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L_∞ ESTIMATES OF OPTIMAL ORDER FOR GALERKIN METHODS TO SECOND ORDER HYPERBOLIC DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper we will consider finite element approximations on the solutions of the second order hyperbolic boundary value problem:

$$\begin{aligned} &\ddot{u} - \Delta u = f \quad \text{in} \quad \Omega \times (0,T] \\ &u = 0 \quad \text{on} \quad \partial\Omega \times (0,T] \\ &u(0) = u_0, \quad \dot{u}(0) = u_1 \quad \text{in} \quad \Omega \end{aligned} \tag{1.1}$$

The aim of this paper is a L_∞-boundedness of Standard–Galerkin–approximations for (1.1).

For the Ritz-approximations on the solution of second order elliptic boundary value problem, in Scott [8] for $N = 2$ dimensions it is proven:

$$\|u - R_h u\|_{L_\infty} \leq ch \inf_{x \in S_h} \|\nabla(u - x)\|_{L_\infty}.$$

The proof is based on a careful analysis of the approximability of the Green's function in the norm of W_1^1.

In Nitsche [3] for arbitrary dimensions the a priori estimate

$$\|R_h u\|_{L_\infty} + h\|\nabla(R_h u)\|_{L_\infty} \leq c\{\|u\|_{L_\infty} + h\|\nabla u\|_{L_\infty}\}$$

was shown.

Generalizing earlier results of Natterer the proof is based on the extensive use of certain weighted norms which are in the case of finite elements strongly connected with L_∞-norms.

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The Galerkin-approximation $u_h = u_h(t) \in \overset{\circ}{S}_h \subseteq \overset{\circ}{H}_1$ on (1.1) is defined by

$$(1.2) \qquad (\ddot{u}_h, x) + D(u_h, x) = (f, x) \quad \text{for} \quad x \in \overset{\circ}{S}_h \text{ and } t \in (0, T]$$

with

$$(1.3) \qquad u_h(0) = Q_h(u_0), \quad \dot{u}_h(0) = Q_h(u_1)$$

where Q_h denote the projection onto S_h, $(\cdot,\cdot)$ is L_2-scalar product and $D(\cdot,\cdot)$ is Dirichlet-integral.

The estimate of the error $e = u - u_h$ will be derived making use of the Ritz-approximation $U_h = R_h\Delta^{-1}f \in \overset{\circ}{S}_h$ of the corresponding Dirichlet problems:

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

The Ritz-approximation $U_h = R_h\Delta^{-1}f \in \overset{\circ}{S}_h$ is defined by

$$D(U_h, x) = (f, x) \quad \text{for} \quad x \in \overset{\circ}{S}_h.$$

In Nitsche [3] the estimates for arbitrary dimension

$$(1.4) \qquad \|u - R_h u\|_{L_\infty} \le ch^k \|u\|_{W^k_\infty}, \; k \le m$$

$$(1.5) \qquad \|u - R_h u\|_{L_2} \le ch^k \|u\|_{W^k_2}, \; k \le m$$

by application of finite element of order $m \ge 3$ are proven.

For our theorems we derive the projection Q_h in (1.6) below:

$$\begin{aligned} u_h(0) &= Q_h(u_0) := R_h(u_0) \\ \dot{u}_h(0) &= Q_h(u_1) := R_h(u_1). \end{aligned}$$

Our main theorem is

$$\|e\|_{L_\infty(L_\infty)} \le ch^m \{\|u\|_{L_\infty(W^m_\infty)} + \|\ddot{u}\|_{L_\infty(W^m_2)} + \|\dddot{u}\|_{L_2(W^m_2)}\}.$$

The method used is similar to Natterer: we first derive some error estimates in weighted Sobolev-norms and then turn over to L_∞-estimates.

