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QUOTIENT MAPPINGS, HELLY EXTENSIONS, HAHN–BANACH EXTENSIONS, TIETZE EXTENSIONS, LIPSCHITZ EXTENSIONS, AND BEST APPROXIMATION

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1. Introduction

In [6], authors established interesting connections between Helly extensions, Hahn-Banach extensions, Tietze extensions, and Lipschitz extensions, and best approximations. In this paper, we want to recover those results by using the inverse mapping φ_M of the restriction of the quotient mapping to the kernel of the metric projection P_M when M is a proximinal subspace of a normed linear space X.

In section 2, we will give general formula which relates the metric projection P_M to the mapping φ_M. Using this formula, we can easily relate continuity and selection properties of the (set-valued) metric projection with the analogous properties of φ_M.

In section 3, we can relate φ_M with the Helly extension mapping, and continuity and selection properties of the metric projection with the analogous properties of the Helly extension mapping. We recover the results of Deutsch, Li, and Mabizela [6].

In section 4, M and X specialized to $M^\perp$ and X^*, respectively. We observe a relation between φ_M and the Hahn-Banach extension mapping. In this case, results of Phelp [10], Lindenstrauss [8], Fakoury [7], Deutsch, Li, and Mabizela [6], and Deutsch, Li, and Park [5] are recovered.

In section 5, X is specialized to $C_0(T)$ and M is the closed ideal of functions vanishing on a given compact subset S of T. We observe a relation between φ_M and the Tietze extension mapping. Results of Deutsch, Li, and Mabizela [6] are recovered.

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In section 6, X is specialized to the normed linear space of all Hilbert space valued Lipschitz functions of order α, $\mathrm{Lip}_\alpha(T)$, on a metric space T, and M is the subspace consisting of all such functions which vanish on a prescribed subset S of T. We observe a relation between φ_M and the Lipschitz extension mapping. Using this formula, we recover the results of Deutsch, Li, and Mabizela [6].

2. Quotient mapping

In this section, let M be a proximinal subspace of a normed linear space X, and let X/M denote the quotient space equipped with the norm $\|x+M\| = d(x,M)$. Let $Q_M : X \to X/M$ be the quotient mapping and let

$$P_M^{-1}(0) = \{x \in X \mid \|Q_M(x)\| = \|x\|\}.$$

PROPOSITION 2.1 [1]. *For a linear subspace M of the normed linear space X, the following statements are equivalent:*

1. *M is proximinal.*
2. *We have*

$$X = M + P_M^{-1}(0) = \{m + x \mid m \in M,\ x \in P_M^{-1}(0)\}.$$

3. *M is closed and for the quotient mapping $Q_M : X \to X/M$ we have*

$$Q_M(P_M^{-1}(0)) = X/M.$$

(In other words, $Q_M|_{P_M^{-1}(0)}$ maps $P_M^{-1}(0)$ onto X/M.)

THEOREM 2.2 [11]. *If M is a closed subspace of X, then for every $x \in X$,*

$$\begin{aligned} P_M(x) &= x - (Q_M|_{P_M^{-1}(0)})^{-1} \circ Q_M(x) \\ &= x - (Q_M|_{P_M^{-1}(0)})^{-1}(x+M). \end{aligned}$$

Proof.

$$\begin{aligned} & y \in (Q_M|_{P_M^{-1}(0)})^{-1}(x+M) \\ & \iff m = x - y \in M \text{ and } \|y\| = d(x,M) \\ & \iff y = x - m \text{ for some } m \in M \text{ and } \|x-m\| = d(x,M) \\ & \iff y \in x - P_M(x). \end{aligned}$$

DEFINITION 2.3 [11]. Let M be a proximinal subspace of a normed linear space X. Define a set-valued mapping $\varphi_M : X/M \to 2^{P_M^{-1}(0)}$ by $\varphi_M(x+M) = x - P_M(x)$ for each $x + M \in X/M$. Then φ_M is well-defined. Moreover, $\varphi_M(x+M) = (Q_M|_{P_M^{-1}(0)})^{-1}(x+M)$ for each $x + M \in X/M$ and $\varphi_M(x+M)$ is nonempty bounded, closed, and convex subset of $P_M^{-1}(0)$.

