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ON THE TYPE OF PLANE CURVE SINGULARITIES ANALYTICALLY EQUIVALENT TO THE EQUATION $z^n + y^k = 0$ WITH $\gcd(n,k) = 1$

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1. Introduction

Let $V = \{(z,y) : f(z,y) = 0\}$ and $W = \{(z,y) : g = z^n + y^k = 0\}$ with $gcd(n,k) = 1$ be germs of analytic subvarities of a polydisc near the origin where f has an isolated singular point at the origin. Assume that V and W are topologically equivalent near the origin. Then by a nonsingular linear change of coordinates f can be written as $z^n + A_2 y^{\alpha_2} z^{n-2} + \cdots + A_i y^{\alpha_i} z^{n-i} + \cdots + A_{n-1} y^{\alpha_{n-1}} z + y^k$ where the $A_i = A_i(y)$ are nonvanishing holomorphic near the origin and $\frac{\alpha_i}{i} > \frac{k}{n}$ for $i = 2, \cdots, n-1$ by [2]. Assume that $n < k$ and $gcd(n,k) = 1$. We are going to prove in each case whether V and W are analytically equivalent near the origin or not as follows : If V and W are analytically equivalent, then denote this relation by $V \approx W$ or $f \approx g$. If not, we write $V \not\approx W$ or $f \not\approx g$.

(1) If $\alpha_i + n - i < k$ and $A_i(0) \neq 0$ for some i with $2 \leq i \leq n-1$, then $f \not\approx g$.

(2) If $\alpha_i + n - i = k$ for some i with $2 \leq i \leq n-1$ and $V \approx W$, then either $A_i(0) = 0$ for $i = 2, \cdots, n-1$ or $A_i(0) \neq 0$ for $i = 2, \cdots, n-1$. If $A_{n-1}(0) \neq 0$, then $A_{n-i}(0) = {}_kC_i(\frac{1}{k}A_{n-1}(0))^i$ for $i = 2, \cdots, n-2$.

(3) Suppose that $\alpha_i + n - i > k$ for $i = 2, \cdots, n-1$ and that there is some t with $2 \leq t \leq n-1$ such that $\alpha_t + n - t = \text{Min}\{\alpha_i + n - i : 2 \leq i \leq n-1\}$ and $\alpha_t \leq k-2$. Then $f \not\approx g$.

(4) If $\alpha_i \geq k-1$ for $i = 2, \cdots, n-1$ (which implies $\alpha_i + n - i \geq k$) and $\alpha_j = k-1$ for some j with $2 \leq j \leq n-1$, then $f \approx g$ if and only if $2(n-l) \geq n-1$ where $l = \text{Max}\{j : \alpha_j = k-1,\ 2 \leq j \leq n-1\}$.

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(5) Assume that $\alpha_i + n - i > k$ for $i = 2, \cdots, n-1$ and $\alpha_t + n - t > \mathrm{Min}\{\alpha_i + n - i : 2 \leq i \leq n-1\}$ whenever $\alpha_t \leq k-2$. Then either $f \approx g$ or $f \not\approx g$.

Moreover, it is interesting to apply this result to some examples which are not analytically equivalent to any weighted homogeneous polynomial by using the blowing-up process.

2. Known Preliminaries

DEFINITION 2.1. Let $V = \{z \in \mathbf{C}^n : f(z) = 0\}$ and $W = \{z \in \mathbf{C}^n : g(z) = 0\}$ be germs of complex anaytic hypersurfaces with isolated singular points at the origin. V and W are said to be topologically equivalent near the origin if there is a germ at the origin of homeomorphisms $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ and $\phi(0) = 0$ where U_1 and U_2 are open subset containing the origin in $\mathbf{C}^2$. Also, V and W are said to be analytically equivalent near the origin if there is a germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $\psi(V) = W$ and $\psi(0) = 0$ where U_1 and U_2 are open subset containing the origin in $\mathbf{C}^2$.

DEFINITION 2.2. The polynomial $f(z_1, \cdots, z_n)$ is weighted homogeneous of type $(\frac{1}{a_1}, \cdots, \frac{1}{a_n})$ if there is some positive rational numbers $a_1, \cdots, a_n$ such that $f(t^{a_1} z_1, \cdots, t^{a_n} z_n) = t f(z_1, \cdots, z_n)$.

Let ${}_n\mathcal{O}$ denote the ring of germs of holomorphic functions at the origin in $\mathbf{C}^n$.

THEOREM 2.3(MATHER-YAU [3] AND SHOSHITAĬSHVILI [4]). *Suppose that $f(z_1, \cdots, z_n)$ has the isolated singular point at the origin. Then the following statements are equivalent.*

(i) *f is analytically equivalent to a weighted homogeneous polynomial.*
(ii) *$f \in m\Delta(f)$ where $m\Delta(f)$ is the ideal in ${}_n\mathcal{O}$ generated by $z_i \frac{\partial f}{\partial z_j}$ for all $i, j = 1, \cdots, n$.*
(iii) *There is a germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $f \circ \psi = g$ for a weighted homogeneous polynomial g where U_1 and U_2 are open neighborhoods of the origin in $\mathbf{C}^n$.*

(iv) *$K(f)$ is isomorphic to $K(g)$ for a weighted homogeneous polynomial g as a* $\mathbf{C}$*-algebra where* $K(f) = {}_n\mathcal{O}/(f, m\Delta(f))$, $K(g) = {}_n\mathcal{O}/(g, m\Delta(g))$ *and* $(f, m\Delta(f))$ *is the ideal in* ${}_n\mathcal{O}$ *generated by* f *and* $m\Delta(f)$.