2. Notations, Finite Elements

In the following $\Omega \subseteq R^N$ for $N = 2, 3$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. For any $\Omega' \subseteq \Omega$ let $W_p^k(\Omega')$ be the Sobolev space of functions having L_p-integrable derivatives of order up to k. The norms are indicated by the corresponding subscripts.

$$\|u\|_{W_p^k(\Omega')} := \left\{ \sum_{|\alpha|\le k} \iint_{\Omega'} |D^\alpha u|^p dx \right\}^{1/p}, \quad 1 \le p < \infty$$

$$\|u\|_{W_\infty^k(\Omega')} := \max_{|\alpha|\le k} \operatorname{ess} - \sup_{\Omega'} |D^\alpha u|, \quad p = \infty.$$

In the case $p = 2$ we also adopt $H_k(\Omega') = W_2^k(\Omega')$. The norms then are written shortly $\|\cdot\|_{k,\Omega'} = \|\cdot\|_{W_2^k(\Omega')}$. In addition we will use the abbreviation for boundary norms $|\cdot|_{k,\Omega'} = \|\cdot\|_{W_2^k(\partial\Omega')}$. Moreover, Ω' is skipped in case of $\Omega' = \Omega$ and k in case of $k = 0$. The use of weighted norms and semi-norms will be essential. They are defined by

$$\|\nabla^k v\|_{\alpha,\Omega'} = \left\{ \sum_{|\xi|=k} \iint_{\Omega'} \mu^{-\alpha} |D^\xi v|^2 dx \right\}^{1/2}$$

with μ given by

$$\mu = \mu(x) = |x - x_0|^2 + \varrho^2 \quad (x_0 \in \overline{\Omega},\ \varrho > 0) \tag{2.1}$$

The boundary semi-norms $|\cdot|_{\alpha,\Omega'}$ are defined in the corresponding way.

By Γ_h a subdivision of Ω into generalized simplicies Δ is meant,; Δ is simplex if Δ intersects $\partial\Omega$ in at most a finite number of points and otherwise one of the faces may be curved. Γ_h is called n-regular if to any $\Delta \in \Gamma_h$ there are two spheres of diameters $n^{-1}h$ and nh such that Δ contains the one and is contained in the other.

The finite element space $S_h = S(\Gamma_h)$ we will work with have the following structure: Let m be an integer fixed. Any element of S_h is continuous in Ω and the restriction to $\Delta \in \Gamma_h$ is a polynomial of degree less than m. In curved elements we use isoparametric modifications as discussed by Zlamal [10]. $\overset{\circ}{S}_h$ is the intersection of S_h and $\overset{\circ}{H}_1$, the closure in H_1 of the functions with compact support. By construction we have $S_h \subseteq H_1$, but in general $S_h \not\subseteq H_k$ for $k \geq 2$. It is useful to introduce the spaces $H_k' = H_k'(\Gamma_h)$ consisting of functions the restriction of which to any Δ is in $H_k(\Delta)$. Obviously $S_h \subseteq H_k'$ for all k. Parallel to the above we use 'broken' seminorms

$$\|\nabla^k v\|_\alpha' = \left\{ \sum_{\Delta \in \Gamma_h} \|\nabla^k v\|_{\alpha,\Delta}^2 \right\}^{1/2}$$

$$|\nabla^k v|_\alpha' = \left\{ \sum_{\Delta \in \Gamma_h} |\nabla^k v|_{\alpha,\Delta}^2 \right\}^{1/2}.$$

Let $L_p(0,T;X)$ denote those vector-valued maps of $[0,T]$ into X such that $\|v\|_{L^p(0,T;X)}^p = \int_0^T \|v(t)\|_X^p dt < \infty$, $1 \leq p < \infty$ and $L_\infty(0,T;X)$ those maps such that $\|v\|_{L_\infty(0,T;X)} = \sup_{0<t<T} \|v(t)\|_X < \infty$.

