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Stability of Dynamical Systems

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In [12], E.C. Zeeman introduces the new definitions of equivalence and stability of diffeomorphisms and vector fields on compact oriented Riemannian C^∞-manifolds without boundary. The new definition has a number of advantages. He compare the new definition of stability with the classical notion of structural stability, and shows that neither implies the other. The results in his paper are only a beginning, and there are a number of open questions. In this paper we extend the results from C^∞ to C^r, $1 \le r \le \infty$.

1. Stability of Diffeomorphisms

Let M be a compact oriented Riemannian C^r-manifold, $1 \le r < \infty$, without boundary, $C(M)$ the Banach space of all continuous functions $f : M \to \mathbf{R}$ with the sup-norm

$$\|f\| = \sup\{|f(x)| : x \in M\},$$

and $U(r)$ the set of all C^r-functions $u : M \to \mathbf{R}$ such that $u \ge 0$ and $\int u = 1$. Then $U(r)$ is a convex affine subspace of $C(M)$.

Let $\theta : M \to M$ be a C^r-diffeomorphism. For $x \in M$ we define the symbol $J\theta(x)$ as Jacobian of θ at x if θ preserves orientation, and as the minus of Jacobian of θ at x if θ alters orientation.

Define a linear operator θ^* on $C(M)$ by

$$\theta^* f(x) = \frac{f(\theta^{-1}(x))}{J\theta(\theta^{-1}(x))}, \qquad f \in C(M), \quad x \in M.$$

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Lemma 1.. *θ^* maps $U(r)$ into $U(r-1)$.*

Proof. Let $u \in U(r)$. It is clear that $\theta^* u \in C^{r-1}(M)$ and $\theta^* u \geq 0$.

$$\begin{aligned} 1 &= \int u(y)\,dy \\ &= \int u(\theta^{-1}(x)) J\theta^{-1}(x)\,dx \quad (\text{putting } x = \theta(y)) \\ &= \int u(\theta^{-1}(x)) \frac{1}{J\theta(\theta^{-1}(x))}\,dx \\ &= \int \theta^* u(x)\,dx. \end{aligned}$$

Thus $\theta^* u \in U(r-1)$.

Given $\varepsilon > 0$, we define

$$K_\varepsilon(x,y) = k(y) \exp\left(-\frac{d(x,y)^2}{2\varepsilon}\right)$$

where, for each $y \in M$, the constant $k(y)$ is determined by

$$\int K_\varepsilon(x,y)\,dx = 1.$$

This is possible by the compactness of M. Thus

$$K_\varepsilon(\ \ ,y) \in U(r) \quad \text{and} \quad K_\varepsilon(x,\ \ ) \in C^r(M).$$

Define an operator S on $C(M)$ by

$$Sf(x) = \int K_\varepsilon(x,y) f(y)\,dy, \qquad f \in C(M), \quad x \in M.$$

LEMMA 2. *S maps $U(r-1)$ into $U(r)$.*

PROOF. Let $u \in U(r-1)$. It is clear that $Su \in C^r(M)$ and $Su \geq 0$.

$$\begin{aligned}\int Su(x)\,dx &= \iint K_\varepsilon(x,y)u(y)\,dydx \\ &= \iint K_\varepsilon(x,y)u(y)\,dxdy \\ &= \int u(y)\int K_\varepsilon(x,y)\,dxdy \\ &= \int u(y)\,dy \\ &= 1.\end{aligned}$$

Thus $Su \in U(r)$.

A relatively compact set in a topological space E is a subset A such that the closure $\overline{A}$ is compact.

An operator on a Banach space is said to be compact if it maps bounded subsets onto relatively compact sets. It is well known that an operator on a Banach space is compact if and only if it maps the unit sphere onto a relatively compact set [5].

Define an operator L on $C(M)$ by $L = S \circ \theta^*$. It is clear that L maps $U(r)$ into $U(r)$.

LEMMA 3. *L is a compact operator on $C(M)$.*

PROOF. Let F be the unit sphere in $C(M)$ given by

$$F = \{f \in C(M) : \|f\| = 1\}.$$

It suffices to show that LF is relatively compact in $C(M)$. By Ascoli's theorem ([3]) it suffices to prove that

(1) for all $x \in M$, $\{Lf(x) : f \in F\}$ is relatively compact in $\mathbf{R}$,
(2) LF is equicontinuous.

To prove (1) it suffices to show that $\{\|Lf\| : f \in F\}$ is bounded. Given $f \in F$,

$$\begin{aligned} Lf(x) &= \int K_\varepsilon(x,y)\theta^* f(y)\,dy \\ &= \int k(y)\exp\left(-\frac{d(x,y)^2}{2\varepsilon}\right)\frac{f(\theta^{-1}(y))}{J\theta(\theta^{-1}(y))}\,dy. \end{aligned}$$

Let $A = \min\{J\theta(x) : x \in M\} > 0$, $B = \int k(y)\,dy$. Then $|Lf(x)| \le B/A$.

