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Remarks on Regularities in Transformation Groups

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ABSTRACT. In this paper we shall study the properties of regular relations. We also define syndetically regular relations as the generalizations of syndetical proximalities.

0. Introduction

Let (X,T) be a transformation group with a compact Hausdorff space X. Two points x and y of X are said to be *proximal* [7] if for every index α of X, there exists $t \in T$ such that $(xt, yt) \in \alpha$. This proximal relation, denoted by $P(X)$, is the set of all pairs of proximal points. The regular relations have been introduced in [11], [12]. The points x and y of X are *regular* if $h(x)$ and y are proximal for some automorphism h of X.

In this paper we investigate the properties of regular pairs and find an equivalent condition the proximal relation $P(X)$ to be closed. We also introduce syndetically regular relations as the analogue concepts of syndetical proximalities.

1. Preliminaries

An arbitrary, but fixed, topological group will be denoted by T throughout this paper and we will consider the transformation group (X,T) with a compact Hausdorff space X. The compact Hausdorff space X carries a natural uniformity U whose indices are the neighborhood of the diagonal in $X \times X$.

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A closed nonempty subset A of X is called *minimal* if for every $x \in A$, the orbit xT is a dense subset of A. If X is itself minimal, we say it is a *minimal transformation group.* A subset A of T is called *syndetic* if there exists a compact subset K of T with $T = AK$. A point $x \in X$ is said to be *almost periodic* if given $\alpha \in U$, there exists a syndetic subset A of T such that $xA \subset x\alpha$. (X, T) is *pointwise almost periodic* if and only if each point of X is almost periodic.

A continuous map h of X to X is called an *endomorphism* if $h(xt) = h(x)t$ for all $x \in X, t \in T$. If h is bijective, h is said to be an *automorphism.* The set of all automorphisms of X is denoted by $A(X)$.

We define $E(X)$, the *enveloping semigroup* of X, to be the closure of T in X^X, providing X^X with its product topology. The *minimal right ideal* M is the nonempty subset of $E(X)$ with $ME(X) \subset M$, which contains no proper nonempty subset of the same property. Two idempotents u and v in $E(X)$ is said to be *equivalent* if $uv = u$ and $vu = v$.

2. Regular relations

DEFINITION 2.1 [12]. Let H be a subset of $A(X)$, and let $R_H(X)$ be the set of all $(x, y) \in X \times X$ such that $h(x)$ and y are proximal for some $h \in H$. The *regular relation* $R(X)$ on (X, T) is defined to be the set $R_{A(X)}(X)$. For a fixed $h \in A(X)$, $R_{\{h\}}(X)$ is denoted simply $R_h(X)$ or R_h.

The regular relations in transformation groups are reflexive, symmetric and invariant, but is not in general transitive or closed.

LEMMA 2.1. *Let (X, T) be a transformation group. If R_h is closed for some $h \in A(X)$, then there exist only one minimal right ideal of $E(X)$, and hence $P(X)$ is an equivalence relation.*

PROOF. Let M and N be minimal right ideals of $E(X)$, and let u and v be equivalent idempotents of M and N, respectively. Let $x \in X$. Then given $h \in A(X)$, $(x, h(xu)) \in R_h$. Since R_h is closed,

$$(xv, h(xu)) = (xv, h(xu)v) = (x, h(xu))v \in R_h,$$

that is, $h(xv)$ and $h(xu)$ are proximal. But

$$\begin{aligned}(h(xv), h(xu)) &= (h(x)v, h(x)u) \\ &= (h(x)v^2, h(x)uv) \\ &= (h(xv), h(xu))v\end{aligned}$$

implies $(h(xv), h(xu))$ is an almost periodic point of $(X \times X, T)$. Therefore

$$h(x)v = h(xv) = h(xu) = h(x)u.$$

Since x is arbitrary, we obtain $v = u$, which shows that $M = N$. By Theorem 2 [7], $P(X)$ is an equivalence relation.

The following theorem gives an equivalent condition $P(X)$ to be closed.

THEOREM 2.2. *Let (X, T) be a transformation group. Then the following are equivalent:*

(1) *$P(X)$ is closed.*
(2) *$R_h(X)$ is closed for all $h \in A(X)$.*
(3) *$R_k(X)$ is closed for some $k \in A(X)$.*

Moreover, if there exists p in a minimal right ideal M of $E(X)$ such that p is continuous, then the above statements are equivalent to

(4) *$P(X)$ is an equivalence relation.*

PROOF. (1) implies (2). Let $h \in A(X)$ and let (x_n, y_n) be a net in $R_h(X)$ such that (x_n, y_n) converges to (x, y). Then $(h(x_n), y_n) \in P(X)$

for all n and $(h(x_n), y_n)$ converges to $(h(x), y)$. Since $P(X)$ is closed, it follows that $(h(x),\ y) \in P(X)$, and hence $(x, y) \in R_h(X)$. Therefore $R_h(X)$ is closed.

