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A Note on the Pettis Integral and the Bourgain Property

JONG SUL LIM, GWANG SIK EUN AND JU HAN YOON

ABSTRACT. In 1986, R. Huff [3] showed that a Dunford integrable function is Pettis integrable if and only if $T : X^* \to L_1(\mu)$ is weakly compact operator and $\{T(K(F, \varepsilon)) \mid F \subset X, F : \text{finite and } \varepsilon > 0\} = \{0\}$. In this paper, we introduce the notion of Bourgain property of real valued functions formulated by J. Bourgain [2]. We show that the class of pettis integrable functions is linear space and if $f$is bounded function with Bourgain property, then $T : X^{**} \to L_1(\mu)$ by $T(x^{**}) = x^{**}f$ is weak* — to — weak linear operator. Also, if operator $T : L_1(\mu) \to X^*$ with Bourgain property, then we show that f is Pettis representable.

I. Introduction

The theory of integration of function with values in a Banach space has long been a fruitful area of study. In the eight years from 1933 to 1940, seminal papers in this area were writen by Bochner, Gilfand, Pettis, Birhoff and Phillips. Out of this flourish of activity, two integrals have proved to be of lastion: the Bochner integral has had a thriving prosperous history. But unfortunately nearly forty years had passed untill 1976 without a significant improvement after B. J. Pettis integral had been achived during 1977–1990 by many authors [1, 3, 4, 5, 6, 7, 8].

In 1986, for a finite measure space (Ω, Σ, μ), R. Huff [3] proved that a Dunford integrable function $f : \Omega \to X$ is Pettis integrable if and only if $T : X^* \to L_1(\mu)$ is weakly compact operator and $\{T(K(F, \varepsilon)) \mid F \subset X, F : \text{finite, and } \varepsilon > 0\} = \{0\}$.

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In this paper, we introduce the notion of Bourgain property of real valued functions formulated by J. Bourgain [2] and Pettis Representable. We investigate some properties of Pettis integral and Pettis Representable by using the Bourgain property.

Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space with continuous dual space X^*. A function $f : \Omega \to X$ is called Dunford integrable provided the composition $T(x^*) = x^*f$ is in $L_1(\mu)$ for every x^* in X^*. In this case, it follows (from the closed graph theorem) that $T : X^* \to L_1(\mu)$ is a bounded linear operator. Hence, for every g in $L_\infty(\mu)$, the map ψ_g, defined by

$$\psi_g(x^*) = \int gT(x^*)\, d\mu,$$

is in X^{**}. In particular, for each E in Σ, $\nu(E) = \int_E f\, d\mu$, defined to equal ψ_E and called the Dunford integral of f over E, is an element of X^{**}. The function $\nu : \Sigma \to X^{**}$ is not necessarily countably additive. R. Huff [3] shown that ν is countably additive if and only if T is a weakly compact operator if and only if $\{x^*f : \|x^*\| \leq 1\}$ is uniformly integrable in $L_1(\mu)$. If f has bounded range, $\{x^*f : \|x\| \leq 1\}$ is uniformly integrable in $L_1(\mu)$. Hence these conditions are automatically satisfied. Let $\widehat{X}$ denote the natural image of X in X^{**}. The function f is said to be Pettis integrable if and only if for every E in Σ, $\nu(E)$ is in $\widehat{X}$.

DEFINITION 1.1. Let (Ω, Σ, μ) be a measure space. A family of real valued function of Ψ is called to be have the Bourgain property if the following condition is satisfied : for each set A of positive measure and for each $\alpha > 0$, there is a finite collection F of subsets of positive measure of A such that for each function f in Ψ, the inequality $\sup f(B) - \inf f(B) < \alpha$ holds for some member B of F.

The following theorem is found in [4], which is due to J. Bourgain [2], essentially allows us to do this for some functions.

THEOREM 1.2. *Let* (Ω, Σ, μ) *be a finite measure space and* Ψ *be a family of real valued function on* Ω *satisfying the Bourgain property. Then :*

(a) *The pontwise closure of* Ψ *satisfies the Bourgain property.*
(b) *Each element in the pointwise closure of* Ψ *is measurable.*
(c) *Each element in the pointwise closure of* Ψ *is the almost every where pointwise limit of a sequence from* Ψ.

DEFINITION 1.3. Let (Ω, Σ, μ) be measure space and let X be a Banach space. A function $f : \Omega \to X$ is called to have the Bourgain property if the family $\{x^*f : x^* \in X^*, \|x^*\| \leq 1\}$ has the Bourgain property.