3. Approximation Theory in Weighted Norms

LEMMA 3.1. [Nitsche 4]. *Let Γ_h be an n-regular Subdivision of Ω and $\varrho \geq \gamma_1 h$ with $\gamma_1 := 2n$. Then*

(i) *To any $v \in H_\ell'$ with $\ell \leq m$ there is a $x \in S_h$ according to*

$$\|\nabla^k(v - x)\|_\alpha' \leq ch^{\ell-k}\|\nabla^\ell v\|_\alpha' \quad 0 \leq k < \ell \leq m$$

(ii) *For $x \in S_h$ and $0 \leq k < \ell < m$ inverse relations of the type*

$$\|\nabla^\ell x\|_\alpha' \leq ch^{-(\ell-k)}\|\nabla^k x\|_\alpha'$$

holds true.

In the subsequent sections we will apply these approximation result to functions v of the structure $v = \mu^{-\alpha}\varphi$ with $\varphi \in S_h$. Then a certain super-approximability property holds:

LEMMA 3.2. [Nitsche 4]. *Let $\varphi \in S_h$ (resp. $\overset{\circ}{S}_h$) be given. The function $\mu^{-\alpha}\varphi$ can be approximated by an element $x \in S_h$ (resp. $\overset{\circ}{S}_h$) according to*

$$\|\nabla^k(\mu^{-\alpha}\varphi - x)\|_\beta' \leq c\{h^{m-k}\|\varphi\|_{\beta+2\alpha+m} + h^{2-k}\|\nabla\varphi\|_{\beta+2\alpha+1}\}.$$

LEMMA 3.3. *For $\beta \geq 0$,*

$$\|v\|_{\alpha+\beta} \leq \rho^{-\beta}\|v\|_\alpha$$

holds true.

Proof.

$$\|v\|^2_{\alpha+\beta} = \iint_\Omega \mu^{-\alpha-\beta}v^2 dx \leq \max_{x\in\Omega}(\mu^{-\beta}) \iint_\Omega \mu^{-\alpha}v^2 dx \leq \rho^{-2\beta}\|v\|^2_\alpha.$$

Weighted Norms are strongly connected with the L_∞-norm.

LEMMA 3.4. [Nitsche 4]. *Let $\alpha > \frac{N}{2}$. Then for any $v \in L_\infty$ it is*

$$\|v\|^2_\alpha \leq c\rho^{-2\alpha+N}\|v\|^2_{L_\infty}.$$

For elements in the space S_h there is the counterpart:

LEMMA 3.5. [Nitsche 4]. *Let $\alpha > \frac{N}{2}$ and $h \leq \rho$. Then for $x \in S_h$ the inequality*

$$\|x\|^2_{L_\infty} \leq c\rho^{2\alpha}h^{-N} \sup_{x_0\in\Omega} \|x\|^2_\alpha$$

holds true.

REMARK 1. The last two lemmas show that the α-norm and the L_∞-norm are equivalent in the space S_h.

For $\phi \in \mathring{S}_h \subseteq \mathring{H}_1$, define $\omega \in \mathring{H}_1 \cap H_3$:

$$-\Delta\omega = \mu^{-2}\phi \quad \text{in} \quad \Omega \tag{3.1}$$
$$\omega = 0 \quad \text{on} \quad \partial\Omega.$$

LEMMA 3.6. [Nitsche 4].

$$\|\nabla^3\omega\|_{-1} \leq ch^{-2}\frac{h}{\rho}\|\phi\|_2 \quad \textit{for} \quad \omega \in \mathring{H}_1 \cap H_3.$$

LEMMA 3.7. [Nitsche 5]. *Let* $N \geq 2$, $\alpha = \frac{N}{2}$. *Then for any* $\omega \in \overset{\circ}{H}_1 \cap H_2$

$$\|\omega\|^2_{-\alpha+1} \leq c\rho^{-2}\|\Delta\omega\|^2_{-\alpha-1}$$

holds true.