Proof. See [3] and [4].

3. On the type of plane curve singularities analytically equivalent to the equation $z^n + y^k = 0$ with $gcd(n,k) = 1$

THEOREM 3.1. *Let* $V = \{(z,y) : f = z^n + A_2 y^{\alpha_2} z^{n-2} + \cdots + A_i y^{\alpha_i} z^{n-i} + \cdots + A_{n-1} y^{\alpha_{n-1}} z + y^k = 0\}$ *and* $W = \{(z,y) : g = z^n + y^k = 0\}$ *be germs of analytic subvarieties of a polydisc near the origin in* $\mathbf{C}^2$ *where* f *and* g *are Weierstrass polynomials, the* $A_i = A_i(y)$ *are nonvanishing holomorphic near* $y = 0$ *for* $i = 2, \cdots, n-1$. *Assume that* $n < k$ *and* $gcd(n,k) = 1$. *If* $\alpha_i + n - i < k$ *and* $A_i(0) \neq 0$ *for some* i *with* $2 \leq i \leq n-1$, *then* $f \not\approx g$.

Proof. Note that since f is irreducible in ${}_2\mathcal{O}$, $\alpha_i + n - i > n$. Note by Theorem 2.3 that $f \approx g$ if and only if $f \circ \phi = g$ for some biholomorphic mapping $\phi : (U_1, 0) \to (U_2, 0)$ with $\phi(V) = W$ where U_1 and U_2 are open subsets containing the origin in $\mathbf{C}^2$. Then we may assume that $\phi(z,y) = (H(z,y), L(z,y))$ as follows :

$$H = H(z,y) = \alpha z + \beta y + H_2 + H_3 + \cdots \quad \text{and}$$
$$L = L(z,y) = \gamma z + \delta y + L_2 + L_3 + \cdots$$

where H_n and L_n are homogeneous polynomials of degree n with $H_n = H_n(z,y) = a_{n,0} z^n + a_{n-1,1} z^{n-1} y + \cdots + a_{0,n} y^n$ and $L_n = L_n(z,y) = b_{n,0} z^n + b_{n-1,1} z^{n-1} y + \cdots + b_{0,n} y^n$.

Observe that $\alpha\delta - \beta\gamma \neq 0$. We suppose that

$$\begin{aligned}
& f \circ \phi(z,y) = (\alpha z + \beta y + H_2 + H_3 + \cdots)^n \\
& + A_2(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{\alpha_2} (\alpha z + \beta y + H_2 + H_3 + \cdots)^{n-2} \\
& + \cdots \\
& + A_i(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{\alpha_i} (\alpha z + \beta y + H_2 + H_3 + \cdots)^{n-i} \\
& + \cdots \\
& + A_{n-1}(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{\alpha_{n-1}} (\alpha z + \beta y + H_2 + H_3 + \cdots) \\
& + A_n(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^k = z^n + y^k.
\end{aligned}$$

Now it is enough to get a contradiction. Let $I = \{j : \alpha_j + n - j = \mathrm{Min}\{\alpha_i + n - i : 2 \le i \le n-1\}\}$ and $m = \mathrm{Max}\, I$. Then $\beta = 0$ because $n < k$ and $n < \alpha_i + n - i$ for any i. Note that $\alpha\delta - \beta\gamma = \alpha\delta \neq 0$. But $A_m(0)\cdot(\delta y)^{\alpha_m}\cdot(\alpha z)^{n-m} = A_m(0)\delta^{\alpha_m}\alpha^{n-m}y^{\alpha_m}z^{n-m}$ is the unique one containing the monomial $y^{\alpha_m}z^{n-m}$ in the expansion of $f\circ\phi(z,y)$ with $\alpha_m + n - m < k$. This would imply that $\alpha\delta = 0$. It is impossible.

Before proving Theorem 3.3, we need the following Lemma.

LEMMA 3.2. *Recall the notation* ${}_nC_k = \binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$. *Then*

$$D = \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-1}C_1 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-1}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_k & {}_{n+1}C_k & \cdots & {}_{n+k-1}C_k \end{vmatrix}$$

$$= \begin{vmatrix} {}_nC_1 & {}_nC_0 & 0 & \cdots & 0 \\ {}_nC_2 & {}_nC_1 & {}_nC_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_nC_{k-2} & {}_nC_{k-3} & \cdots & {}_nC_0 \\ {}_nC_k & {}_nC_{k-1} & {}_nC_{k-2} & \cdots & {}_nC_1 \end{vmatrix}$$

$$= (-1)^{k(k-1)/2} \begin{vmatrix} 0 & \cdots & 0 & {}_{n+k-2}C_0 & {}_{n+k-1}C_1 \\ 0 & \cdots & {}_{n+k-3}C_0 & {}_{n+k-2}C_1 & {}_{n+k-1}C_2 \\ \vdots & & \vdots & \vdots & \vdots \\ {}_nC_0 & \cdots & {}_{n+k-3}C_{k-3} & {}_{n+k-2}C_{k-2} & {}_{n+k-1}C_{k-1} \\ {}_nC_1 & \cdots & {}_{n+k-3}C_{k-2} & {}_{n+k-2}C_{k-1} & {}_{n+k-1}C_k \end{vmatrix}$$

$$= {}_{n+k-1}C_k.$$

Proof. To compute D, subtracting $(k-1)$-column from k-column, we have

$$D = \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-2}C_1 & {}_{n+k-2}C_0 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-2}C_2 & {}_{n+k-2}C_1 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_nC_k & {}_{n+1}C_k & \cdots & {}_{n+k-2}C_{k-1} & {}_{n+k-2}C_{k-1} \end{vmatrix}$$