LEMMA 2.4. *Let M be a proximinal subspace of X, $y_i + M \in X/M$ $(i = 1, 2)$ and $x_1 \in X$ satisfies $x_1 + M = y_1 + M$. Then there exists $x_2 \in X$ such that $x_2 + M = y_2 + M$ and $\|x_1 - x_2\| = \|(y_1 + M) - (y_2 + M)\|$.*

Before establishing the convention between certain continuity properties of P_M with those of φ_M, we recall the relevant definitions. If Y is a metric space, $A \subset Y$, and $\varepsilon > 0$, denote the ε-neighborhood of A by $B_\varepsilon(A) = \{y \in Y \mid d(y, A) < \varepsilon\}$. If $A = \emptyset$, we define $B_\varepsilon(\emptyset) = \emptyset$.

DEFINITION 2.5. Let Y be a metric space, $F : X \to 2^Y$, and $x_0 \in X$. Then F is called:

(1) upper semicontinuous (u.s.c.) at x_0 if for any open set V with $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for each $x \in U$;

(2) lower semicontinuous (l.s.c.) at x_0 if for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for each $x \in U$;

(3) upper Hausdorff semicontinuous (u.H.s.c.) at x_0 if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $F(x) \subset B_\varepsilon(F(x_0))$ for each $x \in U$;

(4) lower hausdorff semicontinuous (l.H.s.c.) at x_0 if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $F(x_0) \subset B_\varepsilon(F(x))$ for each $x \in U$.

LEMMA 2.6 [3]. *Let M be proximinal, $x_0 \in X$, and $\tau = u$, l, u.H., l.H. Then P_M is τ.s.c. at x_0 if and only if $I - P_M$ is τ.s.c. at x_0.*

THEOREM 2.7 [11]. *Let M be proximinal, $x_0 \in X$, and let $\tau = u$, l,l.H., u.H. Then P_M is τ.s.c. at x_0 if and only if φ_M is τ.s.c. at $x_0 + M$.*

The next theorem shows that the existence of continuous, Lipschitz continuous, or linear selections for P_M is equivalent to the analogous property for φ_M.

Recall that H is a Hausdorff metric on $H(X)$ where $H(X)$ is the collection of all closed, bounded and convex subsets of X if for any A, B in $H(X)$, $H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$.

THEOREM 2.8 [9]. *Let M be a proximinal subspace of X. Then the following statements are equivalent:*

(1) *P_M is Lipschitz continuous (resp. linear).*

(2) *φ_M is Lipschitz continuous (resp. linear).*

THEOREM 2.9. *Let M be a proximinal subspace of X. Then*

(1) *P_M has a continuous (resp. linear) selection if and only if φ_M has a continuous (resp. linear) selection.*

(2) *P_M has a linear selection if and only if φ_M has a linear selection* with norm one.

(3) *P_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection which is additive modulo M if and only if φ_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection.*

Proof. (1) If P_M has a continuous selection, then it has a continuous selection p which is also homogeneous and additive modulo M [4; Theorem 3.4]. Define e on X/M by $e(x+M) = x - p(x)$. Then e is well-defined since if $x+M = y+M$, then $m := x-y \in M$ and $x - p(x) = y + m - p(y+m) = y - p(y)$. Moreover, by Theorem 2.2, e is a selection for φ_M. Now if $x+M$ and $y+M$ are in X/M, then Lemma 2.4 implies that there exists $z \in X$ such that $z+M = y+M$ and $\|x-z\| = \|(x+M)-(y+M)\|$. Then

$$\begin{aligned}\|e(x+M)-e(y+M)\| &= \|x-p(x)-(y-p(y))\| \\ &= \|x-p(x)-(z-p(z))\| \\ &\le \|x-z\| + \|p(x)-p(z)\| \\ &\le \|(x+M)-(y+M)\| + \|p(x)-p(z)\|.\end{aligned}$$

Since p is continuous at x, given any $\varepsilon > 0$, choose $0 < \delta < \varepsilon$ such that $\|x-z\| < \delta$ implies $\|p(x)-p(z)\| < \varepsilon$. Thus, if $y \in X$ is chosen so that $\|(x+M)-(y+M)\| < \delta$, then $\|x-z\| < \delta$ so that $\|e(x+M)-e(y+M)\| < 2\varepsilon$. This proves that e is continuous at $x+M$.