LEMMA 3.8. [Nitsche 5]. *For* $N = 3$ *and* $\omega \in \overset{\circ}{H}_1 \cap H_2$

$$\|\omega\|^2 \leq c\rho^{-1}\|\Delta\omega\|^2_{-2} = c\rho^{-1}\|\phi\|^2_2.$$

4. Boundedness in Weighted Norms

For the error

$$e = u - u_h = (u - U_h) - (u_h - U_h) \equiv \epsilon - \phi \tag{4.1}$$

holds the equation

$$(\ddot{e}, x) + D(e, x) = 0 \quad \text{for} \quad x \in \overset{\circ}{S}_h. \tag{4.2}$$

With the choice of Ritz-approximation U_h as appropriate approximation in (1.2)

$$D(\epsilon, x) = 0 \quad \text{for} \quad x \in \overset{\circ}{S}_h$$

holds true.

It is easy to see from (4.1) and (4.2)

$$(\ddot{\phi}, x) + D(\phi, x) = (\ddot{\epsilon}, x) \quad \text{for} \quad x \in \overset{\circ}{S}_h. \tag{4.3}$$

We shall derive estimates in weighted norms for ϕ and $\nabla\phi$. These estimates will be absolutely fundamental in our analysis of the convergence of the Galerkin-approximations to (1.1).

THEOREM 4.1. *Let* $\rho \geq \gamma_2 h$ *with* γ_2 *appropriately chosen. Then* (i) *for* $N = 2$*:*

$$\|\phi\|^2_2 + \|\nabla\phi\|^2_1 \leq c_1\rho^{-2}\|\ddot{\epsilon} - \ddot{\phi}\|^2$$

(ii) *for* $N = 3$:

$$\|\phi\|_2^2 + \|\nabla\phi\|_1^2 \le c_1\rho^{-1}\|\ddot{\epsilon} - \ddot{\phi}\|^2.$$

Proof. Let α and N be appropriately chosen. It is:

$$\begin{aligned}\|\nabla\phi\|_\alpha^2 &= D(\phi, \mu^{-\alpha}\phi) - \iint \phi\nabla\phi\nabla\mu^{-\alpha}dx \\ &= D(\phi, \mu^{-\alpha}\phi) + \frac{1}{2}\iint_\Omega \phi^2\Delta\mu^{-\alpha}dx\end{aligned}$$

with $|\Delta\mu^{-\alpha}| \le c\mu^{-\alpha-1}$,

$$\|\nabla\phi\|_\alpha^2 \le D(\phi, \mu^{-\alpha}\phi) + c\|\phi\|_{\alpha+1}^2.$$

From (4.3) and (4.4), for some $x \in S_h$:

$$\|\nabla\phi\|_\alpha^2 \le D(\phi, \mu^{-\alpha}\phi - x) - (\ddot{\epsilon} - \ddot{\phi}, \mu^{-\alpha}\phi - x) + (\ddot{\epsilon} - \ddot{\phi}, \phi)_\alpha + c\|\phi\|_{\alpha+1}^2.$$

By Schwarz' inequality, we can obtain:

$$(u, v)_\alpha \le \|u\|_{\alpha-\alpha'}\|v\|_{\alpha+\alpha'}$$

and

$$D(u, v) \le \|\nabla u\|_{-\alpha'}\|\nabla v\|_{\alpha'}$$

for some α'.

And because of $\|\mu^{-\alpha}\phi\|_{1-\alpha} = \|\phi\|_{\alpha+1}$,

$$\begin{aligned}\|\nabla\phi\|_\alpha^2 \le \frac{1}{4}\|\nabla\phi\|_\alpha^2 + \|\nabla(\mu^{-\alpha}\phi - x)\|_{-\alpha}^2 + c_2\{\|\ddot{\epsilon} - \ddot{\phi}\|_{\alpha-1}^2 + \|\phi\|_{\alpha+1}^2\} \\ + \|\mu^{-\alpha}\phi - x\|_{1-\alpha}^2.\end{aligned}$$

Lemma 3.2 with $k = 0, 1$ gives

$$\|\mu^{-\alpha}\phi - x\|_{-\alpha+1} \le c\{h^m\|\phi\|_{\alpha+m+1} + h^2\|\nabla\phi\|_{\alpha+2}\}$$

and

$$\|\nabla(\mu^{-\alpha}\phi - x)\|_{-\alpha} \le c\{h^{m-1}\|\phi\|_{\alpha+m} + h\|\nabla\phi\|_{\alpha+1}\}$$

for some $x \in \mathring{S}_h$.

