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On a New Equilibrium Existence Theorem

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ABSTRACT. The purpose of this note is to give a new existence theorem of equilibrium in non-compact generalized games with uncountable number of agents.

In 1952, Nash first proved the existence of equilibrium for games where the player's preferences are representable by continuous quasi-concave utilities and the strategy sets are simplexes. Next Debreu proved the existence of equilibrium for abstract economies. Recently, the existence of Nash equilibrium can be further generalized in more general settings by several authors, e.g. see [3, 6, 8, 9]. In fact, they imposed weaker conditions on the preference correspondences and constraint correspondences, and also they considered the consumer's set in more general spaces, e.g. in locally convex spaces or topological Riesz spaces. In 1990, Tian [8] proved an equilibrium existence theorem for non-compact abstract economy with countable number of agents; however his feasible set is assumed to be metrizable and the set of agent is countable. However, in recent papers (e.g. Ding-Kim-Tan [3], Kim [6]), the underlying spaces are not compact nor metrizable, e.g. the choice set lies in l_p or H^p $(0 < p < 1)$, and the set of agents is uncountable.

In this note, we shall prove a new equilibrium existence theorem for non-compact generalized games with uncountable number of agents with general preference correspondences which do not have open lower

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sections. We also give an example that the previous results due to Borglin-Keiding [2], Yannelis-Prabhakar [9], Ding-Kim-Tan [3] do not work; however our result can be applicable.

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by $cl\,A$ the closure of A in X. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. If A is a non-empty subset of a topological vector space X and S, $T : A \to 2^X$ are correspondences, then $co\,T$, $cl\,T$, $T \cap S : A \to 2^X$ are correspondences defined by $(co\,T)(x) = co\,T(x)$, $(cl\,T)(x) = cl\,T(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

Let X, Y be a non-empty topological spaces and $T : X \to 2^Y$ be a correspondence. A correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a correspondence $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, then there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.

Let I be a (possibly uncountable) set of agents. For each $i \in I$, let X_i be a non-empty set of actions. A *generalized game* (or *an abstract economy*) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i) where X_i is a non-empty topological vector space (a choice set), A_i, $B_i : \prod_{j \in I} X_j \to 2^{X_i}$ are constraint correspondence and $P_i : \prod_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\hat{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in cl\,B_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. Actually, an equilibrium point is both a fixed point of the correspondence $cl\,B_i$ and a separation point of the correspondence P_i and A_i. When $A_i = B_i$ for each $i \in I$, our definitions of an abstract economy (or generalized game) and an equilibrium coincide with the standard definitions, e.g. in Borglin-Keiding [2, p.315] or in Yannelis-Prabhakar [9, p.242].

Now we prove a new equilibrium existence theorem for a non-compact non-metrizable generalized game with proference correspondences which do not have open lower sections.

THEOREM. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be a generalized game where I is a (possibly uncountable) set of agents such that for each $i \in I$,*

(1) X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space and D_i is a non-empty compact subset of X_i,

(2) for each $x \in X = \prod_{i\in I} X_i$, $A_i(x)$ is non-empty, $A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is convex,

(3) the correspondence $cl\, B_i : X \to 2^{D_i}$ is upper semicontinuous,

(4) the correspondence $cl\, P_i : X \to 2^{X_i}$ is upper semicontinuous such that $P_i(x)$ is (possibly empty) convex for each $x \in X$,

(5) the set $W_i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is (possibly empty) closed,

(6) for each $x \in W_i$, $x_i \notin cl\, P_i(x)$.

Then Γ has an equilibrium choice $\hat{x} \in X$, i.e. for each $i \in I$,

$$\hat{x}_i \in cl\, B_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

PROOF. If $W_i = \emptyset$ for all $i \in I$, then by the assumption (3) and Lemma 3 of Fan [4], we can apply Himmelberg's fixed point theorem [5] to the correspondence $\prod_{i\in I} cl\, B_i$ and hence there exists a point $\hat{x} \in X$ such that $\hat{x} = \prod_{i\in I} cl\, B_i(\hat{x})$, i.e. for each $i \in I$, $\hat{x}_i \in cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. Therefore we have done.

Note that by Theorem 3.1.8 in [1], the correspondence $cl\, B_i \cap cl\, P_i$ is also upper semicontinuous. Suppose that I_0 be a maximal non-empty subset of I such that $W_i \neq \emptyset$ for every $i \in I_0$. Then for each $i \in I_0$, we can define a correspondence $\phi_i : X \to 2^{D_i}$ by

$$\phi_i(x) = \begin{cases} cl\, B_i(x), & \text{if } x \notin W_i, \\ (cl\, B_i \cap cl\, P_i)(x), & \text{if } x \in W_i, \end{cases}$$

Then each $\phi_i(x)$ is a non-empty closed convex subset of D_i. We shall show that ϕ_i is upper semicontinuous. Let V be any open subset of D_i containing $\phi_i(x)$. Then we have

$$\begin{aligned} U &= \{x \in X : \phi_i(x) \subset V\} \\ &= \{x \in W_i : \phi_i(x) \subset V\} \cup \{x \in X \setminus W_i : \phi_i(x) \subset V\} \\ &= \{x \in W_i : (cl\, B_i \cap cl\, P_i)(x) \subset V\} \cup \{x \in X \setminus W_i : cl\, B_i(x) \subset V\} \\ &= \{x \in X : (cl\, B_i \cap cl\, P_i)(x) \subset V\} \cup \{x \in X \setminus W_i : cl\, B_i(x) \subset V\}. \end{aligned}$$

Since $X \setminus W_i$ is open and $cl\, B_i \cap cl\, P_i$ is upper semicontinuous, U is open and hence ϕ_i is also upper semicontinuous.

