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Fibrewise Hausdorff Convergence Spaces

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ABSTRACT. In this paper, we introduce T_0, T_1 and Hausdorff axioms in fibrewise convergence spaces as a generalization of fibrewise topological spaces and of convergence spaces. Furthermore we investigate some results about the fibrewise Hausdorff convergence space.

1. Preliminaries

In this section, we collect some known results of the fibrewise topology from I.M. James [2, 3] and the convergence space from E. Binz [1], which we shall need in later sections.

The fibrewise topological space X over B is *fibrewise* T_0 if whenever $x, y \in X_b$, where $b \in B$ and $x \neq y$, either there exists a neighborhood of x which does not contain y, or vice versa. In other words, X is fibrewise T_0 if and only if each fibre X_b is T_0. It is easy to check that subspaces of fibrewise T_0 spaces are fibrewise T_0, also the fibrewise topological product of fibrewise T_0 spaces is fibrewise T_0.

The fibrewise topological space X over B is *fibrewise* T_1 if whenever $x, y \in X_b$, where $b \in B$ and $x \neq y$, there exist a neighborhood U of x which does not contain y and a neighborhood V of y which does not contain x. Clearly, subspaces of fibrewise T_1 spaces are fibrewise T_1, also the fibrewise topological product of fibrewise T_1 spaces is fibrewise T_1. Moreover, every fibrewise T_1 space is fibrewise T_0.

The fibrewise topological space X over B is *fibrewise Hausdorff* if whenever $x, y \in X_b$, where $b \in B$ and $x \neq y$, there exist disjoint

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neighborhoods U, V of x, y in X. Clearly, subspaces of fibrewise Hausdorff spaces are fibrewise Hausdorff, also the fibrewise topological product of fibrewise Hausdorff spaces is fibrewise Hausdorff. Moreover, every fibrewise Hausdorff space is fibrewise T_1.

A convergence space X is said to be T_0 if $x \neq y$ implies either the ultrafilter $\dot{x}$ generated by the set $\{x\}$ does not converge to y, or vice versa.

A convergence space X is said to be T_1 if $x \neq y$ implies the ultrafilter $\dot{x}$ does not converge to y and the ultrafilter $\dot{y}$ does not converge to x.

A convergence space X is said to be *Hausdorff* if every filter on X converges to at most one point.

Now we will extend T_0, T_1 and Hausdorff axioms of convergence spaces to fibrewise convergence spaces, in a natural fashion. Specifically we aim to define, for a convergence space B, a property P_B of convergence spaces over B such that the following three conditions are satisfied. In that case the property P_B is said to be *well-behaved* [2].

CONDITION 1. If X, Y are isomorphic convergence spaces over B and if X has property P_B then so does Y.

CONDITION 2. A convergence space x has property P if and only if the convergence space X over the point $*$ has property P_*.

CONDITION 3. If a convergence space X over B has property P_B then the convergence space ζ^*X over B' has property $P_{B'}$ for each convergence space B' and continuous map $\zeta : B' \to B$, where ζ^*X is the convergence space $(B' \times_B X, pr_1)$ over B'.

It is convenient to refer to P_B as "*P over B*" or as "*fibrewise P*" according to circumstances.

2. Fibrewise T_0 and T_1 convergence spaces

DEFINITION. A convergence space X over B is to be *fibrewise* T_0 if $x \neq y$, where x and y belong to the same fibre, implies either the ultrafilter $\dot{x}$ does not converge to y or the ultrafilter $\dot{y}$ does not converge to x.

Then it is easy to show that the above definition is an extension of the definition of fibrewise topology i.e. for a fibrewise topological space X, X is fibrewise T_0 as a topological space if and only if it is fibrewise T_0 as a convergence space.

THEOREM 1. *The property "fibrewise T_0" is* well-behaved.