To prove (2) we have to show that for each $x \in M$, $\eta > 0$, there exists $\delta > 0$ such that if $x' \in M$, $d(x,x') < \delta$, then $|Lf(x) - Lf(x')| < \eta$ for all $f \in F$. Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(r) = \exp(-r^2/2\varepsilon)$. It is easy to show that $|g'(r)| \le 1/\sqrt{e\varepsilon} \equiv C$. By the Mean Value theorem, $|g(r) - g(r')| \le C|r - r'|$ for all r, $r' \in \mathbf{R}$. Given x, $x' \in M$, let $r = d(x,y)$, $r' = d(x',y)$. Since $|r - r'| \le d(x,x')$, $|g(r) - g(r')| \le C|r - r'| \le Cd(x,x')$. Let $\delta = A\eta/BC > 0$. For all $f \in F$,

$$\begin{aligned} |Lf(x) = Lf(x')| &= \left|\int k(y)(g(r) - g(r'))\frac{f(\theta^{-1}(y))}{J\theta(\theta^{-1}(y))}\,dy\right| \\ &\le (BC/A)d(x,x') \\ &< (BC/A)\delta = \eta \text{ provided } d(x,x') < \delta. \end{aligned}$$

This completes the proof of Lemma 3.

The positive cone P in $C(M)$ is given by

$$P = \{f \in C(M) : f(x) \ge 0 \text{ for all } x \in M\}.$$

The interior of P is given by

$$\text{int}(P) = \{f \in C(M) : f(x) > 0 \text{ for all } x \in M\}.$$

An operator on $C(M)$ is called strongly positive if it maps $P - \{0\}$ into $\text{int}(P)$.

LEMMA 4. *L is strongly positive.*

PROOF. Let $f \geq 0$, $f \neq 0$. There exists $z \in M$ such that $f(z) > 0$. Let $y = \theta(z)$. Since $f(\theta^{-1}(y)) > 0$, $k(y)\exp\left(-\frac{d(x,y)^2}{2\varepsilon}\right)\frac{f(\theta^{-1}(y))}{J\theta(\theta^{-1}(y))} > 0$. Thus $Lf(x) > 0$.

Since L is a compact strongly positive operator on $C(M)$, by the Krein-Rutman theorem([7]), we get the following lemma.

LEMMA 5. *L has a positive real eigenvalue λ of maximum modulus, and $C(M)$ can be written as the sum of L-invariant subspaces*

$$C(M) = E + H$$

where

(1) *E is the eigenspace of λ,*
(2) *E has a dimension 1,*
(3) *E meets* $\text{int}(P)$,
(4) *H has codimension 1, and*
(5) *$L|H$ has spectral radius $< \lambda$.*

LEMMA 6. $\lambda = 1$

PROOF. Choose $f \in E \cap \text{int}(P)$. Let $a = \int f > 0$, $u = a^{-1}f$.

$$\int u = \int a^{-1} f = a^{-1} \int f = 1$$

Since $f \in E$, $a^{-1}f = u \in E$. Thus $Lu = \lambda u$. Since

$$Lu(x) = \int k(y)\exp\left(\frac{-d(x,y)^2}{2\varepsilon}\right)\frac{u(\theta^{-1}(y))}{J\theta(\theta^{-1}(y))}\,dy,$$

$Lu \in C^r(M)$. Since $u \in Lu/\lambda \in U(r)$, $Lu \in L(U(r)) \subset U(r)$. Thus

$$1 = \int Lu = \int \lambda u = \lambda \int u = \lambda.$$

LEMMA 7.

$$\begin{aligned} H &= \{f \in C(M) : L^n f \to 0 \text{ as } n \to \infty\} \\ &= \{f \in C(M) : \int f = 0\} \end{aligned}$$

PROOF. Let $L_1 = L|H$, and let ρ be the spectral radius of L_1. Then $\rho < 1$. We choose r, $\rho < r < 1$. Since

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|L_1^n\|} \qquad ([5])$$

there exists n_0 such that $\sqrt[n]{\|L_1^n\|} < r$ for all $n > n_0$. Thus $\|L_1^n\| < r^n$. If $f \in H$, then

$$\|L^n f\| = \|L_1^n f\| \leq \|L_1^n\| \, \|f\| \leq r^n \|f\|.$$

Thus $L^n f \to 0$ as $n \to \infty$.

Conversely suppose $f \in C(M)$ and $L^n f \to 0$ as $n \to \infty$. There exist $e \in E$, $h \in H$ such that $f = e + h$. Since $Le = e$,

$$e = L^n e = L^n(f - h) = L^n f - L^n h \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $e = 0$. Therefore $f = h \in H$, as required. This completes the proof of the first half of lemma 7, and we begin the proof of the second half.