Conversely, suppose that φ_M has a continuous selection e. Define p on X by $p(x) = x - e(x+M)$. Then p is a selection for P_M by

Theorem 2.2. Given $\varepsilon > 0$ and $x \in X$, choose $0 < \delta < \varepsilon$ so that $\|e(x+M) - e(y+M)\| < \varepsilon$ whenever $\|(x+M) - (y+M)\| < \delta$. Thus if $\|x-y\| < \delta$, then $\|(x+M) - (y+M)\| \le \|x-y\| < \delta$ so that

$$\|p(x) - p(y)\| \le \|x-y\| + \|e(x+M) - e(y+M)\| < \delta + \varepsilon < 2\varepsilon.$$

Thus p is continuous at x.

The proof that P_M has a linear selection if and only if φ_M has a linear selection is similar.

(2) By part (1), it suffices to verify that if p is a linear selection for P_M, then $e(x+M) = x - p(x)$ defines a linear selection for φ_M with norm one. But $\|e(x+M)\| = \|x - p(x)\| = d(x, M) = \|x+M\|$ for each $x \in X$. Thus e has norm one.

(3) Suppose that p is a pointwise Lipschitz continuous selection for P_M which is additive modulo M. Then, just as in the proof of (1), the function e defined in X/M by $e(x+M) = x - p(x)$ is a selection for φ_M. Moreover, given $x+M$, Lemma 2.4 implies that for each $y+M$ in X/M, there exists $z \in X$ such that $z+M = y+M$ and $\|(x+M) - (y+M)\| = \|x-z\|$. Thus

$$\begin{aligned}
\|e(x+M) - e(y+M)\| &= \|e(x+M) - e(z+M)\| \\
&= \|x - p(x) - (z - p(z))\| \\
&\le \|x-z\| + \|p(x) - p(z)\| \\
&\le \|x-z\| + \lambda(x)\|x-z\| \\
&= (1+\lambda(x))\|x-z\| \\
&= (1+\lambda(x))\|(x+M) - (y+M)\|.
\end{aligned}$$

Then e is pointwise Lipschitz continuous at $x+M$ with Lipschitz constant $1+\lambda(x)$.

Conversely, let e be a pointwise Lipschitz continuous selection for φ_M. Defining p on X by $p(x) = x - e(x+M)$, we see that p is a selection for P_M such that for every $x \in X$ and $m \in M$,

$$p(x+m) = x + m - e(x+M) = p(x) + m.$$

That is, p is additive modulo M. Then

$$\begin{aligned}
\|p(x) - p(y)\| &\le \|x-y\| + \|e(x+M) - e(y+M)\| \\
&\le \|x-y\| + \lambda(x)\|(x+M) - (y+M)\| \\
&\le (1+\lambda(x))\|x-y\|.
\end{aligned}$$

This shows that p is pointwise Lipschitz continuous at x with Lipschitz constant $1 + \lambda(x)$.

The proof of the global Lipschitz properties now follows immediately since in this case the Lipschitz constants are independent of the particular points.

COROLLARY 2.10. *Let M be a proximinal subspace of X which is complemented. Then P_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection if and only if φ_M has a Lipschitz (resp. pointwise Lipschitz) continuous selection.*

Proof. In [4; Theorem 3.5], it was shown that, when M is complemented, P_M has a (pointwise) Lipschitz continuous selection if and only if P_M has one which is also homogeneous and additive modulo M. An appeal to theorem 2.9 completes the proof.

LEMMA 2.11 [6]. *Let X, Y, and Z be normed linear space, $F : X \to 2^Y$, let $i : X \to Z$ be a linear isomorphism from X onto Z, and suppose $G : Z \to 2^Y$ is defined $G \circ i = F$; i.e., $G(i(x)) = F(x)$, $x \in X$. Then:*

(1) F is τ.s.c. at x_0 if and only if G is τ.s.c. at $i(x_0)$. (Here $\tau = u, l, u.H, l.H.$)

(2) F is Lipschitz continuous (resp. linear) if and only if G is Lipschitz continuous (resp. linear).

(3) F has a continuous (linear, Lipschitz) selection if and only if G has one of the same type.

(4) If i is an isometry, then F has a linear selection with norm ρ if and only if G does.