Applying the same technique to $(k-1)$–column, $(k-2)$–column,$\cdots$, the second column in order, we get

$$D = \begin{vmatrix} {}_{n}C_1 & {}_{n}C_0 & \cdots & {}_{n+k-3}C_0 & {}_{n+k-2}C_0 \\ {}_{n}C_2 & {}_{n}C_1 & \cdots & {}_{n+k-3}C_1 & {}_{n+k-2}C_1 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n}C_k & {}_{n}C_{k-1} & \cdots & {}_{n+k-3}C_{k-1} & {}_{n+k-2}C_{k-1} \end{vmatrix}$$

Using the same technique, by induction we get

$$D = \begin{vmatrix} {}_{n}C_1 & {}_{n}C_0 & \cdots & 0 & 0 \\ {}_{n}C_2 & {}_{n}C_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n}C_{k-1} & {}_{n}C_{k-2} & \cdots & {}_{n}C_1 & {}_{n}C_0 \\ {}_{n}C_k & {}_{n}C_{k-1} & \cdots & {}_{n}C_2 & {}_{n}C_1 \end{vmatrix}$$

This is the first form which we want.

With respect to the k-th column only, D is linear and so, D can be represented in the following:

$$D = \begin{vmatrix} {}_{n}C_1 & {}_{n}C_0 & \cdots & 0 & 0 \\ {}_{n}C_2 & {}_{n}C_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n}C_{k-1} & {}_{n}C_{k-2} & \cdots & {}_{n}C_1 & {}_{n-1}C_0 \\ {}_{n}C_k & {}_{n}C_{k-1} & \cdots & {}_{n}C_2 & {}_{n-1}C_1 \end{vmatrix}$$

$$+ \begin{vmatrix} {}_{n}C_1 & {}_{n}C_0 & \cdots & 0 \\ {}_{n}C_2 & {}_{n}C_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ {}_{n}C_{k-2} & {}_{n}C_{k-3} & \cdots & {}_{n}C_0 \\ {}_{n}C_{k-1} & {}_{n}C_{k-2} & \cdots & {}_{n}C_1 \end{vmatrix}$$

$= D_1 + D_2$ where D_1 is the $k \times k$ matrix and D_2 is the $(k-1) \times (k-1)$ matrix. Applying the same technique to D_1 as in the beginning of the proof, then we have

$$D_1 = \begin{vmatrix} {}_{n-1}D_1 & {}_{n-1}C_0 & \cdots & 0 & 0 \\ {}_{n-1}C_2 & {}_{n-1}C_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n-1}C_{k-1} & {}_{n-1}C_{k-2} & \cdots & {}_{n-1}C_1 & {}_{n-1}C_0 \\ {}_{n-1}C_k & {}_{n-1}C_{k-1} & \cdots & {}_{n-1}C_2 & {}_{n-1}C_1 \end{vmatrix}$$

Next, applying the same method to D_1 and D_2 reversely as in the beginning of the proof, then we get

$$D = D_1 + D_2 = \begin{vmatrix} {}_{n-1}C_1 & {}_nC_1 & \cdots & {}_{n+k-2}C_1 \\ {}_{n-1}C_2 & {}_nC_2 & \cdots & {}_{n+k-2}C_2 \\ \vdots & \vdots & & \vdots \\ {}_{n-1}C_k & {}_nC_k & \cdots & {}_{n+k-2}C_k \end{vmatrix} + \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-2}C_1 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-2}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_{n+1}C_{k-1} & \cdots & {}_{n+k-2}C_{k-1} \end{vmatrix}$$

By induction on $n+k$, then $D_1 = {}_{n+k-2}C_k$ and $D_2 = {}_{n+k-2}C_{k-1}$. Therefore $D = {}_{n+k-1}C_k$. Now to express D in another way, subtracting the second column from the first column, the third column from the second column,$\cdots$, the k-column from $(k-1)$-column in D in order, then we have

$$D = (-1)^k \begin{vmatrix} {}_nC_0 & {}_{n+1}C_0 & \cdots & {}_{n+k-1}C_1 \\ {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-1}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_{n+1}C_{k-1} & \cdots & {}_{n+k-1}C_k \end{vmatrix}.$$

Using the same process by induction on k, we have the desired result.

THEOREM 3.3. *Let $V = \{(z,y) : f = z^n + A_2y^{k-n+2}z^{n-2} + \cdots + A_iy^{k-n+i}z^{n-i} + \cdots + A_{n-1}y^{k-1}z + y^k = 0\}$ and $W = \{(z,y) : g = z^n + y^k = 0\}$ be germs of analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f and g are Weierstrass polynomials and the $A_i = A_i(y)$ are holomorphic near $y=0$ for $i = 2,\cdots,n-1$. Assume that $n < k$ and $gcd(n,k) = 1$. If $f \approx g$, then either $A_i(0) = 0$ for all $i = 2,\cdots,n-1$ or $A_i(0) \neq 0$ for all $i = 2,\cdots,n-1$. If $A_{n-1}(0) \neq 0$, then $A_{n-i}(0) = {}_kC_i(\frac{1}{k}A_{n-1}(0))^i$ for $i = 2,\cdots,n-2$.*