Given $f \in C(M)$, then

$$\begin{aligned} \int Lf(x)\,dx &= \int S\theta^* f(x)\,dx \\ &= \iint K_\varepsilon(x,y)\theta^* f(y)\,dydx \\ &= \iint K_\varepsilon(x,y)\theta^* f(y)\,dxdy \end{aligned}$$

$$\begin{aligned}
&= \int \theta^* f(y) \int K_\varepsilon(x,y)\, dx dy \\
&= \int \theta^* f(y)\, dy \\
&= \int \frac{f(\theta^{-1}(y))}{J\theta(\theta^{-1}(y))}\, dy \\
&= \int \frac{f(x)}{J\theta(x)} J\theta(x)\, dx \quad \text{(putting } y = \theta(x)\text{))} \\
&= \int f(x)\, dx.
\end{aligned}$$

Thus $\int L^n f = \int f$. If $f \in H$, then $L^n f \to 0$ as $n \to \infty$ by above. Thus $\int f = \int L^n f \to 0$ as $n \to \infty$. Therefore $\int f = 0$.

Conversely suppose $f \in C(M)$ and $\int f = 0$. By lemma 5, we write $f = e + h$, $e \in E$, $h \in H$. Since E is one dimensional, by lemma 5, we can write $e = cu$ where u is the point constructed in lemma 6, and $c \in \mathbf{R}$. Then

$$c = c\int u = \int cu = \int e = \int (f - h) = \int f - \int h = 0$$

Thus $e = cu = 0$. Therefore $f = h \in H$, as required. This completes the proof of lemma 7.

THEOREM 1. (1) *$L|U(r)$ has a unique fixed point u.*

(2) *Any bounded subset of $U(r)$ converges uniformly to u.*

PROOF. (1) In the proof of lemma 6 we constructed a point $u \in E \cap U(r)$. u is a fixed point of $L|U(r)$ because $Lu = u$. Conversely let u' be a fixed point of $L|U(r)$. Then $u \in E$, because it is an eigenvector of the eigenvalue 1. There exists $b \in \mathbf{R}$ such that $u' = bu$. Thus $b = b\int u = \int bu = \int u' = 1$, because $u' \in U(r)$. Therefore $u' = u$, and so u is the unique fixed point of $L|U(r)$.

(2) Let G be a bounded subset of $U(r)$. We have to show for any $\varepsilon > 0$, there exists N such that if $n > N$, then $\|L^n g - u\| < \varepsilon$ for all

$g \in G$. Let $G' = \{g - u : g \in G\}$. Then G' is also bounded, by k say. If $g \in G$, then $g - u \in H$ by lemma 7 because $f(g-u) = \int g - \int u = 1 - 1 = 0$. Thus $G' \subset H$. Using r, n_0 as in the proof of lemma 7, we choose N so that $N > n_0$ and $r^N k < \varepsilon$. Then for all $n > N$, $g \in G$,

$$\begin{aligned}\|L^n g - u\| &= \|L^n g - L^n u\| = \|L^n(g-u)\| = \|L_1^n(g-u)\| \\ &\le \|L_1^n\|\,\|g-u\| \le r^n\|g-u\| \le r^n k < \varepsilon\end{aligned}$$

as required.

This unique fixed point u of $L|U(r)$ is called the steady state for θ, and denoted by $u(\theta, \varepsilon)$.

EXAMPLE 1. Lt $M = \mathbf{R}$, and let $\theta : M \to M$ be the linear contraction given by $x \to kx$, $0 < k < 1$. We shall show that the normal distribution N_σ is mapped by $N_\sigma \xrightarrow{L} N_{\sigma k^2+\varepsilon}$.

$$\begin{aligned}\theta^* N(x) &= \frac{N_\sigma(\theta^{-1}(x))}{J\theta(\theta^{-1}(x))} = \frac{1}{k\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma k^2}\right) = N_{\sigma k^2}(x) \\ k(y) &= \left(\int \exp\left(-\frac{(x-y)^2}{2\varepsilon}\right) dx\right)^{-1} = \frac{1}{\sqrt{2\pi\varepsilon}} \\ K_\varepsilon(x,y) &= \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{(y-x)^2}{2\varepsilon}\right)\end{aligned}$$

$$\begin{aligned}SN_{\sigma k^2}(x) &= \int \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{(y-x)^2}{2\varepsilon}\right) \frac{1}{k\sqrt{2\pi\varepsilon}} \exp\left(-\frac{x^2}{2\sigma k^2}\right) dy \\ &= \exp\left(-\frac{x^2}{2(\sigma k^2+\varepsilon)}\right) \int \frac{1}{2\pi k\sqrt{\varepsilon\sigma}} \exp\left(-\frac{\left(y - \frac{\sigma k^2}{\sigma k^2+\varepsilon}\right)^2}{\frac{2\varepsilon\sigma k^2}{\sigma k^2+\varepsilon}}\right) dy \\ &= \frac{1}{\sqrt{2\pi(\sigma k^2+\varepsilon)}} \exp\left(-\frac{x^2}{2(\sigma k^2+\varepsilon)}\right) = N_{\sigma k^2+\varepsilon}(x).\end{aligned}$$