3. Helly Extensions

In this section, let M be a proximinal subspace of a normed linear space X, and let $M^\perp$ denote the dual cone or annihilator of M in the dual space X^*; that is,

$$M^\perp = \{x^* \in X^* : x^*(y) = 0 \text{ for all } y \in M\}.$$

Further, let $J : X \to X^{**}$ denote the canonical embedding of X into its second dual space $X^{**} : J(x) = \hat{x}$, where $\hat{x}(x^*) = x^*(x)$, $x^* \in X^*$.

DEFINITION 3.1. The set-valued mapping $E_{M^\perp}(x^{**}|_{M^\perp}) : X^{**}|_{M^\perp} \to 2^X$ defined by

$$E_{M^\perp}(x^{**}|_{M^\perp}) = \{y \in X : \hat{y}|_{M^\perp} = x^{**}|_{M^\perp},\ \|y\| = \|x^{**}|_{M^\perp}\|\}$$

is called the Helly extension mapping relative to $M^\perp$.

LEMMA 3.2. *For any $x \in X$,*

(1) $\|\hat{x}|_{M^\perp}\| = d(x, M)$.

(2) $E_{M^\perp}(\hat{x}|_{M^\perp}) = \{y \in X \mid x - y \in M,\ \|y\| = d(x, M)\}$.

(3) $\varphi_M(x + M) = E_{M^\perp}(\hat{x}|_{M^\perp})$.

Proof. In [6], we can find the proofs of (1) and (2).

(3)

$$\begin{aligned} y \in \varphi_M(x+M) &\iff Q_M(y) = x + M \text{ and } y \in P_M^{-1}(0) \\ &\iff x - y \in M \text{ and } \|y\| = d(x, M) \\ &\iff y \in E_{M^\perp}(\hat{x}|_{M^\perp}). \end{aligned}$$

COROLLARY 3.4 [6]. *Let M be a proximinal subspace of X. Then*

(1) *For each $x \in X$,*

$$P_M(x) = x - E_{M^\perp}(\hat{x}|_{M^\perp}).$$

(2) *P_M is τ.s.c. at x if and only if $E_{M^\perp}$ is τ.s.c. at $\hat{x}|_{M^\perp}$. (Here $\tau = u, l, u.H.$, or $l.H$)*

(3) *P_M is Lipschitz continous (resp. linear) if and only if $E_{M^\perp}$ is Lipschitz continuous (resp. linear).*

(4) *P_M has a continuous (resp. Lipschitz continuous, linear) selection if and only if $E_{M^\perp}$ has a continuous (resp. Lipschitz continuous, linear) selection.*

(5) *P_M has a linear selection if and only if $E_{M^\perp}$ has a linear selection with norm one.*

Proof. Note that for $x \in X$, we have

$$d(x, M) = \|\hat{x}|_{M^\perp}\|.$$

The mapping $i : X/M \to \hat{X}|_{M^\perp}$ defined by $i(x + M) = \hat{x}|_{M^\perp}$ is an (linear) isometry and by Lemma 3.2, $\varphi_M = E_{M^\perp} \circ i$. It follows from Theorems 2.8 and 2.9, Lemma 2.11, and Lemma 3.2.

4. Hahn-Banach Extensions

In this section we replace X by X^* and M by $M^\perp$ in section 2 and thereby deduce results relating approximative properties of $M^\perp$ with properties of extending continuous linear functionals on M to all of X.

DEFINITION 4.1. For a subspace of X and $m^* \in M^*$, let $N_M(m^*)$ denote the set of all Hahn-Banach extensions of m^*; that is,

$$N_M(m^*) = \{x^* \in X^* : x^*|_M = m^*,\ \|x^*\| = \|m^*\|\}.$$

By Hahn-Banach theorem, $N_M : M^* \to 2^{X^*}\backslash\{\emptyset\}$.