Proof. Note by Theorem 2.3 that $f \approx g$ if and only if $f \circ \phi = g$ for some biholomorphic mapping $\phi : (U_1, 0) \to (U_2, 0)$ with $\phi(V) = W$ where U_1 and U_2 are open subsets containing the origin in $\mathbf{C}^2$. Now we may assume that $\phi(z,y) = (H(z,y), L(z,y))$ where $H = H(z,y) =$

$\alpha z + \beta y + H_2 + H_3 + \cdots$ and $L = L(z, y) = \gamma z + \delta y + L_2 + L_3 + \cdots$, and H_n and L_n are homogeneous polynomials of degree n with $H_n = H_n(z, y) = a_{n,0}z^n + a_{n-1,0}z^{n-1}y + \cdots + a_{0,n}y^n$ and $L_n = L_n(z, y) = b_{n,0}z^n + b_{n-1,0}z^{n-1}y + \cdots + b_{0,n}y^n$. Then

$$
\begin{aligned}
&f \circ \phi(z, y) = (\alpha z + \beta y + H_2 + H_3 + \cdots)^n \\
&+A_2(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{k-n+2}(\alpha z + \beta y + H_2 + H_3 + \cdots)^{n-2} \\
&+\cdots \\
&+A_i(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{k-n+i}(\alpha z + \beta y + H_2 + H_3 + \cdots)^{n-i} \\
&+\cdots \\
&+A_{n-1}(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{k-1}(\alpha z + \beta y + H_2 + H_3 + \cdots) \\
&+(\gamma z + \delta y + L_2 + L_3 + \cdots)^k = z^n + y^k.
\end{aligned}
$$

Observe that $\beta = 0$, $H_2 = H_3 = \cdots = H_{k-n} = 0$, $\alpha^n = 1$ and $\delta^k = 1$ since $n < k$. If $\gamma = 0$, then it is easy to show that $A_i(0) = 0$ for all $i = 2, \cdots, n-1$ because $\alpha\delta - \beta\gamma = \alpha\delta \neq 0$. Suppose that $A_{n-i}(0) = 0$ for some i with $1 \leq i \leq n-2$. Then consider coefficients of the following monomials $y^{k-i}z^i$, $y^{k-i+1}z^{i-1}, \cdots, y^{k-2}z^2, y^{k-1}z$ in $f \circ \phi(z, y)$: For brevity, put $s_j = A_{n-j}(0)$ for $j = 1, \cdots, i$.

$$
\begin{aligned}
[1] \qquad & y^{k-i}z^i : s_i\delta^{k-i}\alpha^i + s_{i-1}\binom{k-i+1}{1}\gamma\delta^{k-i}\alpha^{i-1} + \cdots \\
& + s_1\binom{k-1}{i-1}\gamma^{i-1}\delta^{k-i}\alpha + \binom{k}{i}\gamma^i\delta^{k-i} \\
& = \delta^{k-i}[s_{i-1}\binom{k-i+1}{1}\gamma\alpha^{i-1} + \cdots + s_1\binom{k-1}{i-1}\gamma^{i-1}\alpha \\
& + \binom{k}{i}\gamma^i] = 0.
\end{aligned}
$$

$$[2] \quad y^{k-i+1}z^{i-1} : \delta^{k-i+1}[s_{i-1}\alpha^{i-1} + s_{i-2}\binom{k-i+2}{1}\gamma\alpha^{i-2} + \cdots + s_1\binom{k-1}{i-2}\gamma^{i-2}\alpha + \binom{k}{i-1}\gamma^{i-1}]$$

$$= \delta^{k-i+1}\gamma^{-1}[s_{i-1}\gamma\alpha^{i-1} + s_{i-2}\binom{k-i+2}{1}\gamma^2\alpha^{i-2} + \cdots + s_1\binom{k-1}{i-2}\gamma^{i-1}\alpha + \binom{k}{i-1}\gamma^i] = 0.$$

$$\cdots$$

$$[i-1] \quad y^{k-2}z^2 : \delta^{k-2}[s_2\alpha^2 + s_1\binom{k-1}{1}\gamma\alpha + \binom{k}{2}\gamma^2]$$

$$= \delta^{k-2}\gamma^{2-i}[s_2\gamma^{i-2}\alpha^2 + s_1\binom{k-1}{1}\gamma^{i-1}\alpha + \binom{k}{2}\gamma^i] = 0.$$

$$[i] \quad y^{k-1}z : \delta^{k-1}[s_1\alpha + \binom{k}{1}\gamma] = \delta^{k-1}\gamma^{1-i}[s_1\gamma^{i-1}\alpha + \binom{k}{1}\gamma^i] = 0.$$

We are going to prove that $\gamma = 0$. Suppose not.

Then considering $s_{i-1}\gamma\alpha^{i-1}, s_{i-2}\gamma^2\alpha^{i-2}, \cdots, s_2\gamma^{i-2}\alpha^2, s_1\gamma^{i-1}\alpha, \gamma^i$ as a solution of the above $[i]$-homogeneous equations, we get an $i \times i$-square matrix Δ consisting of coefficients of $s_{i-1}\gamma\alpha^{i-1}, s_{i-2}\gamma^2\alpha^{i-2}, \cdots$, $s_2\gamma^{i-2}\alpha^2$, $s_1\gamma^{i-1}\alpha, \gamma^i$ in these equations as follows:

$$\Delta = \begin{pmatrix} \binom{k-i+1}{1} & \binom{k-i+2}{2} & \cdots & \binom{k-2}{i-2} & \binom{k-1}{i-1} & \binom{k}{i} \\ 1 & \binom{k-i+2}{1} & \cdots & \binom{k-2}{i-3} & \binom{k-1}{i-2} & \binom{k}{i-1} \\ 0 & 1 & & \binom{k-2}{i-4} & \binom{k-1}{i-3} & \binom{k}{i-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & & 1 & \binom{k-1}{1} & \binom{k}{2} \\ 0 & 0 & & 0 & 1 & \binom{k}{1} \end{pmatrix}$$

