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POINTWISE DECAY ESTIMATES OF SOLUTIONS OF THE GENERALIZED ROSENAU EQUATION

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1. Introduction

Consider the following initial-value problem for the generalized Rosenau equation ([5], [6]):

$$(1.1) \qquad \begin{aligned} &u_t + u_x + F(u)_x + u_{xxxxt} = 0, \quad && x \in \mathbf{R},\ t > 0 \\ &u(x,0) = \varphi(x), && x \in \mathbf{R}. \end{aligned}$$

Here $F(u) = \sum_{i=1}^{n} \frac{c_i}{p_i+1} u^{p_i+1}$ ($c_i \in \mathbf{R}$, $p_i > 0$ integers). It is of interest to investigate the behavior of solutions of the initial value problem (1.1). It has been proved in ([5], Chapter I) that (1.1) possesses global solutions for initial data $\varphi \in H_0^4(\mathbf{R})$.

The purpose of this paper is to prove Theorem 1 below, which says that solutions with small initial data φ decay like $t^{-1/5}$ in the supremum norm (if each $p_i \geq 6$). This is an analogue of Albert's result ([2]) on the L^∞ decay of solutions of the generalized Benjamin-Bona-Mahony equation.

2. Statement and proof of the main result

For $1 \leq p < \infty$, the symbol L^p will denote the space of Lebesgue measurable functions $f : \mathbf{R} \to \mathbf{R}$ with the norm

$$\|f\|_{L^p} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p};$$

while L^∞ denotes the space of measurable functions such that $\|f\|_{L^\infty} = \text{esssup}_{x\in\mathbf{R}}|f(x)|$ is finite.

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The Fourier transform is defined, for smooth functions $f(x)$ with compact support, by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)dx, \quad k \in \mathbf{R}.$$

The definition is extended by continuity to the space of tempered distributions on $\mathbf{R}$, and the Sobolev space H^s ($s \in \mathbf{R}$) is defined to be the subspace of tempered distributions f such that $\widehat{f}(k)$ is a function and the norm $\|f\|_{H^s} = (\int_{-\infty}^{\infty}(1+|k|^2)^s|\widehat{f}(k)|^2 dk)^{1/2}$ is finite. (See [4], [8] or [9].)

THEOREM 1. *Suppose* $p^* = \min\limits_{1\leq i\leq n} p_i \geq 6$. *Then there exists* $\delta > 0$ *such that if* $\|\varphi\|_{L^1} + \|\varphi'\|_{L^1} + \|\varphi\|_{H^4} < \delta$, *then the unique solution* $u(x,t)$ *of (1.1) satisfies*

$$|u(x,t)| \leq C(1+t)^{-1/5}$$

for all $t > 0$ *and* $x \in \mathbf{R}$, *where the constant* C *does not depend on* x *or* t.

The proof of Theorem 1 uses the estimate stated in Proposition 1 below for solutions of the linearized problem

$$\begin{aligned} &u_t + u_x + u_{xxxxt} = 0 \\ &u(x,0) = \varphi(x). \end{aligned} \tag{2.1}$$

This is the linearization of (1.1) about the solution $v \equiv 0$. Zero is a solution since $F(0) = F'(0) = 0$.

Using the Fourier transformation method, one finds that the solution of (2.1) is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ih_\alpha(k)t}\widehat{\varphi}(k)dk,$$

where

$$\begin{cases} h_\alpha(k) = g(k) - \alpha k \\ g(k) = \frac{k}{1+k^4} \quad \text{and} \\ \alpha = \frac{x}{t}. \end{cases} \tag{2.2}$$

PROPOSITION 1. *There exists a constant $C > 0$ such that*

$$|\int_{-\infty}^{\infty} e^{ih_\alpha(k)t}\widehat{\varphi}(k)dk| \leq C(\|\varphi\|_{L^1} + \|\varphi\|_{H^3})(1+t)^{-1/5}$$

for all $\alpha \in \mathbf{R}$ and $t > 0$.

We first prove Theorem 1, assuming Proposition 1. Let $S_t\varphi(x)$ denote the solution $u(x,t)$ of the linearized problem (2.1). Then solutions of the inhomogeneous equation

$$w_t + w_x + w_{xxxxt} = (G(x,t))_x$$

may easily be seen to satisfy the variation-of-parameters formula

$$w(x,t) = S_t(w(x,0)) + \int_0^t S_{(t-\zeta)}(\mathcal{K} * G(x,\zeta))d\zeta,$$

where $\mathcal{K}(x) = \frac{\pi i}{2}(e^{-\frac{\sqrt{2}}{2}|x|+\frac{\sqrt{2}}{2}ix} + e^{-\frac{\sqrt{2}}{2}|x|-\frac{\sqrt{2}}{2}ix})$ and $*$ denotes convolution with respect to the x variable.

Hence the solution of (1.1) satisfies the integral equation

$$u(x,t) = S_t\varphi - \int_0^t S_{(t-\tau)}(\mathcal{K} * F(u))d\tau. \tag{2.3}$$

Moreover, differentiating (1.1) yields

$$(u_x)_t + (u_x)_x + (u_x)_{xxxxt} = -(F'(u)u_x)_x$$

and $v = u_x$ satisfies

$$v(x,t) = S_t(\varphi') - \int_0^t S_{(t-\tau)}(\mathcal{K} * F'(u)v)d\tau. \tag{2.4}$$

Now define

$$q(t) = \sup_{0\leq\tau\leq t}\{(\|u(\tau)\|_{L^\infty} + \|u_x(\tau)\|_{L^\infty})(1+\tau)^{1/5} + \|u\|_{H^3}\}.$$

It will be shown that

$$q(t) \leq C(\|\varphi\|_{L^1} + \|\varphi'\|_{L^1} + \|\varphi\|_{H^4} + \sum_{i=1}^{n} q(t)^{p_i+1}). \tag{2.5}$$

