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Complete Reducibility of some Modules for a Generalized Kac Moody Lie Algebra

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ABSTRACT. Let $G(A)$ denote a generalized Kac Moody Lie algebra associated to a symmetrizable generalized Cartan matrix A. In this paper, we study on representations of $G(A)$. Highest weight modules and the category O are described. In the main theorem we show that some $G(A)$ modules from the category O are completely reducible. Also a criterion for irreducibility of highest weight modules is obtained. This was proved in [3] for the case of Kac Moody Lie algebras.

I. Introduction

We introduce generalized Kac Moody Lie algebras and some basic properties. Let A be a real $n \times n$ matrix (a_{ij}) satisfying the following properties:

(1) either $a_{ii} = 2$ or $a_{ii} \leq 0$

(2) $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in Z$ if $a_{ii} = 2$

(3) $a_{ij} = 0$ implies $a_{ji} = 0$.

And let $(H, \Pi, \Pi^\vee)$ be a realization of A i.e., H is a complex vector space, both of $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ $(\subset H^*)$ and $\Pi^\vee \subset \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}$ $(\subset H)$ are linearly independent, $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ and $\dim H = 2n - l$.

To this matrix we associate a Lie algebra G which is generated by e_i, $f_i (i = 1, 2, \ldots, n)$, H with the following relations:

(1) $[h, h'] = 0$ $(h,\ h' \in H)$

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(2) $[e_i, f_j] = \delta_{ij}\alpha_i^\vee \ (i,\ j = 1, 2, \ldots, n)$
(3) $[h, e_i] = \langle \alpha_i, h \rangle e_i \ [h, f_i] = -\langle \alpha_i, h \rangle f_i, \ (i = 1, 2, \ldots, n; h \in H)$
(4) If $a_{ii} > 0$, $(ade_i)^{1-2a_{ij}/a_{ii}} e_j = 0$ and similarly $(adf_i)^{1-2a_{ij}/a_{ii}} f_j = 0$.
(5) If $a_{ij} = 0$, then $[e_i, e_j] = [f_i, f_j] = 0$.

The Lie algebra G is called a generalized Kac Moody algebra (see [1]), the elements e_i, f_i, $\alpha_i^\vee$, $i = 1, 2, \ldots, n$ are called the canonical generators of G. We denote by Q the lattice generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$ i.e.,

$$Q = \sum_{i=1}^{n} Z\alpha_i$$

and

$$Q_+ = \sum_{i=1}^{n} Z_+\alpha_i.$$

Here Z_+ is the set of nonnegative integers. For α and β in Q, we write $\alpha \geq \beta$ if $\alpha - \beta \in Q_+$. The matrix A is said to be symmetrizable if there exists a diagonal matrix D with positive rational diagonal entries such that DA is symmetric. From now on we shall assume that A is indecomposable and symmetrizable. In this case, G admits a invariant bilinear form $(\cdot, \cdot)$ which is nondegenerate on H. Since the bilinear form $(\cdot, \cdot)$ on H is nondegenerate, we have an isomorphism $\nu : H \to H^*$ defined by

$$\nu(h)h_1 = (h, h_1), \qquad h,\ h_1 \in H.$$

Via ν, we induce a bilinear form $(\cdot, \cdot)$ on H^*, thus also on Q. We let ρ be the element in H^* such that

$$(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2.$$

The subspace $G_\alpha = \{x \mid [h,x] = \alpha(h)x\}$ is the linear span of the elements of the form

$$[\cdots[[e_{i_1}, e_{i_2}], e_{i_3}]\cdots e_{i_n}]$$

such that $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n} = \alpha$. A root is defined by a nonzero element α of Q such that G_α is nonzero. A root α is called real if $(\alpha, \alpha) > 0$ and imaginary otherwise. For any root α clearly we have either $\alpha \geq 0$ or $-\alpha \geq 0$; a root α is called respectively a positive or a negative root. Each α_i, $i = 1, 2, \ldots, n$ is called a simple root. Let $\Delta^+(\Delta^-)$ denote the set of positive (negative respectively) roots of G. Set

$$N_+ = \bigoplus_{\alpha \in \Delta^+} G_\alpha, \quad N_- = \bigoplus_{\alpha \in \Delta^-} G_\alpha.$$

N_+ and N_- are subalgebras of G and as a vector space one has $G = N_- \oplus H \oplus N_+$. For a subalgebrea B of G let $U(B)$ denote the universal enveloping algebra of B. By the Poincaré-Birkhoff-Witt theorem one has

$$U(G) = U(N_-) \bigotimes_{\mathbf{C}} U(H) \bigotimes_{\mathbf{C}} U(N_+).$$

A G module $M(\lambda)$ is called a highest weight module with highest weight $\lambda \in H^*$ if there exists a nonzero vector $v \in M(\lambda)$ such that $N_+(v) = 0$, $h(v) = \lambda(h)v$ for $h \in H$ and $U(G)(v) = M(\lambda)$. A G module M is called H-diagonalizable if

$$M = \bigoplus_{\lambda \in H^*} M_\lambda$$

where $M_\lambda = \{v \in M \mid hv = \lambda(h), h \in H\}$. A nonzero element $\lambda \in H^*$ is called a weight if $M_\lambda \neq 0$ and M_λ is called a weight space.