To compute the determinant $|\Delta|$, subtracting $(i-1)$-column from i-column and $(i-2)$-column from $(i-1)$-column and so on, by induction on i, we get

$$|\Delta| = \begin{pmatrix} \binom{k-i+1}{1} & \binom{k-i+1}{2} & \binom{k-i+1}{3} & \cdots & \binom{k-i+1}{i-2} & \binom{k-i+1}{i-1} & \binom{k-i+1}{i} \\ 1 & \binom{k-i+1}{1} & \binom{k-i+1}{2} & \cdots & \binom{k-i+1}{i-3} & \binom{k-i+1}{i-2} & \binom{k-i+1}{i-1} \\ 0 & 1 & \binom{k-i+1}{1} & \cdots & \binom{k-i+1}{i-4} & \binom{k-i+1}{i-3} & \binom{k-i+1}{i-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & & 1 & \binom{k-i+1}{1} & \binom{k-i+1}{2} \\ 0 & 0 & 0 & & 0 & 1 & \binom{k-i+1}{1} \end{pmatrix}$$

$$= {}_{k-i+1+i-1}C_i = {}_kC_i \neq 0$$

by Lemma 3.2.

Since $|\Delta| \neq 0$, we would have $s_{i-1}\gamma\alpha^{i-1} = s_{i-2}\gamma^2\alpha^{i-2} = \cdots = s_1\gamma^{i-1}\alpha = \gamma^i = 0$, and so γ would be zero. It is impossible. Thus we proved that $\gamma = 0$. Therefore, $A_i(0) = 0$ for each $i = 2, \cdots, n-2$. As a result, either $A_i(0) = 0$ for $i = 2, \cdots, n-2$ or $A_i(0) \neq 0$ for $i = 2, \cdots, n-2$.

Now we assume that $\gamma \neq 0$. Similarly, consider coefficients of the following monomials $y^{k-i}z^i, y^{k-i+1}z^{i-1}, \cdots, y^{k-2}z^2, y^{k-1}z$ in $f \circ \phi(z, y)$: Put $s_j = A_{n-j}(0)$ for $j = 1, \cdots, i$.

$$[1] \qquad y^{k-i}z^i : \delta^{k-i}[s_i\alpha^i + s_{i-1}\binom{k-i+1}{1}\gamma\alpha^{i-1} + \cdots + s_1\binom{k-1}{i-1}\gamma^{i-1}\alpha + \binom{k}{i}\gamma^i] = 0.$$

$$[2] \quad y^{k-i+1}z^{i-1} : \delta^{k-i+1}\gamma^{-1}[s_{i-1}\gamma\alpha^{i-1} + s_{i-2}\binom{k-i+2}{1}\gamma^2\alpha^{i-2} + \cdots + s_1\binom{k-1}{i-2}\gamma^{i-1}\alpha + \binom{k}{i-1}\gamma^i] = 0.$$

$$\cdots$$

$$[i-1] \quad y^{k-2}z^2 : \delta^{k-2}\gamma^{2-i}[s_2\gamma^{i-2}\alpha^2 + s_1\binom{k-1}{1}\gamma^{i-1}\alpha + \binom{k}{2}\gamma^i] = 0.$$

$$[i] \qquad y^{k-1}z : \delta^{k-1}\gamma^{1-i}[s_1\gamma^{i-1}\alpha + \binom{k}{1}\gamma^i] = 0.$$

Note that $s_i \neq 0$ because $\gamma \neq 0$. Considering $s_i\alpha^i, s_{i-1}\gamma\alpha^{i-1}, \cdots, s_2\gamma^{i-2}\alpha^2, s_1\gamma^{i-1}\alpha$ as a solution of the above $[i]$-nonhomogeneous equations, then we get $s_i\alpha^i = (-1)^i|\Delta|\gamma^i = (-1)^i{}_kC_i\gamma^i$ by Lemma 3.2. Thus $s_i = (-\frac{\gamma}{\alpha})^i{}_kC_i = (\frac{1}{k}A_{n-1}(0))^i{}_kC_i$ for $i = 2, \cdots, n-2$.

THEOREM 3.4. *Let $V = \{(z,y) : f = z^n + A_2 y^{\alpha_2} z^{n-2} + \cdots + A_i y^{\alpha_i} z^{n-i} + \cdots + A_{n-1} y^{\alpha_{n-1}} z + y^k = 0\}$ and $W = \{(z,y) : g = z^n + y^k = 0\}$ be germs of analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f is a Weierstrass polynomial and the $A_i = A_i(y)$ are nonvanishing holomorphic near $y = 0$ for $i = 2, \cdots, n-1$. Assume that $n < k$ and $gcd(n,k) = 1$. If $\alpha_i + n - i > k$ for all $i = 2, \cdots, n-1$ and $\alpha_t \le k-2$ for some t such that $\alpha_t + n - t = Min\{\alpha_i + n - i : 2 \le i \le n-1\}$, then $f \not\approx g$.*

