[0,1] \times [0,1] \times R$, we have

$$\begin{aligned} |\int_0^1 k(t,s,(P_nx^*)(s)) - k(t,s,x^*(s))\,ds| &\leq |\int_0^1 k_t(s)(1-P_n)x^*(s)\,ds| \\ &+ \beta \parallel x^* - P_nx^* \parallel^2. \end{aligned}$$

Combine this with (12), we get the estimate (16).

3. Discrete Galerkin Scheme

In this section, we prove the existence of discrete Galerkin and iterated discrete Galerkin approximations z_n , $\tilde{z}_n$ and give the order of convergence of the the approximations.

For the analysis of discrete Galerkin method we firstly discuss the discrete orthogonal projection Q_n introduced by Atkinson, K.E. and Bogomolny,A. [1987]. Using this discrete projection, we will give an error analysis for the discrete Galerkin method. Here we summarize the results in [2]. Let Φ be the martix of order $n \times m_n$ with

$$\Phi_{i,k} = u_{n_i}(t_k), \qquad i=1,...,n\ ,\ k=1,...,m_n,$$

where $\{ u_{n_i} \}$ is a basis of S_n and $\{ t_k \}$ node points. Hereafter we assume that $n \leq m_n$ and $Rank(\Phi) = n$. With these assumptions, we can define the discrete orthogonal projection Q_n. For $z \in C[a,b]$, the discrete orthogonal projection $Q_n : C[a,b] \longrightarrow S_n$ is defined by

$$< Q_nz, x >_n = < z, x >_n, \qquad x \in S_n$$

where $< .,. >_n$ is the discrete inner product (11). We state a relation between orthogonal projection P_n and discrete orthogonal projection Q_n.

LEMMA 2 (ATKINSON, K.E.,AND BOGOMOLNY, A. 1987).

(i) *If the family $\{Q_n : n \geq 1\}$ is uniformly bounded on $C[a,b]$. Then*

$$\| f - Q_n f \| \leq c \| f - P_n f \|, \tag{17}$$

with $c > 0$ independent of n and f.

(ii) *Q_n is selfadjoint on $C[0,1]$, relative to the discrete innerproduct (11);*

$$< Q_n f, g >_n = < f, Q_n g >_n, \qquad f, g \in C[0,1] \tag{18}$$

In fact, the uniform boundedness of $\{Q_n : n \geq 1\}$ depends on the approximating space S_n and the partition of node points. (Atkinson, K.E.,and Bogomolny, A. 1987). Hereafter , for our analysis , we assume that Q_n is uniformly bounded.

Recall the numerical integral operator K_n based on the integration rule (7),

$$(K_n x)(t) = \sum_{i=1}^{m_n} w_i \, k(t, s_i, x(s_i)), \qquad x \in C[0,1].$$

Using the orthogonal projection Q_n and the numerical integral operator K_n, equation (9) may be written in the operator form

$$z_n = Q_n K_n z_n, \qquad z_n \in S_n, \tag{19}$$

and iterated discrete Galerkin approximation $\tilde{z}_n = K_n z_n$ satisfies the equation

$$z = K_n Q_n z, \qquad z \in C[0,1]. \tag{20}$$

PROPOSITION 3. *Suppose that the kernel $k(t,s,u)$ is continuous and has continuous partial derivative $\frac{\partial k}{\partial u}(t,s,u)$ for all $(t,s,u) \in [0,1] \times [0,1] \times R$. Then*

(i) *K_n is a completely continuous operator from $C[0,1]$ into itself.*

(ii) K_n *is Frechet differentiable on* $[0,1] \times [0,1] \times R$ *and its derivative at* $x \in C[0,1]$ *is given by*

$$(21) \qquad (K_n{}'x)y(t) = \sum_{i=1}^{m_n} \frac{\partial k}{\partial u}(t, s_i, x(s_i))y(s_i), \qquad y \in C[0,1].$$

(iii) *Mapping* $x \to K_n{}'(x)$, $n \geq 1$, *is continuous from* $C[0,1]$ *into* $B(C[0,1])$.