We will make use of the two estimates

$$\|\mathcal{K} * \psi\|_{L^1} \leq \|\mathcal{K}\|_{L^1}\|\psi\|_{L^1} \leq C\|\psi\|_{L^1}, \tag{2.6}$$

$$\|\mathcal{K} * \psi\|_{H^s} \leq C\|\psi\|_{H^s}. \tag{2.7}$$

Inequality (2.6) is a standard one. To show (2.7), we have, for all $s \in \mathbf{R}$,

$$\begin{aligned}
\|\mathcal{K} * \psi\|_{H^s} &= (\int_{-\infty}^{\infty} (1+k^2)^s (\widehat{\mathcal{K} * \psi})^2 dk)^{1/2} \\
&= (\int_{-\infty}^{\infty} (1+k^2)^s (\widehat{\mathcal{K}}\widehat{\psi})^2 dk)^{1/2} \\
&= (\int_{-\infty}^{\infty} (1+k^2)^s (\frac{ik}{k^4+1})^2 (\widehat{\psi})^2 dk)^{1/2} \\
&\leq (\int_{-\infty}^{\infty} (1+k^2)^s (\frac{k}{k^4+1})^2 |\widehat{\psi}|^2 dk)^{1/2} \\
&\leq (\int_{-\infty}^{\infty} (1+k^2)^s |\widehat{\psi}|^2 dk)^{1/2} \\
&= \|\psi\|_{H^s}.
\end{aligned}$$

Thus the proof of (2.7) is complete.

Now, from (2.6) and Proposition 1 one has

$$\begin{aligned}
\|v\|_{L^\infty} \leq\ & C(\|\varphi'\|_{L^1} + \|\varphi'\|_{H^3})(1+t)^{-1/5} \\
&+ \int_0^t (\|\mathcal{K} * (F'(u)v)(\tau)\|_{L^1} \\
&+ \|\mathcal{K} * (F'(u)v)(\tau)\|_{H^3})(1+(t-\tau))^{-1/5} d\tau,
\end{aligned}$$

and

$$\|\mathcal{K} * (F'(u)v)(\tau)\|_{L^1} \leq C\|(F'(u)v)(\tau)\|_{L^1}$$

$$
\begin{aligned}
&\leq C\|\sum_{i=1}^{n} u^{p_i-1}(\tau)\|_{L^\infty}\|u(\tau)v(\tau)\|_{L^1}\\
&\leq C\sum_{i=1}^{n}\|u(\tau)\|_{L^\infty}^{p_i-1}\|u(\tau)\|_{L^2}\|v(\tau)\|_{L^2}\\
&\leq C\sum_{i=1}^{n} q(t)^{p_i+1}(1+\tau)^{\frac{1-p_i}{5}}.
\end{aligned}
$$

The last inequality is from the following estimates:

$$
\begin{aligned}
q(t) &\geq (\|u(\tau)\|_{L^\infty}+\|u_x(\tau)\|_{L^\infty})(1+\tau)^{1/5}+\|u\|_{H^3}\\
&\geq (\|u(\tau)\|_{L^\infty}+\|u_x(\tau)\|_{L^\infty})(1+\tau)^{1/5}\\
q(t)(1+\tau)^{-1/5} &\geq \|u(\tau)\|_{L^\infty}+\|u_x(\tau)\|_{L^\infty}\\
&\geq \|u(\tau)\|_{L^\infty},
\end{aligned}
$$

so $\|u(\tau)\|_{L^\infty}^{p_i-1} \leq q(t)^{p_i-1}(1+\tau)^{\frac{1-p_i}{5}}$ and

$$
\begin{aligned}
2\|u(\tau)\|_{L^2}\|v(\tau)\|_{L^2} &\leq \|u(\tau)\|_{L^2}^2+\|v(\tau)\|_{L^2}^2 = \|u(\tau)\|_{L^2}^2+\|u_x(\tau)\|_{L^2}^2\\
&\leq \|u(\tau)\|_{H^3}^2 \leq q(t)^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|\mathcal{K}*(F'(u)v)\|_{H^3} &\leq C\|\sum_{i=1}^{n} u^{p_i}(\tau)v(\tau)\|_{H^3} = C\sum_{i=1}^{n}\|u^{p_i}(\tau)u_x(\tau)\|_{H^3}\\
&\leq C\sum_{i=1}^{n}\{\|u^{p_i-3}(\tau)(u_x(\tau))^4\|_{L^2}\\
&\quad+\|u^{p_i-2}(\tau)(u_x(\tau))^2u_{xx}(\tau)\|_{L^2}\\
&\quad+\|u^{p_i-1}(\tau)(u_{xx}(\tau))^2\|_{L^2}\\
&\quad+\|u^{p_i-1}(\tau)u_x(\tau)u_{xxx}(\tau)\|_{L^2}\\
&\quad+\|u^{p_i}(\tau)u_{xxxx}(\tau)\|_{L^2}+\|u^{p_i}(\tau)u_x(\tau)\|_{L^2}\}\\
&\leq C\sum_{i=1}^{n}\{\|u(\tau)\|_{L^\infty}^{p_i-3}\|u_x(\tau)\|_{L^\infty}^3\|u_x(\tau)\|L^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \|u(\tau)\|_{L^\infty}^{p_i-2}\|u_x(\tau)\|_{L^\infty}^2\|u_{xx}(\tau)\|_{L^2} \\
&+ \|u(\tau)\|_{L^\infty}^{p_i-1}\|u_{xx}(\tau)\|_{L^\infty}\|u_{xx}(\tau)\|_{L^2} \\
&+ \|u(\tau)\|_{L^\infty}^{p_i-1}\|u_x(\tau)\|_{L^\infty}\|u_{xxx}(\tau)\|_{L^2} \\
&+ \|u(\tau)\|_{L^\infty}^{p_i}\|u_{xxxx}(\tau)\|_{L^2} + \|u(\tau)\|_{L^\infty}^{p_i}\|u_\tau(\tau)\|_{L^2}\} \\
\le\ & C\sum_{i=1}^{n}(q(t))^{p_i+1}(1+\tau)^{\frac{1-p_i}{5}}.
\end{aligned}
$$