Proof. Assume the contrary. Note by Theorem 2.3 that $f \approx g$ if and only if $f \circ \phi = g$ for some biholomorphic mapping $\phi : (U_1, 0) \to (U_2, 0)$ with $\phi(V) = W$ where U_1 and U_2 are open subsets containing the origin in $\mathbf{C}^2$. Now we may assume that $\phi(z,y) = (H(z,y), L(z,y))$ as follows:

$$\begin{aligned} H &= H(z,y) = \alpha z + \beta y + H_2 + H_3 + \cdots \\ L &= L(z,y) = \gamma z + \delta y + L_2 + L_3 + \cdots \end{aligned}$$

where H_n and L_n are homogeneous polynomials of degree n with $H_n = H_n(z,y) = a_{n,0} z^n + a_{n-1,1} z^{n-1} y + \cdots + a_{0,n} y^n$ and $L_n = L_n(z,y) = b_{n,0} z^n + b_{n-1,1} z^{n-1} y + \cdots + b_{0,n} y^n$. Observe that $\alpha\delta - \beta\gamma \ne 0$. Then

$$\begin{aligned} & f \circ \phi(z,y) = (\alpha z + \beta y + H_2 + H_3 + \cdots)^n \\ &+ A_2(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{\alpha_2} (\alpha z + \beta y + H_2 + H_3 + \cdots)^{n-2} \\ &+ \cdots \\ &+ A_i(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{\alpha_i} (\alpha z + \beta y + H_2 + H_3 + \cdots)^{n-i} \\ &+ \cdots \\ &+ A_{n-1}(L)(\gamma z + \delta y + L_2 + L_3 + \cdots)^{\alpha_{n-1}} (\alpha z + \beta y + H_2 + H_3 + \cdots) \\ &+ (\gamma z + \delta y + L_2 + L_3 + \cdots)^k = z^n + y^k. \end{aligned}$$

Note that $\beta = 0$, $H_2 = H_3 = \cdots = H_{k-n} = 0$ and $\alpha^n = 1$ since $n < k < \alpha_i + n - i$ for $i = 2, \cdots, n-1$. If $k < \alpha_i + n - i \le k - 2 + n - i$ for some i, then $n > i + 2$. So we may assume that $n \ge 5$. Also, $\alpha_{n-1} > k - 1$ and $\alpha_{n-2} < k - 2$. Let

$$\begin{aligned} m &= \text{Min}\{\alpha_i + n - i : 2 \le i \le n-1\} \text{ and} \\ t &= \text{Min}\{l : \alpha_l + n - l = m,\ \alpha_l \le k-2,\ 2 \le l \le n-3\}. \end{aligned}$$

Then there is a positive integer p such that $\alpha_t + n - t = k + p$. We claim that $H_{k-n+1} = 0$, $H_{k-n+2} = L_2 = 0$, $H_{k-n+3} = L_3 = 0, \cdots, H_{k-n+p} = L_p = 0$. We are going to prove it by induction on the integer p. Since $\alpha_i + n - i \geq k + p > k$, in the expansion of monomials of degree k in $f \circ \phi(z, y)$, ${}_nC_1(\alpha z)^{n-1} H_{k-n+1} + (\gamma z + \delta y)^k = y^k$. That is, $n\alpha^{n-1} z^{n-1}(a_{k-n+1,0} z^{k-n+1} + a_{k-n,1} z^{k-n} y + \cdots + a_{0,k-n+1} y^{k-n+1}) + (\gamma z + \delta y)^k = y^k$. Since $n \geq 5$, the coefficients of $y^{k-1}z, y^{k-2}z^2$ and $y^{k-3}z^3$ are zero, γ must be zero and so $H_{k-n+1} = 0$. If $\alpha_i + n - i > k+1$ for any i, then

$$\begin{aligned}
0 &= {}_nC_1(\alpha z)^{n-1} H_{k-n+2} + {}_kC_1(\delta y)^{k-1} L_2 \\
&= {}_nC_1(\alpha z)^{n-1}(a_{k-n+2,0} z^{k-n+2} + a_{k-n+1,1} z^{k-n+1} y + \cdots \\
&\quad + a_{0,k-n+2} y^{k-n+2}) + {}_kC_1(\delta y)^{k-1}(b_{2,0} z^2 + b_{1,1} zy + b_{0,2} y^2)
\end{aligned}$$

in the expansion of monomials of degree $k+1$ in $f \circ \phi(z, y)$. Since $n \geq 5$, $L_2 = 0$ and so $H_{k-n+2} = 0$. To prove the above claim by induction on the integer p, assume that if $\alpha_i + n - i > k + s$ for any i and some positive integer s with $1 < s < p$, then $H_{k-n+s+1} = L_{s+1} = 0$ with $s + 1 < p$. Now it is enough to show that $H_{k-n+s+2} = L_{s+2} = 0$ with $s + 2 \leq p$. Then

$$\begin{aligned}
0 &= {}_nC_1(\alpha z)^{n-1} H_{k-n+s+2} + {}_kC_1(\delta y)^{k-1} L_{s+2} \\
&= {}_nC_1(\alpha z)^{n-1}(a_{k-n+s+2,0} z^{k-n+s+2} + a_{k-n+s+1,1} z^{k-n+s+1} y + \cdots \\
&\quad + a_{0,k-n+s+2} y^{k-n+s+2}) + {}_kC_1(\delta y)^{k-1}(b_{s+2,0} z^{s+2} + b_{s+1,1} z^{s+1} y \\
&\quad + \cdots + b_{0,s+2} y^{s+2}).
\end{aligned}$$

Note that $n - 1 > s + 2$ because $k - 2 + n - t \geq \alpha_t + n - t > k + s$. So $L_{s+2} = H_{k-n+s+2} = 0$.