REMARK. Let X be Banach space. The $f : \Omega \to X^*$ has the Bourgain property if and only if the family $\{xf : x \in X, \|x\| \leq 1\}$ has the Bourgain property.

II. Pettis integral

PROPOSITION 2.1. *Let* (Ω, Σ, μ) *be a finite measure space and* $f : \Omega \to X$ *is bounded weakly masurable function. Then* x^*f *is Dunford integrable.*

PROOF. Since x^*f is weakly measurable function, it suffices to show that $\int |x^*f|\, d\mu < \infty$. By assumption, $\int |x^*f|\, d\mu \leq \|x^*\|\, \|f\|_\infty \mu(\Omega) < \infty$, therefore $x^*f \in L_1(\mu)$. Hence x^*f is Dunford integrable.

PROPOSITION 2.2. *Let* (Ω, Σ, μ) *be finite measure space. If* f, $g : \Omega \to X$ *are Pettis integrable function. Then* $\alpha f + \beta g$ *is Pettis integrable,* $\int_E (\alpha f + \beta g)\, d\mu = \alpha \int_E f\, d\mu + \beta \int_E f\, d\mu$ *for each* E *and for all scalars* α, β.

PROOF. Let f, g be Pettis integrable. Then $x^* \int_E f\, d\mu = \int_E x^*f\, d\mu$, $x^* \int_E g\, d\mu = \int_E x^*g\, d\mu$, for each $E \in \Sigma$, for each $x^* \in X^*$, thus for each

$E \in \Sigma$, for each $x^* \in X^*$, $x^*(\alpha \int_E f\,d\mu + \beta \int_E g\,d\mu) = x^*(\int_E \alpha f\,d\mu) + x^*(\int_E \beta\,d\mu) = \int_E x^*(\alpha f)\,d\mu + \int_E x^*(\beta g)\,d\mu = \int_E x^*(\alpha f + \beta g)\,d\mu$. Hence $\alpha f + \beta g$ is Pettis integrable and $\int_E (\alpha f + \beta g)\,d\mu = \alpha \int_E f\,d\mu + \beta \int g\,d\mu$.

Thus the class of the Pettis integrable functions is linear space. The following theorem gives a sufficient condition for Pettis integrability. Its converse is not true in general. It is shown in example 14 of [4].

THEOREM 2.3. *Let (Ω, Σ, μ) be finite measure space and let X be a Banach space. If $f : \Omega \to X^*$ is a bounded function that has the Bourgain property. Then $f : \Omega \to X^*$ is Pettis integrable.*

PROOF. While no a priori hypothesis about the measurability of f is assumed, the Bourgain condition does show immediately that xf is measurable for each $x \in X$. Fix x^{**} in the unit ball of X^{**} and fix a set A, so that

$$x_A^*(x) = \int_A xf\,d\mu \quad \text{for all} \quad x \in X \tag{1}$$

We must show that $x^{**}f$ is measurable and that $x^{**}(x_A^*) = \int_A x^{**}f\,d\mu$. Accordingly, let $\alpha > 0$ and set

$$\Psi = \{xf : \|x\| \le 1, |(x^{**} - x)(x_A^*)| < \alpha\}.$$

Goldstine's theorem ensures that $x^{**}f$ lies in the pointwise closure of Ψ. Since Ψ has the Bourgain property, the function $x^{**}f$ is measurable by theorem 1.2. (b), (c) shows that $x^{**}f$ is the almost everywhere limit of sequence $x_n f$ from Ψ : that is,

$$\lim x_n f = x^{**}f \quad a.e. \tag{2}$$

where

$$|x^{**}(x_A^*) - x_A^*(x_n)| < \alpha \quad \text{for each} \quad n \tag{3}$$

It now follows from equations (1), (2), (3) and Dominated convergence theorem that $|x^{**}(x_A^*) - \int_A x^{**}f\,d\mu| \leq \alpha$. Since α is arbitrary, we conclude that x_A^* is Pettis integral of f over the set A. This completes the proof.