Consequently,

$$\int_0^t (1+\tau)^{\frac{1-p_i}{5}}(1+(t-\tau))^{-\frac{1}{5}}d\tau \le C(1+t)^{-\frac{1}{5}}$$

for $p^* = \min_{1\le i\le n} p_i \ge 6$, since (letting $*_t$ denote convolution on $[0,t]$)

$$\int_0^t (1+\tau)^{\frac{1-p_i}{5}}(1+(t-\tau))^{-\frac{1}{5}}d\tau = ((1+\tau)^{-\frac{1}{5}}) *_t ((1+\tau)^{\frac{1-p_i}{5}})$$

and

$$
\begin{aligned}
(2.8) \qquad & \|(1+\tau)^{\frac{1}{5}} *_t (1+\tau)^{\frac{1-p_i}{5}}\|_{L^\infty[0,t]} \\
\le\ & \|(1+\tau)^{-\frac{1}{5}}\|_{L^2[0,t]}\|(1+\tau)^{\frac{1-p_i}{5}}\|_{L^2[0,t]} \\
=\ & (\int_0^t (1+\tau)^{-\frac{2}{5}}d\tau)^{\frac{1}{2}}(\int_0^t (1+\tau)^{\frac{2(1-p_i)}{5}}d\tau)^{\frac{1}{2}} \\
\le\ & C(1+t)^{\frac{3}{10}}(1+t)^{\frac{7-2p_i}{10}} \\
\le\ & C(1+t)^{-\frac{1}{5}} \quad \text{for} \quad p^* \ge 6.
\end{aligned}
$$

It follows that

$$(2.9) \qquad (1+t)^{\frac{1}{5}}\|v\|_{L^\infty} \le C\{\|\varphi'\|_{L^1} + \|\varphi\|_{H^4} + \sum_{i=1}^{n} q(t)^{p_i+1}\}$$

Next, from (2.3) one has

$$\|u\|_{L^\infty} \le C(\|\varphi\|_{L^1} + \|\varphi\|_{H^3})(1+t)^{-\frac{1}{5}}$$

$$+ \int_0^t (\|(\mathcal{K} * F(u))(\tau)\|_{L^1} + \|(\mathcal{K} * F(u))(\tau)\|_{H^3})(1 + (t - \tau))^{-\frac{1}{5}} d\tau.$$

But

$$\begin{aligned}
\|(\mathcal{K} * F(u))(\tau)\|_{L^1} &\le C\|F(u)(\tau)\|_{L^1} \\
&\le C\sum_{i=1}^{n} \|u(\tau)\|_{L^\infty}^{p_i-1} \|u(\tau)\|_{L^2}^2 \\
&\le C\sum_{i=1}^{n} (q(t))^{p_i+1}(1+\tau)^{\frac{1-p_i}{5}}
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathcal{K} * F(u))(\tau)\|_{H^3} &\le C\sum_{i=1}^{n} \|u^{p_i+1}(\tau)\|_{H^3} \\
&\le C\sum_{i=1}^{n} \{\|u^{p_i-2}(\tau)(u_x)^3(\tau)\|_{L^2} \\
&\quad + \|u^{p_i-1}(\tau)u_x(\tau)u_{xx}(\tau)\|_{L^2} \\
&\quad + \|u^{p_i}(\tau)u_{xxx}(\tau)\|_{L^2} + \|u^{p_i+1}(\tau)\|_{L^2}\} \\
&\le C\sum_{i=1}^{n} \{\|u(\tau)\|_{L^\infty}^{p_i-2} \|u_x(\tau)\|_{L^\infty}^2 \|u_x(\tau)\|_{L^2} \\
&\quad + \|u(\tau)\|_{L^\infty}^{p_i-1} \|u_x(\tau)\|_{L^\infty} \|u_{xx}(\tau)\|_{L^2} \\
&\quad + \|u\|_{L^\infty}^{p_i} \|u_{xxx}\|_{L^2} + \|u\|_{L^\infty}^{p_i} \|u\|_{L^2}\} \\
&\le C\sum_{i=1}^{n} (q(t))^{p_i+1}(1+\tau)^{\frac{1-p_i}{5}}.
\end{aligned}$$

Again, for $p^* \ge 6$ it follows that

$$(2.10) \qquad (1+t)^{\frac{1}{5}} \|u\|_{L^\infty} \le C\{\|\varphi\|_{L^1} + \|\varphi\|_{H^3} + \sum_{i=1}^{n} (q(t))^{p_i+1}\}.$$

Finally, from (2.3) one has

$$\begin{aligned}(2.11)\qquad \|u\|_{H^3} &\le \|S_t\varphi\|_{H^3} + \int_0^t \|S_{(t-\tau)}(\mathcal{K} * F(u))\|_{H^3} d\tau \\ &\le C(\|\varphi\|_{H^3} + \int_0^t \|\mathcal{K} * F(u)\|_{H^3} d\tau) \\ &\le C(\|\varphi\|_{H^3} + \int_0^t \|F(u)\|_{H^3} d\tau) \\ &\le C(\|\varphi\|_{H^3} + (\sum_{i=1}^{n} (q(t))^{p_i+1} \int_0^t (1+\tau)^{\frac{1-p_i}{5}} d\tau) \\ &\le C(\|\varphi\|_{H^4} + \sum_{i=1}^{n} (q(t))^{p_i+1}).\end{aligned}$$

Adding (2.9), (2.10) and (2.11) gives (2.5).

Choose a number $\eta > 0$ such that $\eta > C\sum_{i=1}^{n} \eta^{p_i+1}$, where C is the same constant appearing in (2.5). Choose $\delta > 0$ such that if $(\|\varphi\|_{L^1} + \|\varphi'\|_{L^1} + \|\varphi\|_{H^4}) < \delta$ then $q(0) < \eta$ and

$$(2.12)\qquad \eta > C\{\|\varphi\|_{L^1} + \|\varphi'\|_{L^1} + \|\varphi\|_{H^4} + \sum_{i=1}^{n} \eta^{p_i+1}\}$$

Then $\|\varphi\|_{L^1} + \|\varphi'\|_{L^1} + \|\varphi\|_{H^4} < \delta$ must imply $q(t) < \eta$ for all $t \ge 0$. For otherwise, by the continuity of $q(t)$, we would have $q(t) = \eta$ for some t, and (2.12) would contradict (2.5).