Thus the fixed point of L is $u(\theta, \varepsilon) = N_\sigma$ where $\sigma = \sigma k^2 + \varepsilon$ therefore $\sigma = \varepsilon/(1-k^2)$.

We define two C^r-functions $f, g : M \to \mathbf{R}$ to be equivalent, written $f \sim g$, if there exist C^r-diffeomorphisms α of M, β of $\mathbf{R}$ such that $\beta \circ f = g \circ \alpha$. Given $\varepsilon > 0$, we define two C^r-diffeomorphisms θ, θ' of M to be ε-equivalent, written $\theta \overset{\varepsilon}{\sim} \theta'$, if $u(\theta, \varepsilon) \sim u(\theta', \varepsilon)$. We define a C^r-diffeomorphism θ of M to be ε-stable if it has a neighborhood U in the space $\mathrm{Diff}^r(M)$ of all C^r-diffeomorphisms of M with the C^r-topology such that $\theta \overset{\varepsilon}{\sim} \theta'$ for all $\theta' \in U$. We define a C^r-diffeomorphism θ of M to be stable if for any $\varepsilon > 0$, there exists ε', $0 < \varepsilon' < \varepsilon$, such that θ is ε'-stable.

2. Stability of Vector Fields

Given a C^r-vector field X on M, and given $\varepsilon > 0$, the Fokker-Planck equation for X with ε-diffusion is the partial differential equation on M

$$\partial u / \partial t = \varepsilon \Delta u - \nabla \cdot (uX)$$

where $u : M \times \mathbf{R}^+ \to \mathbf{R}$, $u \geq 0$ and $\int u(x,t)\, dx = 1$ for all $t \in \mathbf{R}^+$. Here the divergence operator ∇ and the Laplacian $\Delta = \nabla^2$ are determined by the Riemannian structure. Let $u^t \in U(r+1)$ be given by $u^t(x) = u(x,t)$. Now the Fokker-Planck equation can be regarded as an ordinary differential equation on $U(r)$, called the evolution equation, which, given an initial condition u^0, determines the forward orbit $\{u^t : t \geq 0\}$. Let $\Lambda^t : U(r) \to U(r)$ be the time-t map given by $u^0 \to u^t$. We can extend the evolution flow on $U(r)$ to a flow $\{f^t : t \geq 0\}$ on $C(M)$ for any initial condition $f^0 \in C(M)$. Notice that f^t is C^{r+1} for $t > 0$. The time-t map $\Lambda^t : C(M) \to C(M)$ given by $f^0 \to f^t$ is an operator on $C(M)$, and an affine map on the convex affine subspace $U(r)$.

EXAMPLE 2. Let $M = \mathbf{R}^n$, $X(x) = c + kx$, where c, k are constants, $c \in \mathbf{R}^n$. Let $\{u^t : t \geq 0\}$ be the solution of the Fokker-Planck equation with initial condition the delta function on M at the origin. Then

$$u^t(x) = (2\pi\sigma)^{-n/2} \exp(-|x-\mu|^2/2\sigma)$$

where $\sigma = \varepsilon(e^{2kt}-1)/k,$, $\mu = (e^{kt}-1)c$.

LEMMA 8. *For the flow u^t in example 2,*

(1) *there exists $A > 0$ such that $0 < u^t(x) \leq A$ for all $x \in M$,*
(2) *there exists $B > 0$ such that $|u^t(x) - u^t(x')| \leq B|x-x'|$ for all x, $x' \in M$.*

PROOF. (1) For all $x \in M$, $0 < u^t(x) \leq (2\pi\sigma)^{-n/2} \equiv A$.

(2) Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(r) = (2\pi\sigma)^{-n/2}\exp(-r^2/2\sigma)$. It is easy to show that $|g'(r)| \leq (2\pi)^{-n/2}e^{-1/2}\sigma^{-(n+1)/2} \equiv B$. By the Mean Value theorem, $|g(r)-g(r')| \leq B|r-r'|$ for all r, $r' \in \mathbf{R}$. Given x, $x' \in M$, let $r = |x-\mu|$, $r' = |x'-\mu|$. Since $|r-r'| \leq |x-x'|$,

$$|u^t(x) - u^t(x')| = |g(r)-g(r')| \leq B|r-r'| \leq B|x-x'|.$$

LEMMA 9. *Λ^t is a compact operator on $C(M)$ for $t > 0$.*

PROOF. Let $F = \{f \in C(M) : \|f\| = 1\}$. It suffices to show that $\Lambda^t F$ is relatively compact in $C(M)$. By Ascoli's theorem it suffices to prove that

(1) for all $x \in M$, $\{\Lambda^t f(x) : f \in F\}$ is relatively compact in $\mathbf{R}$.
(2) $\Lambda^t F$ is equicontinuous.