(2) implies (3) is obvious.

(3) implies (1). Let R_k be closed for some $k \in A(X)$ and let $((x_n, y_n))$ be a net in $P(X)$ such that (x_n, y_n) converges to (x, y). Then $(k^{-1}(x_n), y_n)$ is a net in $R_k(X)$ and $(k^{-1}(x_n), y_n)$ converges to $(k^{-1}(x), y)$. Since R_k is closed, we have $(k^{-1}(x), y) \in R_k(X)$, and $(x, y) \in P(X)$. Therefore $P(X)$ is closed.

(3) implies (4) follows from Lemma 2.1.

(4) implies (1) follows from Lemma 2.1 [9].

COROLLARY 2.3. *Let (X, T) be a transformation group and let R_h be closed for some $h \in A(X)$. Then the relations $R(X)$ and $R_H(X)$ are equivalence relations for subgroup H of $A(X)$.*

PROOF. From Theorem 2.6 [12], it suffices to show that $(xu, yu) \in R_H(X)$ for every $(x, y) \in R_H(X)$ and idempotent u of $E(X)$. Let H be a subgroup of $A(X)$ and let $(x, y) \in R_H(X)$, $u^2 = u \in E(X)$. Then $(x, y) \in R_h$ for some $h \in A(X)$.

Since R_h is closed,

$$\begin{aligned}(x, y)u &= \lim(x, y)t_\alpha = \lim(xt_\alpha, yt_\alpha)\\ &= (xu, yu) \in R_h \subset R_H(X).\end{aligned}$$

Therefore, $R_H(X)$ is an equivalence relation, and so is $R(X)$.

DEFINITION 2.2. Let I_M be the set of idempotents of a minimal right ideal M of $E(X)$, and let $I = \cup\{I_M \mid M \text{ a minimal right ideal}\}$. For a fixed $x_0 \in X$ and $h \in A(X)$, define $R_{I_M}(h, x_0)$ (resp. $R_I(h, x_0)$) to be the set $\{h(x_0)u \mid u \in I_M\}$ (resp. $\{h(x_0)u \mid u \in I\}$).

REMARK. Let (X, T) be pointwise almost periodic and let $h = 1_X$. Then $R_I(h, x_0) = P(x_0)$. ($y \in P(x_0)$ iff $(x_0, y) \in P(X)$).

THEOREM2.4. *Given $x_0 \in X$ and $h \in A(X)$, the following hold:*

(1) *y and y' are proximal for all y, y' in $R_{I_M}(h, x_0)$.*

(2) *If x_0 and y_0 are proximal, then $R_I(h, x_0) = R_I(h, y_0)$.*

PROOF. (1) Let y, y' be two points of $R_{I_M}(h, x_0)$. There exist idempotents u and v of M such that $y = h(x_0)u$ and $y' = h(x_0)v$. Then

$$yv = h(x_0)uv = h(x_0)v = h(x_0)v^2 = h(x_0)vv = y'v.$$

This shows that y and y' are proximal.

(2) Let $y \in R_I(h, x_0)$, then $y = h(x_0)u$. Since x_0, y_0 are proximal, there exists a minimal right ideal M of $E(X)$ such that $x_0p = y_0p$ for all $p \in M$, and thus $x_0u = y_0u$ for $u^2 = u \in M$. Therefore,

$$y = h(x_0)u = h(x_0u) = h(y_0u) = h(y_0)u \in R_I(h, y_0).$$

Similarly, $R_I(h, y_0) \subset R_I(h, x_0)$.

THEOREM 2.5. *Let (X, T) be a transformation group. Then the following are equivalent:*

(1) *$(x, y) \in R(X)$ and y is an almost periodic point.*

(2) *$y \in R_I(h, x)$ for some $h \in A(X)$.*

PROOF. (1) implies (2). Suppose that $(x, y) \in R(X)$ and y is an almost periodic point. Then $(h(x), y) \in P(X)$ for some $h \in A(X)$ and therefore $h(x)p = yp$ for all p in a minimal right ideal M. Since y is an almost periodic point, $y = yu$ for some $u^2 = u \in M$, and thus $y = yu = h(x)u$, which implies $y \in R_I(h, x)$.

(2) implies (1). Let $y \in R_I(h, x)$ for some $h \in A(X)$. Then $y = h(x)u$. Since $yu = h(x)u^2 = h(x)u = y$, we have $(x, y) \in R$ and y is an almost periodic point.

3. Syndetically regular relations

The *syndetically proximal relation* of (X,T) [5] denoted by $L(X)$ or L is defined to be the set all $(x,y) \in X \times X$ such that if for every index α of X, there exists a syndetic subset A of T such that $t \in A$ implies $(xt, yt) \in \alpha$.