3. Estimate for the linearized equation

We need Lemmas 1-7 to prove Proposition 1.

The following four lemmas hold under the stated hypotheses. That is, in Lemmas 1-4, g can be any function and $h_\alpha(k) = g(k) - \alpha k$, subject to the stated hypotheses. Beginning with Lemma 5 the particular choice of g given by (2.2) will be used. See [2] for proofs of Lemmas 1-3.

Lemma 1. *Suppose $h'_\alpha(k)$ and $h''_\alpha(k)$ do not vanish for $a \le k \le b$. Then*

$$\left|\int_a^b e^{ith_\alpha(k)} dk\right| \le \frac{2}{t}\left[\frac{1}{|h'_\alpha(a)|} + \frac{1}{|h'_\alpha(b)|}\right]$$

for all $\alpha \in \mathbf{R}$ and $t \ge 1$.

LEMMA 2. *Suppose $g''(k) \neq 0$ for $k_1 \leq k \leq k_2$. Then there exists a constant $C > 0$ such that*

$$|\int_{k_1}^{k_2} e^{ith_\alpha(k)} dk| \leq Ct^{-1/2}$$

for all $\alpha \in \mathbf{R}$, $t \geq 1$.

LEMMA 3. *Suppose $g''(k_1) = 0$, $g'''(k_1) \neq 0$ and $g''(k) \neq 0$ for $k_1 < k \leq k_2$. Then there exists a constant $C > 0$ such that*

$$|\int_{k_1}^{k_2} e^{ih_\alpha(k)t} dk| \leq Ct^{-1/3}$$

for all $\alpha \in \mathbf{R}$, $t \geq 1$. The same result holds if $g''(k_2) = 0$, $g'''(k_2) \neq 0$ and $g''(k) \neq 0$ for $k_1 \leq k < k_2$.

LEMMA 4. *Suppose $g''(k_1) = g'''(k_1) = g^{(iv)}(k_1) = 0$, $g^{(v)}(k_1) \neq 0$ and $g''(k) \neq 0$ for $k_1 < k \leq k_2$. Then there exists a constant $C > 0$ such that*

$$|\int_{k_1}^{k_2} e^{ih_\alpha(k)t} dk| \leq Ct^{-1/5}$$

for all $\alpha \in \mathbf{R}$, $t \geq 1$. The same result holds if $g''(k_2) = g'''(k_2) = g^{(iv)}(k_2) = 0$, $g^{(v)}(k_2) \neq 0$ and $g''(k) \neq 0$ for $k_1 \leq k < k_2$.

Proof. We may assume that $g''(k) > 0$ on $(k_1, k_2]$ and that $g^{(v)}(k_1) \neq 0$. The other cases are similar. Define $\alpha_i = g'(k_i)$ for $i = 1, 2$; and for any $\alpha \in \mathbf{R}$, define $k_\alpha \in [k_1, k_2]$ by

$$\begin{aligned} k_\alpha &= k_1, && \text{if} \quad \alpha \leq k_1 \\ h'_\alpha(k_\alpha) &= 0 && \text{if} \quad \alpha_1 \leq \alpha \leq a_2 \\ k_\alpha &= k_2 && \text{if} \quad \alpha \geq \alpha_2. \end{aligned}$$

If $\alpha \in [\alpha_1, \alpha_2]$, then $h'_\alpha(k_\alpha) = 0$, and the Taylor expansion of $h'_\alpha(k)$ about $k = k_\alpha$ is

$$\begin{aligned} h'_\alpha(k) = {} & g''(k_\alpha)(k - k_\alpha) + \frac{g'''(k_\alpha)}{2}(k - k_\alpha)^2 \\ & + \frac{g^{(iv)}(k_\alpha)}{6}(k - k_\alpha)^3 + \frac{g^{(v)}(k_\alpha)}{24}(k - k_\alpha)^4 \\ & + O((k - k_\alpha)^5). \end{aligned} \tag{3.1}$$

Now expand $g''(k_\alpha), g'''(k_\alpha), g^{(iv)}(k_\alpha)$ and $g^{(v)}(k_\alpha)$ about $k_\alpha = k_1$ to obtain

$$g''(k_\alpha) = \frac{g^{(v)}(k_1)}{6}(k_\alpha - k_1)^3 + O((k_\alpha - k_1)^4) \tag{3.2}$$

$$g'''(k_\alpha) = \frac{g^{(v)}(k_1)}{6}(k_\alpha - k_1)^2 + O((k_\alpha - k_1)^3) \tag{3.3}$$

$$g^{(iv)}(k_\alpha) = g^{(v)}(k_1)(k_\alpha - k_1) + O((k_\alpha - k_1)^2) \tag{3.4}$$

$$g^{(v)}(k_\alpha) = g^{(v)}(k_1) + O(k_\alpha - k_1) \tag{3.5}$$

and substitute (3.2)-(3.5) into (3.1). The result is

$$\begin{aligned} h'_\alpha(k) = &\frac{g^{(v)}(k_1)}{6}(k_\alpha - k_1)^3(k - k_\alpha) \\ &+ \frac{g^{(v)}(k_1)}{4}(k_\alpha - k_1)^2(k - k_\alpha)^2 \\ &+ \frac{g^{(v)}(k_1)}{6}(k_\alpha - k_1)(k - k_\alpha)^3 \\ &+ \frac{g^{(v)}(k_1)}{24}(k - k_\alpha)^4 + \mathcal{R} \end{aligned} \tag{3.6}$$

