JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 5, June 1992

Capillary-Gravity waves on the Interface of a Two Layer Fluid-Derivation of K-dV Equation with Higher Order Terms

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ABSTRACT. The objective of this paper is to study two dimensional waves on the interface between two immiscible, invicid and incompressible fluid bounded by two rigid varing boundaries when gravity and surface tension appear. By using unfied asymptotic method, a K-dV equation with higher order terms from which many model equations for the fluid domain can be obtained, is derived.

1. Introduction

In this paper, we consider two-dimensional flow of two immisible, invicid, and incompressible fluids of different constant densities when surface tension appears. We assume the fluid is bounded by two rigid boundaries which have variations.

Numerical studies of one-layer fluid past an obstruction were carried out by Forbes and Schwartz [1], Vanden-Broeck [2] and Forbes [3]. An asymptotic theory for small-amplitude steady flow over an obstruction were studied by Shen, Shen and Sun [4] and Shen [5]. In [4] and [5] they derived an equation called Forced K-dV equation and found new types of solutions for one layer fluid. For two layer fluid over an obstruction has been studied by Choi, Sun, and Shen [6], in which they derived Forced Modified K-dV equation and many results obtained before were recovered with new types of solutions.

Received by the editors on March 25, 1992.
1980 *Mathematics subject classifications*: Primary 76, 65.

For the problem considered here, we derive move general equation, K-dV equation with higher order terms, from which Forced K-dV and Forced Modified K-dV and other model equations can be derived.

2. Derivation of $K-dV$ Equations with higher order terms

The fluid domain $\Omega^{+*} \cup \Omega^{-*}$ is two dimensional with x^*-aixs to be horizontal direction and z^*-axis to be the vertical in the direction opposite to gravity. Ω^{*+} and Ω^{*-} are separated by the interface $z^* = \eta^*$ in the presence of the surface tension and bounded by two rigid boundaries $z^* = H^{*-}(x^*)$ and $z^* = H^{*-}(x^*)$ (Fig. 1)

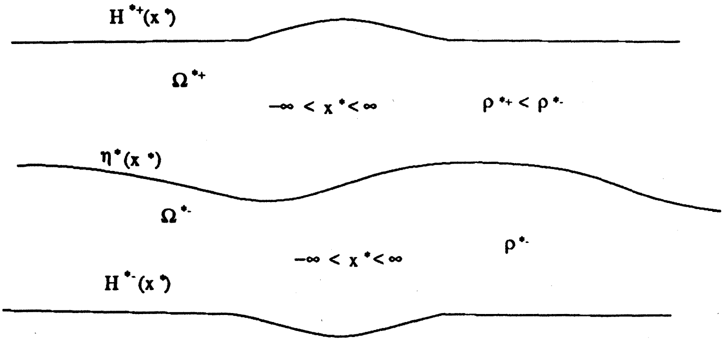

Fig. 1

Then the equation of motion and boundary conditions are :
In $\Omega^{*\pm}$,

(1) $$u^{*\pm}_{x^*} + w^{*\pm}_{z^*} = 0$$

(2) $$u^{*\pm}_{t^*} + u^{*\pm}u^{*\pm}_{x^*} + w^{*\pm}u^{*\pm}_{z^*} = -p^{*\pm}_{x^*}/\rho^{*\pm}$$

(3) $$w^{*\pm}_{t^*} + u^{*\pm}w^{*\pm}_{x^*} + w^{*\pm}w^{*\pm}_{z^*} = -p^{*\pm}_{z^*}/\rho^{*\pm} - g\ ;$$

at the interface $z^* = \eta^*$,

(4) $$\eta^*_{t^*} + u^{*\pm}\eta^*_{x^*} - w^{*\pm} = 0,$$

(5) $$p^{*+} - p^{*-} = T^*\eta^*_{x^*x^*}/(1 + (\eta^*_{x^*})^2)^{3/2}\ ;$$

at the rigid boundaries $z^* = H^{*\pm}(x^*)$,

$$(6) \qquad w^{*\pm} - u^{*\pm} H^{*\pm}_{x^*} = 0, \quad H^{*\pm}(x^*) = h^{*\pm} + b^{*\pm}(x^*)\varepsilon^3.$$