From Lemma 3.3 and $\frac{h}{\rho} < 1$, it follows that

$$\|\mu^{-\alpha}\phi - x\|_{-\alpha+1} \le c_3 \frac{h}{\rho}\{\|\phi\|_{\alpha+1} + \|\nabla\phi\|_{\alpha}\}$$

and

$$\|\nabla(\mu^{-\alpha}\phi - x)\|_{-\alpha} \le c_4 \frac{h}{\rho}\{\|\phi\|_{\alpha+1} + \|\nabla\phi\|_{\alpha}\}.$$

Hence

$$\|\nabla\phi\|_{\alpha}^2 \le [\frac{1}{4} + 2(c_3^2 + c_4^2)(\frac{h}{\rho})^2]\|\nabla\phi\|_{\alpha}^2 + c_5\{\|\ddot{\epsilon} - \ddot{\phi}\|_{\alpha-1}^2 + \|\phi\|_{\alpha+1}^2\}.$$

With the choice $\gamma_3 := \text{Max}\{\gamma_1, 4(c_3 + c_4)\}$ for $\rho \ge \gamma_3 h$,

$$\frac{1}{4} + 2(c_3^2 + c_4^2)(\frac{h}{\rho})^2 < \frac{1}{2}.$$

Then

$$\|\nabla\phi\|_{\alpha}^2 \le c_6\{\|\ddot{\epsilon} - \ddot{\phi}\|_{\alpha-1}^2 + \|\phi\|_{\alpha+1}^2\} \tag{4.5}$$

for some α and N.

In order to estimate for $\|\phi\|_2^2$ for $\phi \in \mathring{S}_h \subseteq \mathring{H}_1$, define $\omega \in \mathring{H}_1 \cap H_3$ by (3.1). It follows from (4.3) that

$$\|\phi\|_2^2 = D(\phi, \omega) = D(\phi, \omega - x) - (\ddot{\epsilon} - \ddot{\phi}, \omega - x) + (\ddot{\epsilon} - \ddot{\phi}, \omega).$$

From Lemma 3.1 it follows that

$$\begin{aligned} D(\phi, \omega - x) &\le \|\nabla\phi\|_1 \|\nabla(\omega - x)\|_{-1} \\ &\le \|\nabla\phi\|_1 ch^2 \|\nabla^3\omega\|_{-1}. \end{aligned}$$

Lemma 3.6 gives

$$D(\phi, \omega - x) \le c\frac{h}{\rho}\{\|\nabla\phi\|_1^2 + \|\phi\|_2^2\}.$$

And by Schwarz' inequality

$$(\ddot{\epsilon} - \ddot{\phi}, \omega - x) \leq \|\ddot{\epsilon} - \ddot{\phi}\| \, \|\omega - x\|$$

and

$$(\ddot{\epsilon} - \ddot{\phi}, \omega) \leq \|\ddot{\epsilon} - \ddot{\phi}\| \, \|\omega\|.$$

Since $\frac{h}{\rho} < 1$, with Lemma 3.1 and 3.6

$$\|\omega - x\| \leq ch^3 \|\nabla^3 \omega\| \leq ch^2 \|\nabla^3 \omega\|_{-1} \leq c\frac{h}{\rho}\|\phi\|_2.$$

Hence

$$\|\phi\|_2^2 \leq c_7 \frac{h}{\rho}\{\|\phi\|_2^2 + \|\nabla\phi\|_1^2\} + \delta\|\omega\|^2 + c_8(1+\delta^{-1})\|\ddot{\epsilon} - \ddot{\phi}\|^2$$

with $\delta > 0$.