LEMMA 4.2. *Let Γ be a $\sigma(X^*, X)$-closed linear subspace of X^* and $f \in X^*$. Then $\varphi_\Gamma(f+\Gamma) = N_{\perp\Gamma}(f\big|_{\perp\Gamma})$.*

Proof. Sinced Γ is a $\sigma(X^*, X)$-closed linear subspace of X^*, Γ is proximinal in X^*. Then

$$\begin{aligned} g \in \varphi_\Gamma(f+\Gamma) &\iff g \in P_\Gamma^{-1}(0) \text{ and } f - g \in \Gamma \\ &\iff f - g \in \Gamma \text{ and } \|g\| = d(f,\Gamma) \\ &\iff f\big|_{\perp\Gamma} = g\big|_{\perp\Gamma} \text{ and } \|g\| = d(f,\Gamma) \\ &\iff f\big|_{\perp\Gamma} = g\big|_{\perp\Gamma} \text{ and } \|g\| = \|f\big|_{\perp\Gamma}\| = d(f,\Gamma) \\ &\iff g \in N_{\perp\Gamma}(f\big|_{\perp\Gamma}). \end{aligned}$$

COROLLARY 4.3 [6]. *Let M be a closed subspace of X. Then:*
(1) *For each $x^* \in X^*$,*

$$\begin{aligned} P_{M^\perp}(x^*) &= x^* - N_{\perp(M^\perp)}(x^*\big|_{\perp(M^\perp)}) \\ &= x^* - N_M(x^*|_M) \end{aligned}$$

(2) *$P_{M^\perp}$ is τ.s.c. at x^* if and only if N_M is τ.s.c. at $x^*|_M$. (Here $\tau = u$, l,u.H, or l.H)*

(3) *$P_{M^\perp}$ is Lipschitz continuous (resp. linear) if and only if N_M is Lipschitz continuous (resp. linear).*

(4) *$P_{M^\perp}$ has a continuous (resp. Lipschitz continuous, linear) selection if and only if N_M has a continuous (resp. Lipschitz continuous, linear) selection.*

(5) $P_{M^\perp}$ *has a linear selection if and only if* N_M *has a linear selection with norm one.*

Proof. Note that for each $x^* \in X^*$, we have

$$d(x^*, M^\perp) = \|x^*|_M\|.$$

The mapping $i : X^*/M^\perp \to X^*|_M$ is an (linear) isometry and, by Lemma 4.2, $\varphi_M = N_M \circ i$. It follows from Theorems 2.8 and 2.9, Lemma 2.11 and Lemma 4.2.

5. Tietze Extensions

In this section we specialize the results of section 2 by taking $X = C_0(T)$, $S \subset T$ a compact subset, and $M = \{f \in C_0(T) : f|_S = 0\}$. In this way we deduce connections between approximative properties of M and Tietze extensions of functions in $C(S)$ to functions in $C_0(T)$.

Let T be a locally compact Hausdorff space and let $C_0(T)$ be the linear space of all real continuous functions f on T "vanshing at infinity"; that is, $\{t \in T : |f(t)| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$. Endowed with the supremum norm, $C_0(T)$ is a Banach space. Fix any compact subset S of T, and define $M = M_S := \{g \in C_0(T) : g|_S = 0\}$. Then M is a closed subspace in $C_0(T)$.

DEFINITION 5.1. The Tietze extension mapping $T_S : C(S) \to 2^{C_0(T)}$ is defined by

$$T_S(f) = \{F \in C_0(T) : F|_S = f,\ \|F\| = \|f\|\}.$$

By the Tietze extension theorem, $T_S(f) \neq \emptyset$ for every $f \in C(S)$.

LEMMA 5.2. *For each* $f \in C_0(T)$,
(1) $\|f|_S\| = d(f, M)$. [5]
(2) $\varphi_M(f + M) = T_S(f|_S)$.

Proof. (2)

$$\begin{aligned} g \in \varphi_M(f+M) &\iff f - g \in M \text{ and } g \in P_M^{-1}(0) \\ &\iff f|_S = g|_S \text{ and } \|g\| = \|f|_S\| = d(f, M) \\ &\iff g \in T_S(f|_S) \end{aligned}$$

COROLLARY 5.3 [5],[6].

(1) *For each* $f \in C_0(T)$,

$$P_M(f) = f - T_S(f|_S).$$

In particular, M is proximinal.

(2) P_M *is* τ*.s.c. at* f *if and only if* T_S *is* τ*.s.c. at* $f|_S$. *(Here* $\tau = u$*, l,u.H., or l.H.)*

(3) P_M *is Lipschitz continuous (resp. linear) if and only if* T_S *is Lipschitz continuous (resp. linear).*

(4) P_M *has a linear selection if and only if* T_S *has a linear selection.*

Proof. Note that for each $f \in X = C_0(T)$, we have

$$d(f, M) = \|f|_S\|.$$

The mapping $i : C_0(T)/M \to C(S)$ defined by $i(f + M) = f|_S$ is an (linear) isometry and by Lemma 5.2,

$$\varphi_M = T_S \circ i.$$

It follows from Theorems 2.8 and 2.9, Lemma 2.11 and Lemma 5.2.