Here $(u^{*\pm}, w^{*\pm})$ are velocities, $\rho^{*\pm}$ are densities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant, T^* is the surface tension constant, and $\varepsilon > 0$ is a small parameter. Now we need to nondimensionalize equations (1) through (6) to solve them mathematically, and let L be the horizontal length scale and $H = |H^{*-}(-\infty)|$ be the vertical scale. Assume that $\varepsilon^{1/2} = H/L \ll 1$, which is so-called longwave assumption and we introduce the following nondimensional variables :

$$\begin{gathered} t = \varepsilon^{3/2}\sqrt{g/H}t^*, \quad \eta = \varepsilon^{-1}\eta^*/H, \\ p^\pm = p^{*\pm}/gH\rho^{*-*}, \quad (x, z) = (x^*/L, z^*/H), \\ (u, w) = \frac{1}{\sqrt{gH}}(u^*, \varepsilon^{-1/2}w^*), \quad T = T^*/\rho^{*-}gH^2, \\ \rho^\pm = \rho^{*\pm}/\rho^{*-}, \quad h_0^\pm = H^{*\pm}/H. \end{gathered}$$

In therms of them, (1) to (6) become :

In $\Omega^\pm$,

$$(7) \qquad u_x^\pm + w_z^\pm = 0$$

$$(8) \qquad \varepsilon u_t^\pm + u^\pm u_z^\pm + w^\pm u_z^\pm = -p_x^\pm/\rho^\pm$$

$$(9) \qquad \varepsilon^2 w_t^\pm + \varepsilon u^\pm w_z^\pm + \varepsilon w^\pm w_z^\pm = -p_z^\pm/\rho^\pm - 1 \,;$$

at the interface $z = \varepsilon\eta$,

$$(10) \qquad \varepsilon^2\eta_t + \varepsilon u^\pm \eta_x - w^\pm = 0$$

$$(11) \qquad p^+ - p^- = \varepsilon^2 T\eta_{xx}/(1 + \varepsilon^2\eta_x^2)^{3/2} \,;$$

at the rigid boundaries $z = h_0^\pm$

$$(12) \qquad w^\pm - u^\pm h_{0z}^\pm = 0,$$

where $\rho^- = 1$. Then we let $h_0^\pm(x) = h^\pm + \varepsilon^3 b^\pm(x)$ with the conditions that $h^\pm$ are constants so that $h^- = -1$ and $h = h^+$.

In the following, we use unified asymptotic method to derive the equation for $\eta(x)$. Assume that u, w, and p possess an asymptotic expansion of the form,

$$\phi(t,x,z) \sim \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots ,$$

and we use the condition

$$\varepsilon u^+ \eta_x - w^- = \varepsilon u^- \eta_x - w^- \quad \text{at} \quad z = \varepsilon\eta(x), \tag{13}$$

instead of (10). By substituting the asymptotic expansion of u, w, and p into (7) and (9) and (11) to (13) and comparing orders of ε, we obtain a sequence of equations and boundary conditions. The solutions for the zeroth approximation are given so that $u_0^\pm = u_0$ (constants), $w_0^\pm = 0$, and $p_0^\pm = -\rho^\pm z$. The equations for the first order approximation are :

$$u_{1x}^\pm + w_{1z}^\pm = 0, \tag{14}$$

$$u_0 u_{1x}^\pm = -p_{1x}^\pm / \rho^\pm, \tag{15}$$

$$p_{1z}^\pm = 0, \tag{16}$$

subject to the boundary conditions at $z = 0$,

$$w_1^+ - w_1^- = 0, \tag{17}$$

$$p_1^+ - p_1^- + \eta(p_{0z}^+ - p_{0z}^-) = 0, \tag{18}$$

at $z = h^\pm$,

$$w_1^\pm = 0. \tag{19}$$

From (16), we can see that $p_1^{\pm}$ are functions of x and t only. We express $p_1^{\pm} = f_1^{\pm}(x,t)$, and from (14), (15), and (19), it follows that