where $\mathcal{R}$ is of fifth order in $(k_\alpha - k_1)$ and $(k - k_\alpha)$. Since $(k_\alpha - k_1) \to 0$ and $(k - k_\alpha) \to 0$ as $\alpha \to \alpha_1$ and $k \to k_1$, it follows from (3.6) that there exist $\alpha_3 > \alpha_1$ and $k_3 = k_{\alpha_3} > k_1$ such that for all $\alpha \in [\alpha_1, \alpha_3]$ and $k \in [k_1, k_3]$,

$$\begin{aligned} |h'_\alpha(k)| \geq C(|k_\alpha - k_1|^3|k - k_\alpha| + |k_\alpha - k_1|^2|k - k_\alpha|^2) \\ \text{if } |k - k_\alpha| \leq |k_\alpha - k_1| \end{aligned} \tag{3.7}$$

$$\begin{aligned} |h'_\alpha(k)| \geq C(|k_\alpha - k_1|\,|k - k_\alpha|^3 + |k - k_\alpha|^4) \\ \text{if } |k - k_\alpha| \geq |k_\alpha - k_1| \end{aligned} \tag{3.8}$$

where C is independent of α. Moreover (3.7) and (3.8) also hold for $\alpha \in [\alpha_1 - \eta, \alpha_3 + \eta]$ for some $\eta > 0$, since $|h'_\alpha(k)| \geq \frac{1}{2}|h'_{\alpha_1}(k)|$ for

$\alpha \in [\alpha_1 - \eta, \alpha_1]$ and $|h'_\alpha(k)| \geq \frac{1}{2}|h'_{\alpha_3}(k)|$ for $\alpha \in [\alpha_3, \alpha_3 + \eta]$ for suitable $\eta > 0$.

By assumption, $g''(k) \neq 0$ on $[k_3, k_2]$ and so the desired estimate for $\int_{k_3}^{k_2} e^{ih_\alpha(k)t} dk$ follows from Lemma 2. Furthermore, if $\alpha \notin [\alpha_1 - \eta, \alpha_3 + \eta]$, then $|h'_\alpha(k)| \geq C$ for all $k \in [k_1, k_3]$, and hence the desired estimate for $\int_{k_1}^{k_3} e^{ih_\alpha(k)t} dk$ follows from Lemma 1. Therefore, it is enough to estimate $\int_{k_1}^{k_3} e^{ih_\alpha(k)t} dk$ when $\alpha \in [\alpha_1 - \eta, \alpha_3 + \eta]$, and we may assume henceforth that (3.7) and (3.8) are valid.

Define $\delta = |k_\alpha - k_1|$, and consider first $\int_{|k-k_\alpha|\leq\delta} e^{ih_\alpha(k)t} dk$. If $\delta \leq t^{-1/5}$, then this integral is majorized by $2t^{-1/5}$. If, on the other hand, $\delta \geq t^{-1/5}$, write

$$\Big|\int_{|k-k_\alpha|\leq\delta} e^{ih\alpha(k)t} dk\Big| \leq \Big|\int_{|k-k_\alpha|\leq t^{-1/5}} e^{ih_\alpha(k)t} dk\Big| + \Big|\int_{t^{-1/5}\leq|k-k_\alpha|\leq\delta} e^{ih_\alpha(k)t} dk\Big|.$$

The first integral on the right-hand side is again majorized by $2t^{-1/5}$, while for the second integral, (3.7) and Lemma 1 yield

$$\Big|\int_{t^{-1/5}\leq|k-k_\alpha|\leq\delta} e^{ih_\alpha(k)t} dk\Big| \leq \frac{2}{t}\left[\frac{2}{C(\delta^3 t^{-1/5} + \delta^2 t^{-2/5})}\right] \leq Ct^{-1/5}.$$

It remains to consider

$$\int_{\substack{k\in[k_1,k_3] \\ |k-k_\alpha|\geq\delta}} e^{ih_\alpha(k)t} dk = \int_{k_\alpha+\delta}^{k_3} e^{ih_\alpha(k)t} dk.$$

Again, look separately at the cases $\delta \geq t^{-1/5}$ and $\delta \leq t^{-1/5}$. In the first case, (3.8) and Lemma 1 imply that

$$\Big|\int_{k_2+\delta}^{k_3} e^{ih_\alpha(k)t} dk\Big| \leq \frac{2}{t}\Big[\frac{2}{C\delta^4}\Big] \leq Ct^{-1/5}.$$

In the second case, write

$$|\int_{k_\alpha+\delta}^{k_3} e^{ih_\alpha(k)t}dk| \leq |\int_{k_\alpha+\delta}^{k_\alpha+t^{-1/5}} e^{ih_\alpha(k)t}dk| + |\int_{k_\alpha+t^{-1/5}}^{k_3} e^{ih_\alpha(k)t}dk|.$$

Then the first integral on the right-hand side is majorized by $t^{-1/5}$, while the desired estimate for the second integral is again obtained from (3.7) and Lemma 1. Thus the proof of Lemma 4 is complete.

Now we assume specifically that h_α is given by (2.2).

LEMMA 5. *There exists a constant $C > 0$ such that*

$$|\int_{2\leq|k|\leq t^{1/10}} e^{ih_\alpha(k)t}dk| \leq Ct^{-1/4}$$

for all $\alpha \in \mathbf{R}$ and $t \geq 0$.