6. Lipschitz Extensions

In this section the results of section 2 are specialized to the case when X is the set of all Lipschitz functions on a metric space T and M is the subspace of those functions vanishing on a prescribed subset S on T. In this case, we deduce a connection between the approximative properties of M with the set of all Lipschitz extensions of Lipschitz functions on S to all of T.

DEFINITION 6.1. Let (T, d) be a metric space, H a Hilbert space, and $0 < \alpha \leq 1$. For a given nonempty subset S of T, a function $f : S \to H$ is said to satisfy a Lipschitz (or Lipschitz-Hölder) condition of order α on S if

$$\|f\|_{d^\alpha,S} = \sup\left\{\frac{\|f(s) - f(t)\|}{d(s,t)^\alpha} : s,\ t \in S,\ s \neq t\right\} < \infty.$$

f is called bounded on S provided

$$\|f\|_S := \sup\{\|f(s)\| : s \in S\} < \infty.$$

The set of all bounded Lipschitz functions of order α on S will be denoted by $\mathrm{Lip}_\alpha(S)$. We endow $\mathrm{Lip}_\alpha(S)$ with the norm $\|f\|_{d^\alpha+S} := \|f\|_{d^\alpha,S} + \|f\|_S$.

DEFINITION 6.2. A Lipschitz (or Lipschitz-Hölder) extension of $f \in \mathrm{Lip}_\alpha(S)$ is any $F \in \mathrm{Lip}_\alpha(T)$ such that $F|_S = f$ and $\|F\|_{d^\alpha+T} := \|f\|_{d^\alpha+S}$. The Lipschitz extension mapping $L_S : \mathrm{Lip}_\alpha(S) \to 2^{\mathrm{Lip}_\alpha(T)}$ is defined by

$$L_S(f) = \{F \in \mathrm{Lip}_\alpha(T) : F \text{ is a Lipschitz extension of } f\}.$$

Fix any subset S of T and define

$$M = M_S := \{f \in \mathrm{Lip}_\alpha(T) : f|_S = 0\}.$$

Then M is a closed subspace in $\mathrm{Lip}_\alpha(T)$.

LEMMA 6.3 [6]. *For each* $f \in Lip_\alpha(T)$,

$$d(f, M) = \|f\|_{d^\alpha+S}.$$

LEMMA 6.4. *For each* $f \in Lip_\alpha(T)$,

$$\varphi_M(f + M) = L_S(f|_S).$$

Proof. By Lemma 6.3,

$$\begin{aligned} g \in \varphi_M(f+M) &\iff f - g \in M \text{ and } g \in P_M^{-1}(0) \\ &\iff f|_S = g|_S \text{ and } \|g\|_{d^\alpha+T} = \|f\|_{d^\alpha+S} \\ &\iff g \in L_S(f|_S). \end{aligned}$$

THEOREM 6.5 [6]. (1) *For each* $f \in Lip_\alpha(T)$,

$$P_M(f) = f - L_S(f|_S).$$

In particular, M *is proximinal.*

(2) P_M *is* τ*.s.c. at* f *if and only if* L_S *is* τ*.s.c. at* $f|_S$*. (Here* $\tau = u$*, l, u.H., or l.H.)*

(3) P_M *is Lipschitz continuous (resp. linear) if and only if* L_S *is Lipschitz continuous (resp. linear).*

(4) P_M *has a continuous (resp. linear) selection if and only if* L_S *has a continuous (resp. linear) selection.*

Proof. Note that for each $f \in \mathrm{Lip}_\alpha(T)$, we have

$$d(f, M) = \|f\|_{d^\alpha+S}.$$

The mapping $i : \mathrm{Lip}_\alpha(T)/M \to \mathrm{Lip}_\alpha(S)$ defined by $i(f+M) = f|_S$ is an (linear) isometry and, by Lemma 6.4,

$$\varphi_M = L_S \circ i.$$

It follows from Theorems 2.8 and 2.9, Lemma 2.11 and Lemma 5.4.

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