For the estimate of $\|\omega\|$ in (4.6), the case of 2 or 3 dimensions have to be treated separately. This will be clearer because of Lemma 3.7 and 3.8. Now let us consider the case of $N = 2$ dimensions. For $\omega \in \overset{\circ}{H}_1 \cap H_2$ in case of $N = 2$ and $\alpha = \frac{N}{2} = 1$:

$$\|\omega\|^2 \leq c_9 \rho^{-2} \|\phi\|_2^2.$$

And in the case of $N = 3$, Lemma 3.8 gives the boundedness of $\|\omega\|$. Hence

$$\|\phi\|_2^2 \leq (c_7 \frac{h}{\rho} + c_9 \delta \rho^{-2})(\|\phi\|_2^2 + \|\nabla\phi\|_1^2) + c_8(1+\delta^{-1})\|\ddot{\epsilon} - \ddot{\phi}\|^2 \quad \text{for } N = 2,$$

and

$$\|\phi\|_2^2 \leq (c_7 \frac{h}{\rho} + c_{10} \delta \rho^{-1})(\|\phi\|_2^2 + \|\nabla\phi\|_1^2) + c_8(1+\delta^{-1})\|\ddot{\epsilon} - \ddot{\phi}\|^2 \text{ for } N = 3.$$

(4.5) gives for $\alpha = 1$ and for some N

$$\|\nabla\phi\|_1^2 \leq c_6\{\|\ddot{\epsilon} - \ddot{\phi}\|^2 + \|\phi\|_2^2\}.$$

Choose in case of $N = 2$:

$$\delta := \frac{\rho^2}{4c_9(1+c_6)}. \tag{4.7}$$

Then for $\rho \geq \gamma_2 h$ with $\gamma_2 := \mathrm{Max}\{\gamma_3, 4c_7(1+c_6)\}$,

$$(c_7\frac{h}{\rho} + c_9\delta\rho^{-2})(1+c_6) < 1$$

holds true.

In the case of $N = 3$: for $\rho \geq \gamma_2 h$, substitute in (4.7) ρ^2 for ρ and c_9 for c_{10}. Then for $N = 2, 3$ $\|\phi\|_2^2$ and $\|\nabla\phi\|_1^2$ in Theorem 4.1 is bounded by $\|\ddot{\epsilon} - \ddot{\phi}\|^2$.

THEOREM 4.2. *Let $N = 2, 3$. Then*

$$\|\phi\|^2_{L_\infty(0,T;H^2)} \leq c\rho^{N-4}\{\|\ddot{\epsilon}\|^2_{L_\infty(0,T;L_2)} + \|\dddot{\epsilon}\|^2_{L_2(0,T;L_2)}\}.$$

Proof. Theorem 4.1 gives for $N = 2, 3$

$$\|\phi\|_2^2(t) \leq c\rho^{N-4}\{\|\ddot{\epsilon}\|^2(t) + \|\ddot{\phi}\|^2(t)\}.$$

Differentiating (4.3) with respect to time:

$$(\dddot{\phi}, x) + D(\dot{\phi}, x) = (\dddot{\epsilon}, x) \quad \text{for} \quad t \in (0, T] \text{ and } x \in \mathring{S}_h.$$

Take $x := \ddot{\phi} \in S_h$.