Now we define the category O as follows. Its objects are G modules M which are H-diagonalizable with finite dimensional weight spaces and such that there exist finite number of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in H^*$ such that

$$P(M) \subset \bigcup_{i=1}^{s} D(\lambda_i)$$

Here, $P(M) = \{\lambda \in H^* \mid M_\lambda \neq 0\}$ and $D(\lambda) = \{\mu \in H^* \mid \mu \leq \lambda\}$. Important examples of modules from the category O are highest weight modules. Any submodules or quotient modules in O are in O.

Given a G module M in O, we introduce a linear operator Ω on M as follows: Let $u_1, u_2 \ldots$ and $u^1, u^2, \ldots$ be dual bases of H. For each positive root α we choose a basis $e_\alpha^{(i)}$ of the space G_α and let $e_{-\alpha}^{(i)}$ be the dual basis of $G_{-\alpha}$. The generalized Casimir operator is defined by

$$\Omega(v) = 2\nu^{-1}(\rho) + \sum u_i u^i + 2 \sum_{\alpha \in \Delta+} + \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}.$$

LEMMA I.1. *Let M be a G module in O. Then Ω commutes with the action of G on M.*

PROOF. See [3, Theorem 2.6].

COROLLARY I.1. *For a highest weight module $M(\lambda)$ with highest weight λ, one has*

$$\Omega = (\lambda + 2\rho, \lambda) id.$$

PROOF. See [3, Theorem 2.6].

PROPOSITION I.1. *Let $M(\lambda)$ be a highest weight module with highest weight λ. If for any weight μ of $M(\lambda)$, $(\lambda+2\rho, \lambda) = (\mu+2\rho, \mu)$ implies $\lambda = \mu$, then the G module $M(\lambda)$ is irreducible.*

PROOF. If $M(\lambda)$ has a nonzero proper submodule U, there exists a vector w in U such that $N_+ w = 0$. Let μ be the weight of w in U. Since U is a proper submodule of $M(\lambda)$, we have $\mu < \lambda$. On the other hand, applying Ω on w, by Corollary I.1 we obtain the following equality:

$(\lambda + 2\rho, \lambda) idw = \Omega w = (\mu + 2\rho, \mu) w$. Hence $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$, which contradicts the assumption.

PROPOSITION I.2. *Let $M \in O$. For any complex number c, set $M_{(c)} = \{v \in M \mid (\Omega - c)^n = 0$ for some $n\}$. Then $M_{(c)}$ is a submodule of M and*

$$M = \sum_{c \in G} \oplus M_{(c)}.$$

PROOF. See [2, Theorem 6.3].

II. Complete reducibility

In this section we prove the complete reducibility theorem. First we introduce the following notion. Let M be a G module. A vector $v \in M_\lambda$ is called primitive if there exists a submodule U in M such that v is not in U and $N_+(v) \subset U$. In this case λ is called a primitive weight.

Now we state the main theorem.

THEOREM II.1. (a) *Let M be a G module from the category O. If for any primitive weight μ of M and for any simple root α_i, one has $(\mu + \rho, \alpha_i) > 0$, then M is completely reducible.* (b) *Highest weight G module $M(\lambda)$ is irreducible if $(\mu + \rho, \alpha_i) > 0$ for any primitive weight μ of M and for any simple root α_i.*

The proof of the theorem is based on following Lemmas.

LEMMA II.1. *Let M be a G module from the category O. If for any two primitive weights λ and μ of M the inequality $\lambda \geq \mu$ implies $\lambda = \mu$, then the module M is completely reducible.*

PROOF. See [3, Proposition 9.9].

LEMMA II.2. *Let $c \in \mathbf{C}$, and $M_{(c)} = \{v \in M \mid (\Omega - cI)^n = 0$ for some $n\}$. Then for any two primitive weights λ and μ of $M_{(c)}$, we have $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$.*

PROOF. It follows immediately from the definition of primitive weights and the action of Ω.

PROOF OF THEOREM II.1.

(a) By Proposition I.2., we may assume $M = M_{(c)}$ for some complex number c. Now, let λ and μ be two primitive weights of M such that $\lambda \geq \mu$ and $\lambda \neq \mu$. Put $\lambda - \mu = \sum k_i \alpha_i$, where k_i's are positive integers. By assumption $(\lambda+2\rho, \lambda) - (\mu+2\rho, \mu) = (\lambda+\rho+\mu+\rho, \lambda-\mu)$ is positive, which contradicts Lemma II.2. Hence, $\lambda = \mu$ and the theorem follows from Lemma II.1.

(b) If $M(\lambda)$ is reducible, there exists a primitive weight μ such that λ is strictly greater than μ. Since both λ and μ are primitive weights of M, similar to the case (a), we have that $(\lambda + 2\rho, \lambda) - (\mu + 2\rho, \mu)$ is positive, which also contradicts Lemma II.2.

REMARK. Let M be an integrable G module i.e., e_i and f_i are locally nilpotent when $a_{ii} = 2$. Then SL_2 theory implies $(\lambda+\rho, \alpha_i) > 0$ for any primitive weight λ and for any real simple root α_i.

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