For $y \in M$, let δ_y be the delta function on M at y. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\int \Lambda^t\delta_y(x)f(y)\,dy &= \int \frac{\partial}{\partial t}\Lambda^t\delta_y(x)f(y)\,dy \\
&= \int(\varepsilon\Delta\Lambda^t\delta_y(x) - \nabla\cdot(\Lambda^t\delta_y(x)X(x))f(y)\,dy \\
&= \varepsilon\Delta\int \Lambda^t\delta_y(x)f(y)\,dy \\
&\quad - \nabla\cdot\left(\int \Lambda^t\delta_y(x)f(y)\,dyX(x)\right),
\end{aligned}$$

$$\begin{aligned}
\int \Lambda^0\delta_y(x)f(y)\,dy &= \int \delta_y(x)f(y)\,dy \\
&= f(x).
\end{aligned}$$

Thus $\Lambda^t f(x) = \int \Lambda^t\delta_y(x)f(y)\,dy$. Any vector field Y on $\mathbf{R}^n$ has linear approximation at the origin as in example 2, and by comparing Y with its approximation one can show that lemma 8 also holds for the flow u^t arising from Y. Furthermore since M can be locally approximated by a Euclidean space, lemma 8 will also hold for each $\Lambda^t\delta_y$. By compactness of M there exist A, B that will hold uniformly for $\Lambda^t\delta_y$, for all $y \in M$.

(1) For all $f \in F$, $|\Lambda^t f(x)| = |\int \Lambda^t\delta_y(x)f(y)\,dy| \leq AV$ where V is the volume of M.

(2) Given $x \in M$, $\eta > 0$, let $\delta = \eta/BV > 0$. If $f \in F$, $x' \in M$, $d(x, x') < \delta$, then

$$\begin{aligned}
|\Lambda^t f(x) - \Lambda^t f(x')| &= \left|\int(\Lambda^t\delta_y(x) - \Lambda^t\delta_y(x'))f(y)\,dy\right| \\
&\leq BVd(x, x') \\
&< BV\delta = \eta.
\end{aligned}$$

LEMMA 10. *For all $t > 0$, Λ^t is strongly positive.*

PROOF. Let $f \geq 0$, $f \neq 0$. There exists $z \in M$ such that $f(z) > 0$. For all $x \in M$, since $\Lambda^t\delta_y(x) > 0$, $\Lambda^t f(x) = \int \Lambda^t\delta_y(x)f(y)\,dy > 0$. Thus $\Lambda^t f > 0$. Therefore Λ^t maps $P - \{0\}$ into $\mathrm{int}(P)$, as required.

Since Λ^t is a compact strongly positive operator on $C(M)$, by the Krein-Rutman theorem, we get the following lemma.

LEMMA 11. *For all $t > 0$, Λ^t has a positive real eigenvalue λ of maximum modules, and $C(M)$ can be written as the sum of Λ^t-invariant subspaces $C(M) = E + H$ where*

(1) *E is the eigenspace of λ*
(2) *E has a dimension* 1,
(3) *E meets* $\operatorname{int}(P)$,
(4) *H has codimension* 1 *and*
(5) *$\Lambda^t|H$ has spectral radius $< \lambda$.*

LEMMA 12. $\lambda = 1$.

PROOF. We shall show that Λ^t maps $U(r)$ into $U(r)$. Given $u \in U(r)$,

$$\begin{aligned}\int \Lambda^t u(x)\,dx &= \iint \Lambda^t \delta_y(x)u(y)\,dydx\\ &= \iint \Lambda^t \delta_y(x)u(y)\,dxdy\\ &= \int u(y)\int \Lambda^t \delta_y(x)\,dxdy\\ &= \int u(y)\,dy\\ &= 1.\end{aligned}$$

Thus $\Lambda^t u \in U(r)$. Choose $f \in E \cap \operatorname{int}(P)$. Let $a = \int f > 0$, $u = a^{-1}f \in E$. Then $\int u = \int a^{-1}f = a^{-1}\int f = 1$, $\Lambda^t u = \lambda u$. Since $\Lambda^t u \in C^r(M)$, $u = \Lambda^t u/\lambda \in C^r(M)$. Thus $u \in U(r)$. Since $\lambda u = \Lambda^t u \in \Lambda^t(U(r)) \subset U(r)$, $\lambda = \lambda \int u = \int \lambda u = 1$.