Proof. By symmetry, it is enough to prove the result for $\int_2^{t^{1/10}} e^{ih_\alpha(k)t}dk$. For $\alpha < 0$, define $k_\alpha \in [2,\infty)$ by

$$\begin{aligned} k_\alpha &= 2 && \text{if } \alpha \leq g'(2) \\ h'_\alpha(k_\alpha) &= 0 && \text{if } g'(2) < \alpha < 0. \end{aligned}$$

Note first of all that

$$\lim_{k\to\infty} k^4|g'(k)| = 3, \quad \lim_{k\to\infty} k^5|g''(k)| = 12 \quad \text{and}$$
$$\lim_{k\to\infty} k^6|g'''(k)| = 60.$$

In particular, there exists $C > 0$ such that for all $k \in [2,\infty)$,

$$|g'(k)| \leq \frac{C}{k^4}, \ |g''(k)| \leq \frac{C}{k^5} \quad \text{and} \quad |g'''(k)| \leq \frac{C}{k^6}.$$

Next, let r be any number such that $0 < r < 1$ and $(1-r)^4 > \frac{5}{6}$; and choose ρ so that $0 < \rho < 1$ and $\frac{1-\rho}{1+\rho} > \max[(1-r)^4, \frac{5}{6(1-r)^4}]$. Then there exists $M > 0$ such that if $k \geq M$, then

$$\begin{aligned} 3(1-\rho) &\leq k^4|g'(k)| \leq 3(1+\rho) \\ 12(1-\rho) &\leq k^5|g''(k)| \leq 12(1+\rho) \\ 60(1-\rho) &\leq k^6|g'''(k)| \leq 60(1+\rho) \end{aligned} \tag{3.9}$$

By the way how k_α is defined, $h'_\alpha(k_\alpha) = 0$ for $g'(2) < \alpha < 0$. So $\frac{1-3k_\alpha^4}{(1+k_\alpha^4)^2} - \alpha = 0$ and

$$k_\alpha^4 = \frac{-(2\alpha+3) \pm \sqrt{(2\alpha+3)^2 - 4\alpha(\alpha-1)}}{2\alpha}.$$

Since $k_\alpha \geq 2$,

$$k_\alpha = \sqrt[4]{\frac{-(2\alpha+3) - \sqrt{16\alpha - 9}}{2\alpha}} \quad \text{and} \quad \lim_{\alpha \to 0^-} k_\alpha = \infty.$$

Finally, $\eta > 0$ can be chosen so small that if $\alpha \in [-\eta, 0)$, then $k_\alpha > \frac{M}{1-r}$. Then the following estimates hold:

$$(3.10) \qquad |h'_\alpha(k)| \geq \frac{C}{k_\alpha^5}|k - k_\alpha| \quad \text{if} \quad |k - k_\alpha| \leq rk_\alpha$$

$$(3.11) \qquad |h'_\alpha(k)| \geq \frac{C}{k^4} \quad \text{if} \quad |k - k_\alpha| \geq rk_\alpha$$

where $C > 0$ is independent of α.

To prove (3.10), assume $|k-k_\alpha| < rk_\alpha$ and write $h'_\alpha(k) = g''(k_\alpha)(k-k_\alpha) + \frac{g'''(z)}{2}(k-k_\alpha)^2$ where $|z - k_\alpha| \leq |k - k_\alpha| \leq rk_\alpha$. From (3.9) it follows that

$$\begin{aligned}
|h'_\alpha(k)| &\geq |g''(k_\alpha)|\,|k - k_\alpha| - \frac{|g'''(z)|}{2}(k - k_\alpha)^2 \\
&\geq \frac{12(1-\rho)}{k_\alpha^5}|k - k_\alpha| - \frac{60(1+\rho)}{2z^6}(k - k_\alpha)^2 \\
&\geq \frac{|k - k_\alpha|}{k_\alpha^5}\{12(1-\rho) - \frac{30(1+\rho)}{(1-r)^6 k_\alpha}|k - k_a|\} \\
&\geq \frac{|k - k_\alpha|}{k_\alpha^5}\{12(1-\rho) - \frac{10(1+\rho)}{(1-r)^4} \cdot \frac{3r}{(1-r)^2}\} \\
&\geq \frac{|k - k_\alpha|}{k_\alpha^5}\{12(1-\rho) - \frac{10(1+\rho)}{(1-r)^4}\},
\end{aligned}$$

since r satisfies $(1-r)^4 > \frac{5}{6}$, so $\frac{3r}{(1-r)^2} \leq 1$, and the constant in brackets is positive.

For (3.11), consider separately the cases $2 \le k \le M$, $M \le k \le k_\alpha(1-r)$ and $k_\alpha(1-r) \le k$. Since $h'_\alpha(k) = \frac{1-3k^4}{(1+k^4)^2} - \alpha$ is bounded away from zero when $k \in [2, M]$ and $\alpha \in [-\eta, 0)$, (3.11) is obvious in the first case. In case $M \le k \le k_\alpha(1-r)$, it follows from (3.9) and the definition of ρ that

$$\begin{aligned}
|h'_\alpha(k)| &= |h'_\alpha(k) - h'_\alpha(k_\alpha)| \ge |g'(k)| - |g'(k_\alpha)| \\
&\ge \frac{3(1-\rho)}{k^4} - \frac{3(1+\rho)}{k_\alpha^4} \\
&\ge \frac{3(1-\rho)}{k^4} - \frac{3(1+\rho)(1-r)^4}{k^4} \\
&\ge \frac{3((1-\rho) - (1+\rho)(1-r)^4)}{k^4} \\
&\ge \frac{C}{k^4}.
\end{aligned}$$

Finally, if $k_\alpha(1+r) \le k$, we have, for some z in $[k_\alpha, k_\alpha(1+r)]$,

$$\begin{aligned}
|h'_\alpha(k)| &\ge |h'_\alpha(k_\alpha(1+r))| = |h'_\alpha(k_\alpha(1+r)) - h'_a(k_\alpha)| \\
&= |g''(z)| r k_\alpha \ge (\frac{C}{z^5}) r k_\alpha \ge \frac{C}{z^5} r z \ge \frac{C}{k^4}.
\end{aligned}$$

To estimate $\int_2^{t^{1/10}} e^{ih_\alpha(k)t} dk$, first consider the case $t^{1/10} \le k_\alpha(1-r)$. Then from (3.11) and Lemma 1, it follows that

$$\begin{aligned}
\left|\int_2^{t^{1/10}} e^{ih_\alpha(k)t} dk\right| &\le \frac{2}{t}\left[\frac{1}{|h'_\alpha(2)|} + \frac{1}{|h'_\alpha(t^{1/10})|}\right] \\
&\le \frac{2}{t}[C + Ct^{2/5}] \\
&\le C(t^{-1} + t^{-3/5}) \\
&\le Ct^{-3/5}.
\end{aligned}$$