Then

$$(\dddot{\phi}, \ddot{\phi})(z) + D(\dot{\phi}, \ddot{\phi})(z) = (\dddot{\epsilon}, \ddot{\phi})(z). \tag{4.8}$$

Integrating (4.8):

$$\int_0^t (\dddot{\phi}, \ddot{\phi})(z)dz + \int_0^t D(\dot{\phi}, \ddot{\phi})(z)dz = \int_0^t (\dddot{\epsilon}, \ddot{\phi})(z)dz.$$

Since

$$\int_0^t (\dddot{\phi}, \ddot{\phi})(z)dz = \frac{1}{2}\int_0^t \frac{d}{dt}\|\ddot{\phi}\|^2(z)dz = \frac{1}{2}\|\ddot{\phi}\|^2(t) - \frac{1}{2}\|\ddot{\phi}\|^2(0)$$

and

$$\int_0^t D(\dot{\phi}, \ddot{\phi})(z)dz = \frac{1}{2}\int_0^t \frac{d}{dt}\|\nabla\dot{\phi}\|^2(z)dz,$$

we can obtain with $\dot{\phi}(0) = 0$

$$\begin{aligned}\|\ddot{\phi}\|^2(t) - \|\ddot{\phi}\|^2(0) + \|\nabla\dot{\phi}\|^2(t) &= 2\int_0^t (\dddot{\epsilon}, \ddot{\phi})(z)dz \\ &\leq 2\int_0^t \|\dddot{\epsilon}\|(z)\|\ddot{\phi}\|(z)dz \\ &\leq \int_0^t \|\dddot{\epsilon}\|^2(z)dz + \int_0^t \|\ddot{\phi}\|^2(z)dz.\end{aligned}$$

Hence

$$\|\ddot{\phi}\|^2(t) \leq \|\ddot{\phi}\|^2(0) + \int_0^t \|\dddot{\epsilon}\|^2(z)dz + \int_0^t \|\ddot{\phi}\|^2(z)dz.$$

By Gronwall's lemma,

$$\|\ddot{\phi}\|^2(t) \leq c\{\|\ddot{\phi}\|^2(0) + \int_0^T \|\ddot{\epsilon}\|^2(z)dz\}.$$

From (4.3) with $\phi(0) = 0$,

$$\begin{aligned}\|\ddot{\phi}(0)\|^2 &= (\ddot{\phi}(0), \ddot{\phi}(0)) = -D(\phi(0), \ddot{\phi}(0)) + (\ddot{\epsilon}(0), \ddot{\phi}(0)) \\ &\leq \|\ddot{\epsilon}(0)\|\,\|\ddot{\phi}(0)\|.\end{aligned}$$

Since $\|\ddot{\phi}(0)\| \leq \|\ddot{\epsilon}(0)\|$,

$$\|\ddot{\phi}\|^2(t) \leq c\{\|\ddot{\epsilon}\|^2_{L_\infty(0,T;L_2)} + \|\dddot{\epsilon}\|^2_{L_2(0,T;L_2)}\}.$$

5. Boundedness in $L_\infty(0,T;L_\infty(\Omega))$-Norms

For fixed $t \in [0,T]$ there exists $\hat{x} = \hat{x}_t \in \Omega$ such that

$$\phi(\hat{x},t) = \pm\|\phi(t)\|_{L_\infty}.$$

For the weight function $\mu = |x - x_0|^2 + \rho^2$. Choose $x_0 := \hat{x}$ and $\rho := \gamma_2 h$.

Let $\Delta \in \Gamma_h$ be the simplex with $\hat{x} \in \overline{\Delta}$. Since Γ_h is n-regular subdivision, $|x - \hat{x}| \leq nh$ and

$$\rho^2 = \gamma_2^2 h^2 \leq \mu \leq \rho^2 + n^2h^2 = (\gamma_2^2 + n^2)h^2. \tag{5.1}$$

Since ϕ is a polynomial of degree less than m on Δ, ϕ is an element of finite dimensional spaces and in this spaces two norms are equivalent.

With $K := K(N,m,n)$

$$\|\phi\|^2_{L_\infty(\Delta)} \leq Kh^{-N}\|\phi\|^2_{L_2(\Delta)} \tag{5.2}$$

holds true.