Proof. Part (ii) is well known (see [1] , p. 97). To prove (i) , note that the function $k(t,s,u)$ is uniformly continuous on $[0,1] \times [0,1] \times [-r,r]$, $0 \leq r < \infty$. Thus for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$| \; k(t,s,u) \; - \; k(t+\Delta t, s, u) \; | \; < \; \epsilon$$

when $(t,s,u), (t+\Delta t, s, u) \in [0,1] \times [0,1] \times [-r,r]$ and $|\Delta t| < \delta$. Now let $t \in [0,1]$ and $x \in C[0,1]$. We suppose that $\|x\| \leq r$. Then given $\epsilon > 0$

$$(22) \qquad \begin{aligned} &|(K_n x)(t+\Delta t) - (K_n x)(t)| \\ &\leq \sum_{i=1}^{m_n} w_i \; (|k(t+\Delta t, s_i, x(s_i)) - k(t, s_i, x(s_i))| \\ &\leq \epsilon \sum_{i=1}^{m_n} w_i \; \leq \; \epsilon \; w \end{aligned}$$

if $|\Delta t| < \delta$. Final inequality is due to (8). This shows that $\{K_n x : x \in \Omega_r, n \geq 1\}$ is a family of equicontinuous functions. Now the following implication

$$x \in \Omega_r \; \Rightarrow \; \|K_n x\| \; \leq \sup_{(t,s,u) \in [0,1] \times [0,1] \times [-r,r]} |k(t,s,u)| \; w$$

shows that $\{K_n(\Omega_r) : n \geq 1\}$ is precompact by the Arzela-Ascoli theorem. Now we show that K_n is continuous from $C[0,1]$ into $C[0,1]$. Let $x, y \in C[0,1]$ and $\epsilon > 0$ be given. Then from the uniform continuity of $k(t,s,u)$ there exists $\delta > 0$ such that

$$| \; k(t,s,a) \; - \; k(t,s,b) \; | \; < \; \frac{\epsilon}{w},$$

when $|a - b| < \delta$. Hence if $\|x - y\| < \delta$ then

$$| (K_n x)(t) - (K_n y)(t) | < w \frac{\epsilon}{w} = \epsilon.$$

This proves part (i). From the uniform continuity of $\frac{\partial k}{\partial u}(t, s, u)$ on any compact subset of $[0,1] \times [0,1] \times R$, part (iii) can be proved in similar manner to the above argument.

From A1 and (17), discrete orthogonal projection Q_n is uniformly bounded on $C[0,1]$. Hence from the arguments in [12 , p74], $Q_n K_n$ and $K_n Q_n$ are completely continuous.

PROPOSITION 4. *Suppose that the hypothesis of proposition 2 holds. Then*

(i) $x \to Q_n K_n'(x), n \geq 1$, *is a family of equicontinuous mappings on* $C[0,1]$. *Moreover,* $\{Q_n K_n'(x^*) : n \geq 1\}$ *form a pointwise convergent and collectively compact family of operators on* $C[0,1]$.

(ii) $\{x \to K_n'(Q_n x) Q_n \;: n \geq 1\}$ *is a family of equicontinuous mappings. Moreover* $\{K_n'(Q_n x^*) Q_n : n \geq 1\}$ *form a pointwise convergent and collectively compact family of operators on* $C[0,1]$.

Proof. We prove part (ii). Part (i) can be dealt with in similar manner. Let $x, y \in C[0,1]$, $t \in [0,1]$ and $\sup_n \|Q_n\| \leq q$. Given $\epsilon > 0$ there is $\delta > 0$ suct that

$$| \frac{\partial k}{\partial u}(t, s, x(s)) - \frac{\partial k}{\partial u}(t, s, y(s)) | < \frac{\epsilon}{q\, w},$$

when $|x(s) - y(s)| \leq \delta$. So if $u \in \Omega_1$ and $\|x - y\| < \delta$ then

$$\begin{aligned} & | K_n'(Q_n x) Q_n u(t) - K_n'(Q_n y) Q_n u(t) | \\ & \leq \sum_{i=1}^{m_n} | \frac{\partial k}{\partial u}(t, s_i, (Q_n x)(s_i)) - \frac{\partial k}{\partial u}(t, s_i, (Q_n y)(s_i)) | \, |(Q_n u)(s_i)| \\ & < \epsilon. \end{aligned}$$