Therefore, it suffices to consider the case

$$t^{1/10} \ge k_\alpha(1-r). \tag{3.12}$$

We write the integral to be estimated as

$$\int_2^{t^{1/10}} e^{ih_\alpha(k)t}dk = \int_{\substack{2\leq k\leq t^{1/10}\\ |k-k_\alpha|\leq rk_\alpha}} e^{ih_\alpha(k)t}dk + \int_{\substack{2\leq k\leq t^{1/10}\\ |k-k_\alpha|\geq rk_\alpha}} e^{ih_\alpha(k)t}dk. \tag{3.13}$$

Again, using (3.11) and Lemma 1, the second integral on the right-hand side of (3.13) is dominated by

$$\frac{2}{t}[C + Ct^{2/5}] \leq Ct^{-3/5}.$$

Now, let $\lambda = (\frac{k_\alpha^5}{t})^{1/2}$ and write the first integral on the right-hand side of (3.13) as

$$\int_{|k-k_\alpha|\leq\lambda} e^{ih_\alpha(k)t}dk + \int_{\lambda\leq|k-k_\alpha|\leq rk_\alpha} e^{ih_\alpha(k)t}dk. \tag{3.14}$$

The first integral in (3.14) is dominated by 2λ, whereas for the second integral, (3.10) and Lemma 1 give

$$\begin{aligned}|\int_{\lambda\leq|k-k_\alpha|\leq rk_\alpha} e^{ih_\alpha(k)t}dk| &\leq \frac{C}{t}[\frac{1}{\lambda/k_\alpha^5}] = \frac{C}{t}\frac{(k_\alpha)^5}{(k_\alpha^5/t)^{1/2}} \\ &= C(k_\alpha^5/t)^{1/2} = C\lambda.\end{aligned}$$

Therefore, it is seen that (3.14) is dominated by $C\lambda$. However, from (3.12), we have

$$\begin{aligned}\lambda = (\frac{k_\alpha^5}{t})^{1/2} &\leq C\left(\frac{(t^{1/10})^5}{t}\right)^{1/2} \\ &\leq Ct^{-1/4}.\end{aligned}$$

Thus, it shown that

$$|\int_2^{t^{1/10}} e^{ih_\alpha(k)t}dk| \leq Ct^{-1/4}$$

and the proof of Lemma 5 is complete.

LEMMA 6. *There exists a constant* $C > 0$ *such that* $|\int_{|k|\le t^{1/10}} e^{ih_\alpha(k)t}dk| \le Ct^{-1/5}$ *for any* $\alpha \in \mathbf{R}$, $t \ge 1$.

Proof. Since $g''(\pm\sqrt[4]{\frac{5}{3}}) = 0$, $g'''(\pm\sqrt[4]{\frac{5}{3}}) \ne 0$ and $g''(k) \ne 0$ for $-2 \le k < -\sqrt[4]{\frac{5}{3}}$, $-\sqrt[4]{\frac{5}{3}} < k \le -1$, $1 \le k < \sqrt[4]{\frac{5}{3}}$ and $\sqrt[4]{\frac{5}{3}} < k \le 2$, we apply Lemma 3 to the integral of $e^{ih_\alpha(k)t}$ over the intervals $[-2, -\sqrt[4]{\frac{5}{3}}]$, $[-\sqrt[4]{\frac{5}{3}}, -1]$, $[1, \sqrt[4]{\frac{5}{3}}]$ and $[\sqrt[4]{\frac{5}{3}}, 2]$ and since $g''(0) = g'''(0) = g^{(iv)}(0) = 0$, $g^{(v)}(0) \ne 0$ and $g''(k) \ne 0$ for $-1 \le k < 0$ and $0 < k \le 1$, apply Lemma 4 to the integral of $e^{ih_\alpha(k)t}$ over the intervals $[-1, 0]$ and $[0, 1]$ and apply Lemma 5 on the intervals $[-t^{1/10}, -2]$ and $[2, t^{1/10}]$.

LEMMA 7. *There exists a constant* $C > 0$ *such that*

$$|\int_{|k|\le t^{1/10}} e^{ih_\alpha(k)t}\widehat{\varphi}(k)dk| \le C\|\varphi\|_{H^3}t^{-1/5}$$

for all $\varphi \in H^3$, $\alpha \in \mathbf{R}$ *and* $t \ge 1$.

Proof. For $k > 0$, we have

$$\begin{aligned}
&|\int_{t^{1/10}}^{\infty} e^{ih_\alpha(k)t}\widehat{\varphi}(k)dk| \\
&\le \int_{t^{1/10}}^{\infty} |\widehat{\varphi}(k)|dk \\
&\le (\int_{t^{1/10}}^{\infty} k^6|\widehat{\varphi}(k)|^2dk)^{1/2}(\int_{t^{1/10}}^{\infty} k^{-6}dk)^{1/2} \\
&\le (\int_{t^{1/10}}^{\infty} (1+k^2)^3|\widehat{\varphi}(k)|^2dk)^{1/2}([-\frac{1}{5}k^{-5}]_{t^{1/10}}^{\infty})^{1/2} \\
&\le C\|\varphi\|_{H^3}t^{-1/5}.
\end{aligned}$$

Now we prove Proposition 1.

Proof of Proposition 1. For $0 < t \le 1$, we have

$$|\int_{-\infty}^{\infty} e^{ih_\alpha(k)t}\widehat{\varphi}(k)dk|$$

$$\leq \int_{-\infty}^{\infty} |\widehat{\varphi}(k)|dk$$
$$\leq (\int_{-\infty}^{\infty} (1+|k|^2)^2 |\widehat{\varphi}(k)|^2 dk)^{1/2} (\int_{-\infty}^{\infty} \frac{1}{(1+|k|^2)^2} dk)^{1/2}$$
$$\leq C\|\varphi\|_{H^2} \leq C(\|\varphi\|_{L^1} + \|\varphi\|_{H^3}).$$

It suffices to consider the case $t > 1$.