PROOF. (1) Suppose X, Y are isomorphic convergence spaces over B and Y is a fibrewise T_0 convergence space. Let $f : X \to Y$ be an isomorphism over B, and $x \neq y$, where x and y belong to the same fibre X_b. Then $f(x) \neq f(y)$ since f is a bijection. And $f(x)$, $f(y) \in f(X_b) \subset Y_b$. Since Y is fibrewise T_0, either the ultrafilter $\widehat{f(x)}$ generated by $\{f(x)\}$ does not converge to $f(y)$ or the ultrafilter $\widehat{f(y)}$ does not converge to $f(x)$. But $\widehat{f(x)} = f(\dot{x})$ and $\widehat{f(y)} = f(\dot{y})$. Thus either the filter $\dot{x}$ does not converge to y or the filter $\dot{y}$ does not converge to x, since f is a continuous map. Hence X is fibrewise T_0.

(2) Obvious.

(3) Suppose X is fibrewise T_0 as a convergence space. Let B' be a convergence space and $\zeta : B' \to B$ a continuous map. Note that for any $b' \in B'$, $(\zeta^* X)_{b'} = (B' \times_B X)_{b'} = \{b'\} \times X_{\zeta(b')}$. Let $(b', x) \neq (b', y)$, where (b', x), $(b', y) \in (\zeta^* X)_{b'}$. Then $x \neq y$ and x, $y \in pr_2((\zeta^* X)_{b'}) \subset X_{\zeta(b')}$. Since X is fibrewise T_0, either the filter $\dot{x}$ does not converge to y or the filter $\dot{y}$ does not converge to x. But $\dot{x} = pr_2(\widehat{(b', x)})$, $y = pr_2(\widehat{(b', y)})$. Thus either the filter $\widehat{(b', x)}$ does not converge to

(b', y) or the filter $\dot{\widehat{(b', y)}}$ does not converge to (b', x), since pr_2 is a continuous map. Hence $\zeta^* X$ is fibrewise T_0 as a convergence space.

Clearly B is fibrewise T_0 over itself, and each subspace of a fibrewise T_0 convergence space is fibrewise T_0. For a family $\{(X_i, p_i)\}_{i \in I}$ of fibrewise T_0 convergence spaces, the fibre product $\prod_B X_i$ is also fibrewise T_0.

DEFINITION. A convergence space X over B is said to be *fibrewise T_1* if $x \neq y$, where x and y belong to the same fibre, implies the ultrafilter $\dot{x}$ does not converge to y and the ultrafilter $\dot{y}$ does not converge to x.

Then it is easy to show that for a fibrewise topological space X, X is fibrewise T_1 as a topological space if and only if it is fibrewise T_1 as a convergence space. Similarly as in Theorem 1, we can show that the property "fibrewise T_1" is well-behaved. Clearly B is fibrewise T_1 over itself, and each subspace of a fibrewise T_1 convergence space is fibrewise T_1. Also every fibrewise T_1 convergence space is fibrewise T_0. For a family $\{(X_i, p_i)\}_{i \in I}$ of fibrewise T_1 convergence spaces, the fibre product $\prod_B X_i$ is also fibrewise T_1.

3. Fibrewise Hausdorff convergence spaces

DEFINITION. A convergence space X over B is said to be *fibrewise Hausdorff* if a filter $\mathcal{F}$ on X converges to x and y, where x and y belong to the same fibre, implies $x = y$.

THEOREM 2. *For a fibrewise topological space X over B, the following are equivalent:*

(1) *X is fibrewise Hausdorff as a topological space.*

(2) *X is fibrewise Hausdorff as a convergence space.*

PROOF. (1) $\Longrightarrow$ (2) Suppose that X is not fibrewise Hausdorff as a convergence space. Let $x \neq y$, where x and y belong to the same fibre. Then there exists a filter $\mathcal{F}$ on X such that $\mathcal{F}$ converges to x and y. And then $\mathcal{N}_x \subset \mathcal{F}$ and $\mathcal{N}_y \subset \mathcal{F}$. Thus there are no disjoint neighborhoods U, V of x, y in X, since $\mathcal{F}$ is a filter. Therefore X is not fibrewise Hausdorff as a topological space. Hence the result follows.