J. P. Clay [5] proved the following theorem.

THEOREM 3.1. *Let (X,T) be a transformation group. Then*

(1) *L is an invariant equivalence relation in X.*
(2) $P = LP \cup PL$.
(3) $L = \cup\{\overline{zT} : z \in X \times X, \overline{zT} \subset P\}$.

Now we generalize the syndetically proximal relations as follows.

DEFINITION 3.1. Let (X,T) be a transformation group, and let $x, y \in X$. Then x and y are said to be *syndetically regular* if there exists $h \in A(X)$ such that $h(x)$ and y are syndetically proximal. that is, $(h(x), y) \in L$. The set of all syndetically regular pairs is called *syndetically regular relation*, and is denoted by $S(X)$ or S.

Let $H \subset A(X)$. Then $S_H(X)$ is defined to be the set of all $(x,y) \in X \times X$ such that $(h(x), y) \in L$ for some $h \in H$. If $h = \{h\}$, then $S_{\{h\}}(X)$ is simply denoted S_h. Given S_h and S_k, S_hS_k is defined obviously.

REMARK. (1) If $A(X)$ is the trivial, then $S(X)$ coincides with $L(X)$.

(2) $S_{1_X}(X) = L(X)$ and $S_{A(X)}(X) = S(X)$.
(3) $L \subset S_H(X) \subset S(X)$ for subgroup H of $A(X)$.

LEMMA 3.1. *Let (X,T) be a transformation group. Then the following hold.*

(1) $S_h S_k = S_{hk}$ *for* $h, k \in A(X)$.

(2) $S_h^{-1} = S_{h^{-1}}$ *for* $h \in A(X)$.

(3) $LS_h = S_h L$ *for* $h \in A(X)$.

PROOF. (1) Let $(x,z) \in S_h S_k$. Then $(x,y) \in S_k$ and $(y,z) \in S_h$ for some $y \in X$. That is, $(k(x),y) \in L$ and $(h(y),z) \in L$. We also have $(hk(x),h(y)) \in L$. Since L is an equivalence relation, $(hk(x),z) \in L$, which shows that $(x,z) \in S_{hk}$.

Conversely, let $(x,z) \in S_{hk}$. Then $(hk(x),z) \in L$. Put $y = k(x)$. Then $(k(x),y) \in L$ and $(h(y),z) \in L$. That is, $(x,y) \in S_k$ and $(y,z) \in S_h$, which implies $(x, z) \in S_h S_k$.

(2) and (3) are verified similarly.

From Lemma 3.1 we obtain the following theorem.

THEOREM 3.2. *Let (X,T) be a transformation group and let $G(X) = \{S_h : h \in A(X)\}$. Then $G(X)$ forms a group.*

LEMMA 3.3. *Let H be a subgroup of $A(X)$. If $S_h S_k = S_{hk}$ fot $h, k \in H$, then $S_h(X)$ is transitive.*

PROOF. Let $(x,y) \in S_H$ and $(y,z) \in S_H$. Then $(x,y) \in S_k$ and $(y,z) \in S_h$ for some $h,k \in H$. Since $S_h S_k = S_{hk}$ and $hk \in H, (x,z) \in S_{hk} \subset S_H$. Therefore, $S_H(X)$ is transitive.

THEOREM 3.4. *Let (X,T) is a transformation group. Then $S_H(X)$ is an invariant equivalence relation for every subgroup H of $A(X)$.*

PROOF. Obviously $S_H(X)$ is invariant, reflexive and symmetric. Transitivity follows from Lemma 3.3.

COROLLARY 3.5. *Let (X,T) be a transformation group. Then the syndetically regular relation $S(X)$ is an invariant equivalence relation.*

REMARK. We can verify that $R(X) = R(X)S(X) \cup S(X)R(X)$ as the analogue of Theorem 1 [2].

Let (X,T) be a minimal transformation group, and let $(x,y) \in S_h \cap S_k$. Then $(h(x),y) \in L$ and $(k(x),y) \in L$. Since L is transitive, $(h(x),k(x)) \in L$. But $(h(x),k(x)) \in P$, because $L \subset P$. Therefore, there exists a net (t_α) in T such that $\lim h(x)t_\alpha = \lim k(x)t_\alpha$. We may assume $\lim xt_\alpha$ exists, and since h,k are continuous, $h(\lim xt_\alpha) = \lim(h(x)t_\alpha) = \lim(k(x)t_\alpha) = k(\lim xt_\alpha)$. From the minimality of X, we obtain $h = k$.

THEOREM 3.6. *Let (X,T) be a minimal transformation group. Then $A(X)$ and $G(X)$ are isomorphic.*

PROOF. Define a map $A(X) \longrightarrow G(X)$ by $h \longmapsto S_h$. This corespondence is clearly surjective and one-to-one from the preceeding paragraph, and Lemma 3.1 (1) shows that $A(X)$ and $G(X)$ are homomorphic.

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