This proves equicontinuity. Now we prove $\|K_n{}'(Q_n x^*) - K'(x^*)\| \to 0$ as $n \to \infty$. Given $\epsilon > 0$ we can take $\delta > 0$ such that, if $|u_1 - u_2| < \delta$ then

$$(23) \qquad \left| \frac{\partial k}{\partial u}(t,s,u_1) - \frac{\partial k}{\partial u}(t,s,u_2) \right| < \frac{\epsilon}{2w}.$$

Since $\|Q_n x^* - x^*\| \to 0$ and $\|K_n{}'(x^*) - K'(x^*)\| \to 0$ as $n \to \infty$, we can take n_0 such that $n \geq n_0$ implies

$$(24) \qquad \| Q_n x^* - x^* \| < \delta,$$

$$(25) \qquad \| K_n{}'(x^*) - K'(x^*) \| < \frac{\epsilon}{2}.$$

So, from (23) , (24) and (25) , we get

$$(26) \qquad \| K_n{}'(Q_n x^*) - K'(x^*) \| \longrightarrow 0, \quad as\, n \to \infty.$$

Pointwise convergent directly follows from (26). Also from (26), we may suppose that

$$(27) \qquad \| K_n{}'(Q_n x^*) \| \leq \alpha, \qquad \alpha > 0.$$

Now if $x \in \Omega_1$ then

$$\|K_n{}'(Q_n x^*) Q_n x\| \leq \alpha\, q$$

and given $\epsilon > 0$ there is $\delta > 0$ such that

$$\left|\frac{\partial k}{\partial u}(t, s_i, Q_n x^*(s_i)) - \frac{\partial k}{\partial u}(t', s_i, Q_n x^*(s_i))\right| \leq \frac{\epsilon}{q\, w}.$$

Hence if $|t - t'| \leq \delta$ and $y \in \Omega_1$ then

$$| K_n{}'(Q_n x^*)(Q_n y)(t) - K_n{}'(Q_n x^*)(Q_n y)(t') | < \epsilon.$$

So $\{K_n{}'(Q_n x^*) Q_n : n \geq 1\}$ form a collectively compact family of operators on $C[0,1]$ by the Arzela-Ascoli theorem.

THEOREM 2. *Suppose that the hypothesis of proposition 2 holds. If 1 is not an eigenvalue of* $K'(x^*)$. *Then*

(i) *Equation (19) has a unique solution* z_n *in some neighborhood of* x^* *and* z_n *satisfies the estimate*

$$\| z_n - x^* \| \le \alpha_1 \| x^* - P_n x^* \| + \alpha_2 \| K_n x^* - K x^* \| \quad \alpha_1, \alpha_2 > 0. \tag{28}$$

(ii) *Equation (20) has a unique solution* $\tilde{z}_n$ *in some neighborhood of* x^*. *Furthermore, if* $\frac{\partial^2 k}{\partial u^2}(t,s,u)$ *exists and continuous on* $[0,1] \times [0,1] \times R$, *then* $\tilde{z}_n$ *satisfies the estimate*

$$\begin{aligned} &\| \tilde{z}_n - x^* \| \\ &\le \beta_1 \| K x^* - K_n x^* \| + \beta_2 \| x^* - P_n x^* \|^2 \\ &+ \beta_3 \sup_{t \in [0,1]} | < x^* - Q_n x^* , k_t - Q_n k_t >_n | \\ &\quad \beta_1 , \beta_2 , \beta_3 > 0. \end{aligned} \tag{29}$$

Proof. Existence and uniqueness of the approximations z_n , $\tilde{z}_n$ follows from lemma 1. So we prove (28) , (29). From (12), we get

$$\begin{aligned} \| x^* - z_n \| &\le \alpha \| K x^* - Q_n K_n x^* \| \\ &\le \alpha_1 \| x^* - Q_n x^* \| + \alpha_2 \|Q_n\| \| K_n x^* - K x^* \| \end{aligned}$$

This is the result (28). We now prove (29). From (12), we have

$$\begin{aligned} \| x^* - \tilde{z}_n \| &\le \beta \| x^* - K_n Q_n x^* \| \\ &\le \beta \| K x^* - K_n x^* \| \\ &\quad + \beta \| K_n Q_n x^* - K_n x^* \| \end{aligned} \tag{30}$$