(2) $\Longrightarrow$ (1) Suppose that X is not fibrewise Hausdorff as a topological space. Let $x \neq y$, where x and y belong to the same fibre. Then for all neighborhood U, V of x and y in X, $U \cap V \neq \emptyset$. Let $\mathcal{F}$ be the filter generated by the set $U \cap V$. Then $\mathcal{N}_x \subset \mathcal{F}$ and $\mathcal{N}_y \subset \mathcal{F}$. Thus $\mathcal{F}$ converges to x and y. Hence X is not fibrewise Hausdorff as a convergence space. Hence the result follows.

THEOREM 3. *The property "fibrewise Hausdorff" is* well-behaved.

PROOF. (1) Suppose X, Y are isomorphic convergence spaces over B and Y is a fibrewise Hausdorff convergence space. Let $f : X \to Y$ be an isomorphism over B, and $x \neq y$, where x and y belong to the same fibre X_b. Then $f(x) \neq f(y)$ since f is a bijection. And $f(x)$, $f(y) \in f(X_b) \subset Y_b$. Suppose that there exists a filter $\mathcal{F}$ on X such that $\mathcal{F}$ converges to x and y. Since f is a continuous map, the filter $f(\mathcal{F})$ converges to $f(x)$ and $f(y)$. This is impossible, because Y is fibrewise Hausdorff. Thus there is no filter $\mathcal{F}$ on X which converges to x and y simultaneously. Hence X is fibrewise Hausdorff.

(2) Obvious.

(3) Suppose X is fibrewise Hausdorff as a convergence space. Let B' be a convergence space and $\zeta : B' \to B$ a continuous map. Note that for any $b' \in B'$, $(\zeta^*X)_{b'} = (B' \times_B X)_{b'} = \{b'\} \times X_{\zeta(b')}$. Let $(b', x) \neq (b', y)$, where (b', x), $(b', y) \in (\zeta^*X)_{b'}$. Then $x \neq y$ and x, $y \in pr_2((\zeta^*X)_{b'}) \subset X_{\zeta(b')}$. Suppose that there exists a filter $\mathcal{F}$ on ζ^*X such that $\mathcal{F}$ converges to (b', x) and (b', y). Since pr_2 is a continuous map,

the filter $pr_2(\mathcal{F})$ converges to x and y. This is impossible, because X is fibrewise Hausdorff. Thus there is no filter $\mathcal{F}$ on ζ^*X which converges to x and y. Hence ζ^*X is fibrewise Hausdorff as a convergence space.

Clearly B is fibrewise Hausdorff over itself, and each subspace of a fibrewise Hausdorff convergence space is fibrewise Hausdorff. Also every fibrewise Hausdorff convergence space is fibrewise T_1.

THEOREM 4. *Let $\{(X_i, p_i)\}_{i\in I}$ be a family of convergence spaces which are fibrewise Hausdorff. Then $\prod_B X_i$ is also fibrewise Hausdorff.*

PROOF. If $\prod_B X_i = \emptyset$, then it is obvious. Suppose $\prod_B X_i \neq \emptyset$. Let $(x_i) \neq (y_i)$, where (x_i) and (y_i) belong to the same fibre $(\prod_B X_i)_b = \bigcup_{b\in B}(\prod X_i)$. Then $x_j \neq y_j$ for some $j \in I$ and $x_j, y_i \in pr_j((\prod_B X_i)_b) \subset (X_j)_b$. Suppose that there exists a filter $\mathcal{F}$ on $\prod_B X_i$ such that $\mathcal{F}$ converges to (x_i) and (y_i). Since pr_j is a continuous map, the filter $pr_j(\mathcal{F})$ converges to x_j and y_j. This is impossible, because X_j is fibrewise Hausdorff. Thus there is no filter $\mathcal{F}$ on $\prod_B X_i$ which converges to x and y. Hence $\prod_B X_i$ is fibrewise Hausdorff.

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