(5.1) implies for $N = 2$:

$$h^{-2}\iint_\Delta \phi^2 dx \leq c\rho^2 \iint_\Delta \mu^{-2}\phi^2 dx = c\rho^2\|\phi\|_2^2,$$

and for $N = 3$:

$$h^{-3}\iint_\Delta \phi^2 dx \leq c\rho \iint_\Delta \mu^{-2}\phi^2 dx = c\rho\|\phi\|_2^2.$$

From (5.2) for $N = 2,3$

$$\|\phi\|^2_{L_\infty(\Delta)}(t) \leq Kh^{-N}\iint_\Delta \phi^2 dx \leq Kc\rho^{4-N}\|\phi\|_2^2.$$

Then Theorem 4.2 implies

$$\|\phi\|^2_{L_\infty}(t) \leq c\{\|\ddot{\epsilon}\|^2_{L_\infty(0,T;L_2)} + \|\dddot{\epsilon}\|^2_{L_2(0,T;L_2)}\}.$$

For the error ϵ of Ritz-approximation, (1.4) and (1.5) give.

We collect the $L_\infty(0,T;L_\infty(\Omega))$-estimates in he following way:

THEOREM 5.1. *For finite element of order $m \geq 3$, the error between the solution of a second order hyperbolic boundary value problem and the Galerkin approximation u_h in case of $N = 2, 3$ under the assumptions*

$$u \in L_\infty(0,T; W^k_\infty(\Omega))$$
$$\dot{u} \in L_\infty(0,T; W^k_2(\Omega))$$
$$\ddot{u} \in L_2(0,T; W^k_2(\Omega))$$

is of the order h^k with $k \leq m$.

That is,

$$\|e\|_{L_\infty(0,T;L_\infty(\Omega))} \leq ch^m\{\|u\|_{L_\infty(0,T;W^m_\infty)} + \|\dot{u}\|_{L_\infty(0,T;W^m_2)} + \|\ddot{u}\|_{L_2(0,T;W^m_2)}\}.$$

References

1. Robert A. Adams, *Sobolev Space*, Academic Press, New York, San Francisco, London, 1975.
2. O. Axellson and V. A. Barker, *Finite Element Solution of Boundary VAlue Problem*, Acad. pr. 1984, XVIII.
3. J. Nitsche, *L_∞-convergence of Finite Element Approximation*, 2nd Conference on Finite Elements, Rennes (May 1975).
4. ———, *L_∞-convergence of Finite Element Galerkin Approximations*, for Parabolic Problems, R.A.I.R.O., Anal. Num. **13** (1979), 31–54.
5. ———, *Schauder Estimates for Finite Element Approximations on second order Elliptic Boundary Value Problems*, Proceedings of the Special Year in Numerical Analysis, Lecture Notes No. 20., Univ. of Maryland, I. Babuska, T.-P.Liv, J. Osborn eds. **4** (1981), 290–343.
6. ———, and M. Wheeler, *L_∞-Boundedness of the Finite Element Galerkin Operator for parabolic Problems*, Numer. Funct. Anal. And Optimiz., (1981–1982), 325–353.
7. A. H. Schatz and L. B. Wahlbin, *Maximum Norm Estimates in the Finite Element Method on Plane Polygonal Domains*, Math. Comp., **32**, (1978), 73–109.
8. R. Scott, *Optimal L_∞-Estimates for the Finite Element Method on Irregular Meshes*, Math. Comp., **30**, Nr. **136** (1976), 681–697.
9. M. F. Wheeler, *L_∞-Estimates of Optimal Orders for Galerkin Methods for one-dimensional second order parabolic and hyperbolic Equations*, SIAM. J. Number. Anal. **10**, (1973), 908–913.

10. M. Zlamal, *Curved Elements in the Finite Element Method*, Part I: SIAM J. Num. Anal. **10** (1973), 229–240, Part II: SIAM J. Num. Anal. **11** (1974), 347–362.

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