By Taylor's formula, we have

$$\begin{aligned} &k(t,s,(Q_n x^*)(s)) \\ &= k(t,s,x^*(s)) + \frac{\partial k}{\partial u}(t,s,x^*(s))(x^*(s) - Q_n x^*(s)) \\ &+ \frac{1}{2}\frac{\partial^2 k}{\partial u^2}(t,s,x^*(s) + \theta Q_n x^*(s))(x^*(s) - Q_n x^*(s))^2, \\ &(0 < \theta < 1). \end{aligned}$$

From the uniform continuity of $\frac{\partial^2 k}{\partial u^2}$ on any compact subset of $[0,1] \times [0,1] \times R$, we have

$$\begin{aligned} & | \ (K_n x^*)(t) \ - \ (K_n Q_n x^*)(t) \ | \\ = & \ | \ \sum_{i=1}^{m_n} k(t, s_i, x^*(s_i)) \ w_i \ - \ \sum_{i=1}^{m_n} k(t, s_i, (Q_n x^*)(s_i)) \ w_i \ | \\ \leq & \ \beta_2 \ | \ < \ k_t \ , \ (1 - Q_n) x^* \ >_n \ | \ + \ \beta_3 \ \| \ x^* \ - \ Q_n x^* \ \|^2 \end{aligned}$$

Combine this with (18) and (30), we obtain (29).

4. Numerical examples

We illustrate the convergence results that were given in Theorem 1, 2 for the Galerkin and discrete Galerkin method.

Now let $X = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$, where $0 = x_0 < x_1 < \ \ldots \ < x_{n-1} < x_n = 1$, be a partition of $[0,1]$ with

$$h \ = \ \max_{1 \leq i \leq n} \ (\ x_i \ - \ x_{i-1} \ ).$$

We shall assume that the mesh is quasi-uniform, that is, there is a constant c such that $h \ / \ \min_{1 \leq i \leq n}(x_i - x_{i-1}) \ \leq \ c$. We take S_n to be the set of piecewise linear functions and we take the solution x^* from (1) is of class $C^\alpha[0,1]$. Then it is well-known that , (see [5], corollary 7)

$$\| \ P_n x^* \ - \ x^* \ \| \ = \ O(h^{\min(\alpha, 2)}). \tag{31}$$

Further, we suppose that $g(t,s) \ = \ k(t,s,x^*(s))$ and $k(t,s) \ = \ \frac{\partial k}{\partial u}(t,s,x^*(s))$ are the Chatelin and Lebbar class of $\Xi(\alpha,\gamma)$:

A function $w : [0,1] \times [0,1] \to R$ is of class $\Xi(\alpha,\gamma)$ (with $\alpha \geq \gamma, \alpha \geq 0, \gamma \geq -1$) if and only if

$$w(t,s) \ = \ \begin{cases} w_1(t,s), & 0 \leq s < t \leq 1, \\ w_2(t,s), & 0 \leq t \leq s \leq 1 \end{cases}$$

with $w_1 \ \in \ C^\alpha(\{0 \leq s \leq t \leq 1\})$, $w_2 \ \in \ C^\alpha(\{0 \leq t \leq s \leq 1\})$ and $w \ \in \ C^\gamma([0,1] \times [0,1])$ for $\gamma \geq 0$. In cases $\gamma = -1$, w may have a discontinuity of the first kind on $\{s = t\}$.

If we take the m -points Gauss-Legendre quadrature formula for the numerical integration (6). Then from the arguments in [7] (p. 50), we get

$$\| K_n x^* - K x^* \| = O(h^{\min(\alpha,\gamma+2,2m)}). \tag{32}$$

Also, using Atkinson and Potra (1987 , Lemma 1) and Chatelin and Lebbar (1984 , Lemma 9), we can show that

$$| < k_t , (1-Q_n)x^* >_n | = O(h^{(\beta_1+\beta_2)}) \tag{33}$$

where, $\beta_1 = \min(\alpha, 2)$ and $\beta_2 = \min(\beta_1, \gamma + 2)$.

So, from (15),(16),(28),(29) and (31), we expect that

$$\| x_n - x^* \| = O(h^{\beta_1}) \tag{34}$$

$$\| \tilde{x}_n - x^* \| = O(h^{\beta_1+\beta_2}) \tag{35}$$

$$\| z_n - x^* \| = O(h^{\min(\beta_1,\beta_3)}) \tag{36}$$

$$\| \tilde{z}_n - x^* \| = O(h^{\min(\beta_1+\beta_2,\beta_3)}) \tag{37}$$

where, $\beta_1 = \min(\alpha, 2)$, $\beta_2 = \min(\alpha, \gamma + 2, 2)$, $\beta_3 = \min(\alpha, \gamma + 2, 2m)$.