Now consider the nonvanishing monomials of degree $\alpha_t + n - t = k + p$ in the expansion of $f \circ \phi(z, y)$ as follows :

$$\begin{aligned}
&f \circ \phi(z, y) = (\alpha z + H_{k-n+p+1} + H_{k-n+p+2} + \cdots)^n \\
&+ A_2(L)(\delta y + L_{p+1} + L_{p+2} + \cdots)^{\alpha_2} \\
&\quad (\alpha z + H_{k-n+p+1} + H_{k-n+p+2} + \cdots)^{n-2} \\
&+ \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ A_i(L)(\delta y + L_{p+1} + L_{p+2} + \cdots)^{\alpha_i} \\
&\quad (\alpha z + H_{k-n+p+1} + H_{k-n+p+2} + \cdots)^{n-i} \\
&+ \cdots \\
&+ A_{n-1}(L)(\delta y + L_{p+1} + L_{p+2} + \cdots)^{\alpha_{n-1}} \\
&\quad (\alpha z + H_{k-n+p+1} + H_{k-n+p+2} + \cdots) \\
&+ (\delta y + L_{p+1} + L_{p+2} + \cdots)^k.
\end{aligned}
$$

Observe that $\alpha_i + n - i \geq \alpha_t + n - t = k + p$ for $i = 2, \cdots, n-1$ and $\alpha_t \leq k-2$. So $y^{\alpha_t} z^{n-t}$ is one of nonvanishing monomials in the expansion of $f \circ \phi(z, y)$. Therefore $f \not\approx g$.

PROPOSITION 3.5. *Let* $V = \{(z,y) : h = z^n + y^{k-1}z^j + y^k = 0\}$ *and* $W = \{(z,y) : g = z^n + y^k = 0\}$ *where* $j = 1, \cdots, n-1$. *Assume that* $n < k$ *and* $\gcd(n,k) = 1$. *Then* $h \approx g$ *if and only if* $2j \geq n-1$.

Proof. We know by Theorem 2.3 that $h \approx g$ if and only if $K(h)$ is isomorphic to $K(g)$ as a **C**-algebra. Compute $\mu = \dim K(h)$ and $\nu = \dim K(g)$ over the complex field **C** as vector spaces. Then it is easy to find that $\nu = nk - n - k + 3$. Now consider the ideal $(h, m\Delta(h))$ in ${}_2\mathcal{O}$ generated by h and $m\Delta(h)$ as follows:

$$
\begin{aligned}
h &= z^n + y^{k-1}z^j + y^k \\
zh_z &= nz^n + jy^{k-1}z^j, \\
yh_y &= (k-1)y^{k-1}z^j + ky^k, \\
yh_z &= nyz^{n-1} + jy^k z^{j-1}, \\
zh_y &= (k-1)y^{k-2}z^{j+1} + ky^{k-1}z.
\end{aligned}
$$

Then $(h, m\Delta(h)) = (z^n, y^{k-1}z^j, y^k, yz^{n-1}, zhy)$. Now it is enough to consider two cases:

Case (i) : $2j < n-1$.

Then $zh_y = (k-1)y^{k-2}z^{j+1} + ky^{k-1}z \equiv 0 \mod(h, m\Delta(h))$.

So

$$\begin{aligned} yzh_y &= (k-1)y^{k-1}z^{j+1} \equiv ky^k z \equiv 0. \\ z^2h_y &= (k-1)y^{k-2}z^{j+2} + ky^{k-1}z^2 \equiv 0. \\ z^3h_y &= (k-1)y^{k-2}z^{j+3} + ky^{k-1}z^3 \equiv 0. \\ &\vdots \\ z^{j-1}h_y &= (k-1)y^{k-2}z^{2j-1} + ky^{k-1}z^{j-1} \equiv 0. \\ z^jh_y &= (k-1)y^{k-2}z^{2j} \equiv ky^{k-1}z^j \equiv 0. \end{aligned}$$

Using the above equations, we get $\mu = nk - k - 2n + 2j + 4$. But $\mu = \nu$ would imply $2j = n-1$, which is impossible.

Case (ii): $n - 1 \leq 2j$ (put $n-1 = j+l$ with $0 < l \leq j$).
Then $zh_y = (k-1)y^{k-2}z^{j+1} + ky^{k-1}z \equiv 0 \mod(h, m\Delta(h))$.
So

$$\begin{aligned} yzh_y &= (k-1)y^{k-1}z^{j+1} \equiv ky^k z \equiv 0. \\ z^2h_y &= (k-1)y^{k-2}z^{j+2} + ky^{k-1}z^2 \equiv 0. \\ z^3h_y &= (k-1)y^{k-2}z^{j+3} + ky^{k-1}z^3 \equiv 0. \\ &\vdots \\ z^{j+l-1}h_y &= (k-1)y^{k-2}z^{2j+l-1} + ky^{k-1}z^{l-1} \equiv 0. \\ z^{j+l}h_y &= (k-1)y^{k-2}z^{2j+l} \equiv ky^{k-1}z^l \equiv 0. \end{aligned}$$

Using the above equations, we get $\mu = nk - n - k + 3$. Therefore, it is enough to show that if $2j \geq n-1$, then $h \in m\Delta(h)$ by Theorem 2.3. From the ideal $m\Delta(h)$ we have $z^n \equiv ay^{k-1}z^j \equiv by^k \equiv cy^{k-2}z^{2j}$ $(\mathrm{mod}(m\Delta(h)))$ for suitable nonzero constants a, b and c. If $2j \geq n$, then $z^n \equiv cy^{k-2}z^{2j}(\mathrm{mod}(m\Delta(h)))$ implies that z^n belongs to $m\Delta(h)$, and so $h \in m\Delta(h)$. If $2j = n-1$, then $z^n \equiv ay^{k-1}z^j \equiv by^k \equiv cy^{k-2}z^{n-1} \equiv dy^{2k-3}z^{j-1}(\mathrm{mod}(m\Delta(h)))$ for some nonzero constant d. Thus $by^k \equiv dy^{2k-3}z^{j-1}$ and $k \geq 3$ imply $y^k \in m\Delta(h)$ and so $h \in m\Delta(h)$. Thus if $2j \geq n-1$, then we prove that $h \approx g$.