THEOREM 2.4. *Let (Ω, Σ, μ) be finite measure space and let $X = Y^*$, Y be a Banach space and let $f : \Omega \to X$ be a bounded function that has the Bourgain property. If $T : X^* \to L_1(\mu)$ is defined by $T(x^*) = x^*f$ for each x^* in X^*, then T is a weak* — to — weak linear operator and a weakly compact operator.*

PROOF. Let $f : \Omega \to X$ be bounded function with the Bourgain property. For each $x^* \in X^*$, x^*f is measurable function. For $g \in L_\infty(\mu)$, define $\langle T(x^*), g\rangle = \int gT(x^*)\,d\mu$. By theorem 2.3, f is Pettis integrable. Thus for each E, there exists x_E in X such that $x^*(x_E) = \int_E x^*f\,d\mu$ for each x^* in X^*. Thus $\langle T(x^*), g\rangle = \int_E T(x^*)\,d\mu = \int_E x^*f\,d\mu$ $= x^*(x_E)$. Hence $\langle T(x^*), g\rangle$ is weak*-continuous on X^* for any simple function g. By the boundedness of f, it follows that $\langle T(x^*), g\rangle$ is weak*-continuous on X^* for any $g \in L_\infty(\mu)$. Therefore T is weak* — to — weak continuous linear property. Also, it follows from theorem 2.3 that f is Pettis integrable. By Proposition 1 in [3], T is a weakly compact operator.

III. Pettis representable

In this chapter, we shall assume that (Ω, Σ, μ) is a finite separable measure space. This means that there is a sequence (π_n) of finite partitions of consisting of sets in Σ satsifying :

(a) each member of π_{n+1} is contained in a member of π_n (i.e., π_{n+1} refines π_n), and

(b) the union of the σ-algebras generated by the partitions π_n is dense in Σ.

For example, if $\Omega = [0,1]$ and μ is Lebesgue measure on the Borel sets, then the dyadic partitions of $[0,1]$ would satisfy these assumption. Let π_n denote the σ-algebra generated by Σ_n and let $\sigma = \bigcup_{n=1}^{\infty} \Sigma_n$.

Let $T : L_1(\mu) \to X^*$ be a bounded linear operator. Separability assumption that guarantees the existence of an increasing sequence (π_n) of finite patitions of Ω that generate the σ-algebra Σ. For each integer n define a function f_n from Ω into X^* by

$$f_n(\cdot) = \sum_{A \in \pi_n} \frac{T(\chi_A)}{\mu(A)} \chi_A(\cdot).$$

The sequence (f_n, Σ_n) forms a uniformly bounded X^* valued martigale. Moreover, for each x in the ball of X and for each set A in $\bigcup_{n=1}^{\infty} \Sigma_n$, we have $\lim \int_A x f_n \, d\mu = T(\chi_A)x$. Since the limit is eventually constant. Therefore $\lim \int x f_n g \, d\mu = T(g)x$ for each x in the ball of X and for each g in a dense set, and hence the limit exists for each g in $L_1(\mu)$.

We shall say that the operator T has the Bourgain property if the family $\{x f_n : n \in N, \|x\| \leq 1\}$ has the Bourgain property.

DEFINITION 3.1. Let (Ω, Σ, μ) be finite measure space and $T : L_1(\mu) \to X$ be a bounded linear operator. T is said to be Pettis representable if there exists a Pettis integrable function $g : \Omega \to X$ such that $x^*T(f) = \int f x^* g \, d\mu$ for every $f \in L_1(\mu)$ and for all $x^* \in X^*$. g is called the Pettis kernel of T.

THEOREM 3.2. *Let (Ω, Σ, μ) be finite separable measure space. If $T : L_1(\mu) \to X^*$ is a bounded linear operator with Bourgain property. Then T is Pettis representable.*

PROOF. Because the sequence (f_n) is pointwise uniformly bounded in X^*, we can choose a pointwise weak*-cluster point $f : \Omega \to X^*$ of

(f_n). Let x be in the unit ball of X. Because (xf_n) is a uniformly bounded real valued martingale, there is a function $h_X : \Omega \to R$ such that xf_n converges to h_X almost everywhere. However, the function xf is a pointwise cluster point of the sequence (xf_n) and therefore $xf = h_X$ almost everywhere. The Dominated convergence theroem ensures that $T(g)x = \lim \int xf_n g\, d\mu = \int xfg\, d\mu$ for all g in $L_1(\mu)$. Hence f is a Gel'fand derivative of T. On the other hand, the function f has the Bourgain property because the family $\{xf : \|x\| \leq 1\}$ lies in the pointwise closure of the family $\{xf : \|x\| \leq 1\}$, so that f is Pettis integrable by Lemma 5 in [5]. Therefore f is Pettis derivative of T and completes the proof.

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Department of Mathematics Education
Chungbuk National University
Cheongju, 360–763, Korea