LEMMA 13. $H = \{f \in C(M) : \Lambda^{nt} f \to 0 \text{ as } n \to \infty\} = \{f \in C(M) : \int f = 0\}$.

PROOF. Let $\Lambda_1 = \Lambda^t | H$, and let ρ be the spectral radius of Λ_1. Then $\rho < 1$. We choose r, $\rho < r < 1$. Since $\rho = \lim_{n\to\infty} \sqrt[n]{\|\Lambda_1^n\|}$, there exists n_0 such that $\sqrt[n]{\|\Lambda_1^n\|} < r$ for all $n > n_0$. Thus $\|\Lambda_1^n\| < r^n$. If $f \in H$, then $\|\Lambda^{nt} f\| = \|\Lambda_1^n f\| \leq \|\Lambda_1^n\| \, \|f\| \leq r^n \|f\|$. Thus $\Lambda^{nt} f \to 0$ as $n \to \infty$.

Conversely suppose $f \in C(M)$ and $\Lambda^{nt} f \to 0$ as $n \to \infty$. There exist $e \in E$, $h \in H$ such that $f = e + h$. Since $\Lambda^t e = e$,

$$e = \Lambda^{nt} e = \Lambda^{nt}(f - h) = \Lambda^{nt} f - \Lambda^{nt} h \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $e = 0$. Therefore $f = h \in H$, as required. This completes the proof of the first half of lemma 13, and we begin the proof of the second half.

Given $f \in C(M)$ then

$$\begin{aligned}\int \Lambda^t f(x)\, dx &= \iint \Lambda^t \delta_y(x) f(y)\, dy dx \\ &= \iint \Lambda^t \delta_y(x) f(y)\, dx dy \\ &= \int f(y) \int \Lambda^t \delta_y(x)\, dx dy \\ &= \int f(y)\, dy.\end{aligned}$$

Thus $\int \Lambda^{nt} f = \int f$. If $f \in H$, then $\Lambda^{nt} f \to 0$ as $n \to \infty$ by above. Thus $\int f = \int \Lambda^{nt} f \to 0$ as $n \to \infty$. Therefore $\int f = 0$.

Conversely suppose $f \in C(M)$ and $\int f = 0$. By lemma 11, we write $f = e + h$, $e \in E$, $h \in H$. Since E is one dimensional, by lemma 11 we can write $e = cu$ where u is the point constructed in lemma 12, and $c \in \mathbf{R}$. Then $c = c \int u = \int cu = \int e = \int (f - h) = \int f - \int h = 0$. Thus $e = cu = 0$. Therefore $f = h \in H$, as required. This completes the proof of lemma 13.

LEMMA 14. (1) *For all* $t > 0$, $\Lambda^t|U(r)$ *has a unique fixed point* u. (2) *any bounded subset of* $U(r)$ *converges uniformly to* u.

PROOF. (1) In the proof of lemma 12 we constructed a point $u \in E \cap U(r)$. u is a fixed point of $\Lambda^t|U(r)$ because $\Lambda^t u = u$. Conversely let u' be a fixed point of $\Lambda^t|U(r)$. Then $u' \in E$, because it is an eigenvector of the eigenvalue 1. There exist $b \in \mathbf{R}$ such that $u' = bu$. Thus $b = b\int u = \int bu = \int u' = 1$, because $u' \in U(r)$. Therefore $u' = u$, and so u is the unique fixed point of $\Lambda^t|U(r)$.

(2) Let G be a bounded subset of $U(r)$. We have to show for any $\varepsilon > 0$, there exists N such that if $n > N$ then $\|\Lambda^{nt} g - u\| < \varepsilon$ for all $g \in G$. Let $G' = \{g - u : g \in G\}$. Then G' is also bounded, by k say. If $g \in G$, then $g - u \in H$ by lemma 13 because $\int(g-u) = \int g - \int u = 1 - 1 = 0$. Thus $G' \subset H$. Using r, n_0 as in the proof of lemma 13, we choose N so that $N > n_0$ and $r^N k < \varepsilon$. Then for all $n > N$, $g \in G$,

$$\begin{aligned}\|\Lambda^{nt} g - u\| &= \|\Lambda^{nt} g - \Lambda^{nt} u\| = \|\Lambda^{nt}(g-u)\| \\ &= \|\Lambda_1^n (g-u)\| \le \|\Lambda_1^n\| \, \|g-u\| \\ &\le r^n \|g-u\| \le r^n k < \varepsilon\end{aligned}$$

as required.

This unique fixed point u of $\Lambda^t|U(r)$ is called the steady state solution of the Fokker-Planck equation for X with ε-diffusion given by $\partial u/\partial t = 0$, and denoted by $u(X, \varepsilon)$.