With these facts, we give results for two integral equations. Our first equation is

$$x(t) = f(t) + \int_0^1 \frac{1}{t+s+x^2(s)}\, ds, \qquad 0 \le t \le 1$$

where f is so chosen that

$$x^*(t) = e^{-t}, \qquad t \in [0,1]$$

is a solution of the equation. The function K is given by

$$K(t,s,u) = f(t) + \frac{1}{t+s+u^2}.$$

In this case, the constants α and γ can be chosen as large as desired. From (33),(34),(35) and (36), we expect that

$$\| x^* - x_n \| = O(h^2), \qquad \| x^* - \tilde{x}_n \| = O(h^4),$$
$$\| x^* - z_n \| = O(h^2), \qquad \| x^* - \tilde{z}_n \| = O(h^4),$$

for all $m \geq 2$. This is confirmed by the results in table 1 and table 2.

Table 1
$\| x^* - z_n \|_\infty$.

n	m = 2	ratio	m = 4	ratio
3	0.19 E-1	—	0.19 E-1	—
5	0.49 E-2	3.87	0.49 E-2	3.87
9	0.18 E-2	2.72	0.17 E-2	2.88
17	0.32 E-3	5.62	0.32 E-3	5.31
33	0.81 E-4	3.95	0.81 E-4	3.95
65	0.20 E-4	4.05	0.20 E-4	4.05

Table 2
$\| x^* - \tilde{z}_n \|_\infty$.

n	m = 2	ratio	m = 4	ratio
3	0.18 E-3	—	0.19 E-3	—
5	0.73 E-5	24.6	0.90 E-5	21.1
9	0.41 E-6	17.8	0.52 E-6	17.3
17	0.25 E-7	16.4	0.32 E-7	16.3
33	0.16 E-8	15.6	0.20 E-8	16.0
65	0.10 E-9	16.0	0.12 E-9	16.7

Our second example is

$$x(t) = \int_0^1 H(t,s)(\sin(x(s)) + f(s)) \, ds$$
$$H(t,s) = \begin{cases} -s(1-t) & (s \leq t), \\ -t(1-s) & (t \leq s), \end{cases}$$

with $f(s)$ so chosen that

$$x^*(t) = \frac{t(1-t)}{(t+2)}.$$

For this equation, $\gamma = 0$ and α can be chosen as large as desired. From (34),(35),(36) and (37), we expect that

$$\| x^* - x_n \| = O(h^2), \quad \| x^* - \tilde{x}_n \| = O(h^4),$$
$$\| x^* - z_n \| = O(h^2), \quad \| x^* - \tilde{z}_n \| = O(h^2),$$

for all $m \geq 2$. The results for $m = 2, 4, 6$ are given in table 3 and 4.

Table 3
$\| x^* - z_n \|_\infty$.

n	m=2	ratio	m=4	ratio
3	0.26 E-1	—	0.25 E-1	—
5	0.72 E-2	3.61	0.71 E-2	3.52
9	0.19 E-2	3.78	0.18 E-2	3.94
17	0.48 E-3	3.95	0.48 E-3	3.75
33	0.12 E-3	4.00	0.12 E-3	4.00
65	0.30 E-4	4.00	0.30 E-4	4.00

Table 4
$\| x^* - \tilde{z}_n \|_\infty$

n	m=2	ratio	m=4	ratio
3	0.58 E-3	—	0.30 E-3	—
5	0.14 E-3	4.14	0.43 E-4	6.97
9	0.34 E-4	4.11	0.98 E-5	4.38
17	0.85 E-5	4.00	0.24 E-5	4.08
33	0.18 E-5	4.72	0.44 E-6	5.45
65	0.34 E-6	5.29	0.14 E-6	3.14

In each cases, the partition X is given by $\{\frac{i}{n} : 0 \leq i \leq n\}$. The maximum errors listed in Table 1,2,3 and 4 were estimated by taking the largest of the computed errors at $t = \frac{i}{128}, i = 0, ..., 128$. All computations were carried out in double precision on a Sun 4/20 computer.

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