Let $q(k,t) = e^{i(\frac{k}{k^4+1})t} \chi_{\{|k| \leq t^{1/10}\}}$. Then

$$\begin{aligned} |\int_{-\infty}^{\infty} e^{ih_\alpha(k)t} \widehat{\varphi}(k)dk| &= |\int_{-\infty}^{\infty} e^{i(\frac{k}{1+k^4} - \frac{x}{t}k)t} \widehat{\varphi}(k)dk| \\ &= |\int_{-\infty}^{\infty} e^{-ikx} q(k,t) \widehat{\varphi}(k)dk \\ &\quad + \int_{|k|\geq t^{1/10}} e^{ih_\alpha(k)t} \widehat{\varphi}(k)dk| \\ &\leq |\int_{-\infty}^{\infty} e^{-ikx} (\check{q}(x,t) * \varphi(x))^\wedge dk| \\ &\quad + C\|\varphi\|_{H^3} t^{-1/5} \quad \text{by Lemma 7,} \\ &\leq \frac{1}{2\pi} \|\check{q}(x,t) * \varphi(x)\|_{L^\infty} + C\|\varphi\|_{H^3} t^{-1/5}. \end{aligned}$$

However, Lemma 6 asserts that

$$\|\check{q}(x,t)\|_{L^\infty} = |\int_{-\infty}^{\infty} e^{-ikx} e^{i(\frac{k}{k^4+1})t} dk| \leq Ct^{-1/5}.$$

Therefore,

$$\begin{aligned} \|\check{q}(x,t) * \varphi(x)\|_{L^\infty} &\leq \|\check{q}(x,t)\|_{L^\infty} \|\varphi\|_{L^1} \\ &\leq C\|\varphi\|_{L^1} t^{-1/5}. \end{aligned}$$

Thus it is proved that

$$|\int_{-\infty}^{\infty} e^{ih_\alpha(k)t} \widehat{\varphi}(k)dk| \leq C(\|\varphi\|_{L^1} + \|\varphi\|_{H^3}) t^{-1/5}.$$

Since $t \geq 1$, one has $t^{-1/5} \leq C(1+t)^{-1/5}$ for $C > 2$ and the proof of Proposition 1 is complete.

REMARK 1. The assumption that $\min_{1\le i\le n} p_i \ge 6$ in Theorem 1 cannot be lessened, because we want $(1+t)^{1-\frac{p_i}{5}} \le (1+t)^{-1/5}$ in (2.8), so $1-\frac{p_i}{5} \le -\frac{1}{5}$, that is, $p_i \ge 6$ for each i. $(1+t)^{-1/5}$ in the inequality is from Proposition 1.

REMARK 2. The estimate $\|u(\cdot,t)\| \le C(1+t)^{-1/5}$ is basically from Proposition 1. Proposition 1 follows from Lemma 1-7. To find the best estimate in the proof of Lemma 4(p.269), set $\delta \le t^\beta$, then $\int_{|k-k_\alpha|\le\delta} e^{ih_\alpha(k)t}dk$ is majorized by 2δ. When $\delta \ge t^\beta$, $|\int_{|k-k_\alpha|\le\delta} e^{ih_\alpha(k)t}dk| \le 2t^\beta + 4t^{-1-4\beta} \le 8t^\sigma$, $\sigma = \max\{\beta, -1, -4\beta\}$. We minimize the estimate on the right hand side by choosing $\beta = -\frac{1}{5}$. The same estimate holds for $|\int_{k_2+\delta}^{k_3} e^{ih_\alpha(k)t}dk|$. Therefore $\beta = -\frac{1}{5}$ can not be decreased.

4. Example**

$u(x,t) = v(x-kt)$, where $v(y) = \operatorname{sech} y = \frac{1}{\cosh y}$ satisfies

$$cu_t + au_x + \sum_{i=1}^{2} b_i u^{p_i} u_x + u_{xxxxt} = 0$$

where $c \ne -1$, $k = \frac{a}{c+1}$, $p_1 = 2$, $p_2 = 4$, $b_1 = -60k$ and $b_2 = 120k$. In particular, $u(x,t) = \operatorname{sech}(x-t)$ is a solution of

$$2u_t + 3u_x - 60u^2u_x + 120u^4u_x + u_{xxxxt} = 0.$$

Clearly,

$$\|u(\cdot,t)\|_\infty = \sup_{x\in\mathbf{R}} |u(x,t)| = 1 \quad \text{for } t \ge 0.$$

Therefore, no decay estimate holds. This supports $p^* \ge 6$ on Theorem 1.

We verify this. Now, let $v(y) = \operatorname{sech} y = \frac{1}{\cosh y}$. By using the fact that $\cosh^2 - \sinh^2 = 1$. $\sinh' = \cosh$, $\cosh' = \sinh$, so $1 - \tanh^2 =$

** This example is provided by Jerry Goldstein.

sech^2,

$$\begin{aligned} v' &= (\cosh^{-1})' = -(\cosh^{-2})\sinh \\ v'' &= \cosh^{-1} - 2\cosh^{-3} \\ v''' &= -\cosh^{-2}\sinh + 6\cosh^{-4}\sinh \\ v'''' &= \cosh^{-1} - 20\cosh^{-3} + 24\cosh^{-5} \end{aligned}$$

$cu_t + au_x + \sum_{i=1}^{2} b_i u^{p_i} u_x + u_{xxxxt} = 0$ is equivalent to

$$-ckv' + av' + \sum_{i=1}^{2} b_i v^{p_i} v' - kv''''' = 0.$$

So $(a-ck)\cosh^{-1} y + \frac{b_1}{p_1+1}(\cosh^{-1} y)^{p_1+1} + \frac{b_2}{p_2+1}(\cosh^{-1})^{p_2+1} - k(\cosh^{-1} y$ $-20\cosh^{-3} y + 24\cosh^{-5} y) = 0$ gives

$$\begin{aligned} &a - ck = k, \ p_1 = 2, \ p_2 = 4 \\ &\frac{b_1}{3} = -20k \quad \text{and} \quad \frac{b_2}{5} = 24k. \end{aligned}$$

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