Finally we define $\Psi : X \to 2^D$, where $D = \prod_{i \in I} D_i$, by

$$\Psi(x) := \prod_{i \in I} \psi_i(x), \quad \text{for each} \quad x \in X,$$

where

$$\psi_i = \begin{cases} \phi_i, & \text{if } i \in I_0, \\ cl\, B_i, & \text{if } i \notin I_0. \end{cases}$$

Then each $\Psi(x)$ is a non-empty closed convex subset of compact set D and Ψ is upper semicontinuous by Lemma 3 of Fan [4]. Therefore by applying Himmelberg's fixed point theorem [5], there exists a point $\hat{x} \in X$ such that $\hat{x} \in \Psi(x)$, i.e. for each $i \in I$, $\hat{x}_i \in \psi_i(\hat{x})$. For $i \in I_0$, $\hat{x}_i \in \psi_i(\hat{x})$. If $\hat{x} \in W_i$, then

$$\hat{x} \in \psi_i(\hat{x}) = cl\,(B_i \cap P_i)(\hat{x}) \subset cl\, P_i(\hat{x}),$$

which is a contradiction. Therefore for each $i \in I_0$, $\hat{x} \notin W_i$, i.e. $\hat{x}_i \in \psi_i(\hat{x}) = cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. Next, in case $i \notin I_0$, then $W_i = \emptyset$ and $\psi_i = cl\, B_i$. Therefore $\hat{x}_i \in \psi_i(\hat{x}) = cl\, B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. This completes the proof.

REMARK. In the assumption (5), W_i must be a proper subset of X. In fact, if $W_i = X$, then by applying Himmelberg's fixed point theorem to $\prod_{i \in I}(cl\, B_i \cap cl\, P_i)$, we can get a fixed point $\hat{x} \in \prod_{i \in I}(cl\, B_i \cap cl\, P_i)(\hat{x}) \subset \prod_{i \in I} cl\, P_i(\hat{x})$, which contradicts the assumption (6). When $A_i = B_i$ for each $i \in I$, Theorem is reduced to Theorem 1 in [7].

When $A_i = B_i$, $P_i(x) = \emptyset$ for each $i \in I$ and $x \in X$, we can obtain the following generalization of Himmelberg's fixed point theorem as a consequence.

COROLLARY. *Let I be a (possibly uncountable) index set. For each $i \in I$, let X_i be a non-empty convex subset of a locally convex Hausdorff topological vector space, D_i a non-empty compact subset of X_i and let $A_i : X = \prod_{i \in I} X_i \to 2^{D_i}$ be upper semicontinuous such that for each $x \in X$, $A_i(x)$ is a non-empty convex subset of D_i. Then there exists a point $\hat{x} \in X$ such that for each $i \in I$, $\hat{x}_i \in cl\, A_i(\hat{x})$.*

PROOF. By letting $P_i(x) = \emptyset$ for each $x \in X$ and $i \in I$, all hypotheses of Theorem are automatically satisfied. Therefore we obtain the conclusion.

Finally we give an example of a non-compact 1-person game where Theorem can be applicable but the previous results in Borglin-Keiding [2], Yannelis-Prabhakar [9], Tian [8] can not be applicable :

EXAMPLE. Let $X = [0, \infty)$ be the non-compact choice set and the preference correspondence P and the constraint correspondence

$A = B$ be defined as follows :

$$A(x) = B(x) := \begin{cases} [0, 2-x], & \text{if } x \in [0,2), \\ \{0\}, & \text{if } x \in [2,\infty), \end{cases}$$

$$P(x) := \begin{cases} [1 + x/2, 2), & \text{if } x \in [0,2), \\ [2, 8/3], & \text{if } x = 2, \\ \{x + 2/(1+x)\}, & \text{if } x \in (2,\infty). \end{cases}$$

Then the 1-person game (X, A, B, P) satisfies the whole assumptions of Theorem ; in fact, the set $W = [0, 2/3]$ is closed and the image of A is contained in a compact set $D = [0,2]$. Therefore, by Theorem, we can obtain an equilibrium point $\hat{x} = 1 \in X$ such that $\hat{x} \in cl\, B(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Here we note that Theorem 6.1 in Yannelis-Prabhakar [9], Theorem 2 in Tian [8] can not be applicable in this setting since the correspondences A, P and $A \cap P$ do not have open lower sections.

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