THEOREM 2. *Let* X *be a* C^r*-vector field on a compact oriented Riemannian* C^r*-manifold* M *without boundary, and let* $\varepsilon > 0$. *Then the Fokker-Planck equation for* X *with* ε*-diffusion has a unique steady state solution, and all solutions tend to that steady state solution.*

PROOF. Choose $\tau > 0$, and let $\{u^t : t \ge 0\}$ be any solution in $U(r)$. By compactness of M, $\{u^t : 0 \le t \le \tau\}$ is a bounded subset of

$U(r)$, and therefore converges uniformly under Λ^τ to the fixed point u by lemma 14. Therefore for each $\varepsilon > 0$ there exists N such that if $n \geq N$, $0 \leq t \leq \tau$, then $\|(\Lambda^\tau)^n u^t - u\| < \varepsilon$. But $(\Lambda^\tau)^n u^t = u^{n\tau+t}$. Let $T = n\tau$. Therefore we have shown for each $\varepsilon > 0$ there exists T such that if $t \geq T$, then $\|u^t - u\| < \varepsilon$. In other words $u^t \to u$ as $t \to \infty$, as required.

EXAMPLE 3. For any manifold M, let $X = -\nabla f$, where $f : M \to \mathbf{R}$. Then the steady state solution of the Fokker-Planck equation is given by $u = ke^{-f/\varepsilon}$ where the constant k is determined by $\int u = 1$. It is easy to show that this is a steady state because $\varepsilon\nabla u = -u\nabla f = uX$. Therefore $\varepsilon\Delta u - \nabla \cdot (uX) = \nabla \cdot (\varepsilon\nabla u - uX) = 0$.

Given $\varepsilon > 0$, we define two C^r-vector fields X, Y on M to be ε-equivalent if $u(X, \varepsilon) \sim u(Y, \varepsilon)$. We define a C^r-vector field to be ε-stable if it has a neighborhood of ε-equivalents in the space $V(r)$ of all C^r-vector field X on M with the c^r-topology. We define a c^r-vector field x on M to be stable if for any $\varepsilon > 0$, there exists ε', $0 < \varepsilon' < \varepsilon$, such that X is ε'-stable.

3. Density of Stable Vector Fields

In this section we prove that stable vector fields are dense in $V(r)$. In theorem 2 in the last section we constructed, for each $\varepsilon > 0$, a map $\pi^\varepsilon : V(r) \to U(r+1)$ by assigning to each $X \in V(r)$ the steady state $\pi^\varepsilon(X) = u(X, \varepsilon)$ of the Fokker-Planck equation for X with ε-diffusion. Let

$$G = \{X \in V(r) : X = -\nabla f \text{ for some } f \in C^{r+1}(M)\},$$
$$W = \{X \in V(r) : \nabla \cdot X = 0\}.$$

Let $\pi_1 : G \times W \to G$ denote projection onto the first factor.

Lemma 15. *For all $\varepsilon > 0$, there exist homeomorphisms $\pi^\varepsilon|G$, φ^ε, $\bar\pi^\varepsilon$, $\bar\varphi^\varepsilon$ such that the following diagram commutes :*

$$\begin{array}{ccc} G \times W & \underset{\bar\phi^\varepsilon}{\overset{\bar\pi^\varepsilon}{\rightleftarrows}} & V(r) \\ \pi_1 & & \pi^\varepsilon \\ G & \underset{\phi^\varepsilon}{\overset{\pi^\varepsilon|G}{\rightleftarrows}} & U(r+1) \end{array}$$

Proof. For convenience we shall omit the superscript ε from π^ε, $\bar\pi^\varepsilon$, φ^ε, $\bar\varphi^\varepsilon$ throughout this proof. We shall show that for each $X \in G$ there exists uniquely $f \in C^{r+1}(M)$ such that $X = -\nabla f$, $\int f = 1$. Given $X \in G$, there exists $g \in C^{r+1}(M)$ such that $X = -\nabla g$. Let $k = \left(\int \exp(-g/\varepsilon)\right)^{-1}$, $f = g - \varepsilon \log k$. Then $X = -\nabla f$, $\int \exp(-f/\varepsilon) = 1$. Suppose f, $f' \in C^{r+1}(M)$, $X = -\nabla f = -\nabla f'$, $\int \exp(-f/\varepsilon) = \int \exp(-f'/\varepsilon) = 1$. Since $\nabla(f - f') = \nabla f - \nabla f' = -X + X = 0$, $f = f' + c$ for some $c \in \mathbf{R}$. Since $1 = \int \exp(-f/\varepsilon) = \int \exp(-f'/\varepsilon - c/2) = \exp(-c/2) \int \exp(-f'/\varepsilon) = \exp(-c/\varepsilon)$, $c = 0$. Thus $f = f'$. By example 3, $\pi(X) = \exp(-f/\varepsilon)$. Conversely given $u \in U(r+1)$, we define

$$\varphi(u) = (\varepsilon/u)\nabla u = \varepsilon \nabla \log u$$

which is continuous in u. Then $\varphi(\pi|G) = 1$ because

$$\varphi\pi(X) = \varphi(\exp(-f/\varepsilon)) = \varepsilon\nabla \log \exp(-f/\varepsilon) = -\nabla f = X.$$