THEOREM 3.6. *Let* $V = \{(z,y) : f = z^n + A_2y^{\alpha_2}z^{n-2} + \cdots + A_iy^{\alpha_i}z^{n-i} + \cdots + A_{n-1}y^{\alpha_{n-1}}z + y^k = 0\}$ *and* $W = \{(z,y) : g =$

$z^n + y^k = 0\}$ be germs of analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f is a Weierstrass polynomial and the $A_i = A_i(y)$ are nonvanishing holomorphic near $y = 0$ for $i = 2, \cdots, n-1$. Assume that $n < k$ and $\gcd(n,k) = 1$. If $\alpha_i \geq k-1$ for $i = 2, \cdots, n-1$ and $\alpha_j = k-1$ for some j, then $f \approx g$ if and only if $2(n-l) \geq n-1$ where $l = \mathrm{Max}\{j : \alpha_j = k-1,\ 2 \leq j \leq n-1\}$.

Proof. If $\alpha_i \geq k$ for $i = 2, \cdots, n-1$, then there is nothing to prove. Suppose that there is some j such that $\alpha_j = k-1$. Let $l = \mathrm{Max}\{j : \alpha_j = k-1, 2 \leq j \leq n-1\}$. Then we rewrite f as $f = z^n + b_l(y,z)y^{k-1}z^{n-l} + b_k(y,z)y^k = 0$ where $b_l(y,z)$ and $b_k(y,z)$ are nonvanishing holomorphic functions at the origin in $\mathbf{C}^2$. By a nonsingular linear change of coordinates, f is analytically equivalent to $f_1 = z^n + y^{k-1}z^{n-l} + c_k(y,z)y^k = 0$ near the origin where $c_k(y,z)$ is a nonvanishing holomorphic function at the origin in $\mathbf{C}^2$. So it is enough to show that $f_1 \approx g$. Using the similar techniques as we have used in the proof of Proposition 3.5, we can prove that $f_1 \approx g$ if and only if $2(n-l) \geq n-1$ where $l = \mathrm{Max}\{j : \alpha_j = k-1,\ 2 \leq j \leq n-1\}$.

THEOREM 3.7. *Let $V = \{(z,y) : f = z^n + A_2 y^{\alpha_2} z^{n-2} + \cdots + A_i y^{\alpha_i} z^{n-i} + \cdots + A_{n-1} y^{\alpha_{n-1}} z + y^k = 0\}$ and $W = \{(z,y) : g = z^n + y^k = 0\}$ be germs of analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f is a Weierstrass polynomial and the $A_i = A_i(y)$ are nonvanishing holomorphic near $y = 0$ for $i = 2, \cdots, n-1$. Assume that $n < k$ and $\gcd(n,k) = 1$. If $\alpha_i + n - i > k$ for $i = 2, \cdots, n-1$ and $\alpha_t + n - t > \mathrm{Min}\{\alpha_i + n - i : 2 \leq i \leq n-1\}$ whenever $\alpha_t \leq k-2$, then either $f \approx g$ or $f \not\approx g$.*

Proof. It is enough to construct a different kind of two examples satisfying the conclusion of the theorem as follows :

(1) Let $f = z^{11} + {}_{25}C_3 y^{22} z^9 + {}_{25}C_2 y^{23} z^6 + {}_{25}C_1 y^{24} z^3 + y^{25}$.

Then

$$f \approx u(z,y)z^{11} + (y + z^3)^{25}$$
$$\approx z^{11} + y^{25} \text{ near the origin where } u(z,y) \text{ is a unit in } {}_2\mathcal{O}.$$

(2) Let $f = z^{11} + y^{23} z^7 + y^{25} z + y^{25}$.

Then

$$\begin{aligned} f &= z^{11} + y^{23}z^7 + y^{25}(z+1) \\ &\approx z^{11} + y^{23}z^7 + y^{25} \\ &\not\approx z^{11} + y^{25} \text{ near the origin by Theorem 3.4.} \end{aligned}$$

Finally we can apply the previous results to some examples which are not analytically equivalent to weighted homogeneous polynomials as follows:

Let $V = \{(z, y) : f = (z^3 + y^4)^5 + y^{10}z^7(z^3 + y^4) + y^9z^{11} = 0\}$ and $W = \{(z, y) : g = (z^3 + y^4)^5 + y^9z^{11} = 0\}$. We claim that $f \not\approx g$. Since blowing-up process preserves analytical equivalence, it is enough to prove that the proper transforms of V and W are not analytically equivalent after a finite number of blowing-ups. Note that after four times of blowing-ups, the proper transform of V is analytically equivalent to $\{(u, v) : f_4 = u^5 + v^{10}u + v^{11} = 0\}$ and the proper transform of W is analytically equivalent to $\{(u, v) : g_4 = u^5 + v^{11} = 0\}$. By Proposition 3.5, $f_4 \not\approx g_4$. Thus $f \not\approx g$.

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