$$w_1^{\pm} = (z - h^{\pm}) f_{1x}^{\pm}(x,t)/u_0\rho^{\pm}. \tag{20}$$

From (17) and (18), and using the fact that $\eta|_{x=-\infty} = 0$, $u_i^{\pm}|_{x=-\infty} = \lambda_i$, $i = 1, 2$, we obtain

$$\begin{aligned} f_1^- &= (h^+(\rho^+ - \rho^0)/(-h^+ + h^-\rho^+/\rho^-))\eta, \\ p_1^{\pm} &= c_1^{\pm}\eta, \quad c_1^- = h^+(\rho^+ - \rho^0)/(-h^+ + h^-\rho^+/\rho^-), \\ &\qquad c_1^+ = c_1^- + (\rho^+ - \rho^-) \end{aligned} \tag{21}$$

$$u_1^{\pm} = -c_1^{\pm}\eta/u_0\rho^{\pm} + \lambda_1, \tag{22}$$

$$w_1^{\pm} = (z - h^{\pm}) c_1^{\pm}\eta_x/u_0\rho^{\pm}. \tag{23}$$

We use the same method to find $u_2^{\pm}$, $w_2^{\pm}$, $p_2^{\pm}$, $w_3^{\pm}$, $u_3^{\pm}$, and $p_3^{\pm}$ in terms of η, and we can derive an equation of η. As we can see in (17), we did not use the equations

$$\varepsilon^2\eta_t + \varepsilon u^{\pm}\eta_x - w^{\pm} = 0,$$

but only used

$$\varepsilon^2\eta_t + \varepsilon u^+\eta_x - w^+ = \varepsilon^2\eta_t + \varepsilon u^-\eta_x - w^-.$$

Hence, by substituting $u_0^{\pm}$, $w_0^{\pm}$, ..., $u_2^{\pm}$, $w_3^{\pm}$ in the kinematic condition :

$$\varepsilon^2\eta_t + \varepsilon u^-\eta_x - w^- = 0,$$

we obtain

(24)

$$\begin{aligned} &A_1\eta_t + A_2\eta_x + A_3\eta\eta_x + A_4\eta_{xxx} \\ &\quad + \varepsilon\Big(B_1\eta_x + B_2\eta\eta_x + B_3\eta^2\eta_x + B_4\eta_{xxxxx} \\ &\quad + B_5\eta_{xxt} + B_6\eta_x\eta_{xx} + B_7\eta\eta_{xxx} + B_8 b_x^- + B_9 B_x^+ \\ &\quad + B_{10}\int^x \eta_{tt}\,dx + B_{11}\eta\eta_t + B_{12}\eta_t + B_{13}\eta_x\int^x \eta_t\,dx\Big) \\ &\quad = O(\varepsilon^2), \end{aligned}$$

where

$$\begin{aligned}
A_1 &= 1 + h(1 - \rho(u_0^2(\rho + h)),\\
A_2 &= 1 + (1 - \rho)/(u_0^2(\rho + h)),\\
A_3 &= 2(1 - \rho)(2u_0^2(\rho + h)(\rho - h^2)\\
&\quad + (1 + h)(1 - \rho)(h^2 - \rho))/(u_0^3(\rho + h)^3),\\
A_4 &= (-1/u_0)(1 + \rho(h^2 u_0 - 1)/(3(\rho_h)u_0))\\
&\quad + (1/u_0)(hT/(\rho(\rho + h)))\\
B_1 &= (1 + h(1 - \rho).(u_0^2(\rho + h)))\lambda_2 - h(1 - \rho)\lambda_1^2/(u_0^3(\rho + h)),
\end{aligned}$$

B_2 through B_{13} are given explicitly in [7].

The equation (24) is called the K-dV equation with higher order term, from which one can derive model equations of the two layer fluid over an obstruction by changing the order of $\eta(x)$ and t. In particular, the case when $\rho \approx h^2$, $\lambda_1 \approx 0$, and the fluid flow is steady has been studied extensively in [6].

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