Hence $\pi|G$ and φ are homeomorphisms. Given $X \in G$ and $Y \in W$, define $\bar\pi(X, Y) = X + Y/u$ where $u = \pi(X)$, which is continuous. Conversely given $X \in V(r)$, define $\bar\varphi(X) = ((\varepsilon/u)\nabla u,\ uX - \varepsilon\nabla X)$ where $u = \pi(X)$, which is continuous. We shall show that if $u = \pi(X)$, then $\pi(X + Y/u) = u$ for all $Y \in W$.

$$\begin{aligned}\nabla \cdot (u(X + Y/u)) &= \nabla \cdot (uX) + \nabla \cdot Y \\ &= \nabla \cdot (uX) \\ &= \varepsilon \Delta u.\end{aligned}$$

Thus $\pi(X + Y/u) = u$, by the uniqueness of theorem 2, $\bar{\varphi}\bar{\pi} = 1$ because

$$\begin{aligned}\bar{\varphi}\bar{\pi}(X, Y) &= \bar{\varphi}(X + Y/u) \text{ where } u = \pi(X) = \pi(X + Y/u) \\ &= ((\varepsilon/u)\nabla u, u(X + Y/u) - \varepsilon \nabla u) \\ &= (X, Y) \text{ since } X = \varphi(u) = \varepsilon \nabla u/u.\end{aligned}$$

$\bar{\pi}\bar{\varphi} = 1$ because

$$\begin{aligned}\bar{\pi}\bar{\varphi}(X, Y) &= \bar{\pi}((\varepsilon/u)\nabla u, uX - \varepsilon \nabla u) \text{ where } u = \pi(X) \\ &= (\varepsilon/u)\nabla u + (uX - \varepsilon \nabla u)/u \\ &= X.\end{aligned}$$

Hence $\bar{\pi}$ and $\bar{\varphi}$ are homeomorphisms.

Finally $\pi\bar{\pi} = (\pi|G)\pi_1$ because

$$\begin{aligned}\pi\bar{\pi}(X, Y) &= \pi(X + Y/u) \text{ where } u = \pi(X) \\ &= u \\ &= \pi(X) \\ &= (\pi|G)\pi_1(X, Y),\end{aligned}$$

and $\pi_1\bar{\varphi} = \varphi\pi$ because

$$\begin{aligned}\pi_1\bar{\varphi}(X) &= \pi_1((\varepsilon/u)\nabla u, uX - \varepsilon \nabla u) \text{ where } u = \pi(X) \\ &= (\varepsilon/u)\nabla u \\ &= \varphi(u) \\ &= \varphi\pi(X).\end{aligned}$$

This completes the proof of lemma 15.

COROLLARY. *π^ε is open.*

PROOF. $\pi^\varepsilon|G$ and $\bar{\varphi}^\varepsilon$ are open because they are homeomorphisms. π_1 is open because it is a projection. Thus $\pi^\varepsilon = (\pi^\varepsilon|G)\pi_1\bar{\varphi}^\varepsilon$ is open, because it is the composition of three open maps.

LEMMA 16. *For all $\varepsilon > 0$, ε-stable vector fields are open dense in $V(r)$.*

PROOF. Let U_0 be the set of stable functions in $U(r)$. By Thom-Mather theory ([10]), U_0 is dense in $U(r)$. It is easy to show that U_0 is open. Let V^ε be the inverse image of U_0 under π^ε. Then V^ε is open. Thus any $X \in V^\varepsilon$ is ε-stable. Conversely if $X \in V(r)$ is ε-stable then since π^ε is open $\pi^\varepsilon(X)$ must be a stable function, and hence in U_0. Therefore V^ε is the set of ε-stable vector fields. Finally V^ε is dense in $V(r)$ because U_0 is dense in $U(r)$ and π^ε is open.

THEOREM 3. *Stable vector fields are residual and therefore dense in $V(r)$.*

PROOF. Let $W = \bigcap_{n=1}^{\infty} V^{1/n}$. Then any vector field in W is $1/n$-stable for all n, and hence stable. Now W is residual in $V(r)$ because it is a countable intersection of open dense subsets, and $V(r)$ is a Baire space because it is complete metrizable ([1]).

Therefore W is dense in $V(r)$, and since the set of stable vector fields contains W, the stables must also be dense.

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