JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 5, June 1992

Lower and Upper Bounds to Frequencies of Rotating Uniform Beams

GYOU-BONG LEE

ABSTRACT. A new method(EVF method) is applied to get lower bounds to frequencies of rotating uniform beams which are clamped or simply supported at one end and free at the other. For the upper bounds, the Rayleigh-Ritz method is employed. Numerical results are presented.

1. Introduction

It is important to compute accurately the vibration frequencies of elastic structures to analyze their structural development. However they are not explicitly known in most cases, and thus sevral approximation methods have been presented over many years. For precise error estimation in approximation, one may be to use two methods that give upper and lower bounds, respectively, to the frequencies considered. In their analysis, we meet equations of the style $Au = \lambda u$ in Ω, where A is considered as a semi-bounded self-adjoint operator on a Hilbert space, having eigenvalues of finite multiplicity below the lowest limit point of the spectrum.

The most popular method for obtaining numerical upper bounds is the *Rayleigh-Ritz method*([6]). for which we can easily compute their approximations. In contrast, a method finding accurate lower bounds, which is called the *method of intermediate eigenvalue problems*, is much more difficult. However the combination of upper and

Received by the editors on May 22, 1992.
1980 *Mathematics subject classifications*: Primary 65D; Secondary 65N.

lower bounds produces frequencies estimates as well as a measure of the accuracy of these estimates. In a method of intermediate eigenvalue problems, it requires practically not only explicit knowledge of reducing spaces and spectrum of the base operator but also special choices for the range space of the approximating finite rank operators. This makes the resulting matrix problem have dense coefficient and Gram matrices so that it may be difficult to be handled on available computational resources.

The so-called *eigenvector free method*(EVF) which has been developed by Beattie and Goerisch([1]) may overcome such problems since it does not need information of eigenvectors of the base problem and permits the effective use of finite-element trial functions so that it yields final computational matrices which are sparse and well-structured.

In this paper we introduce the EVF method and compute lower and upper bounds to frequencies of rotating uniform beams clamped or simply supported at one end and free at the other. The goal of this paper is to illustrate how to use this new technique to obtain lower bounds which may be applicable to many varieties of eigenvalue problems.

2. On the EVF method of Beattie and Goerisch

This section has a brief description of the EVF method to calculate lower bounds to eigenvalues of self-adjoint operators on a separable Hilbert space $\mathcal{H}$ with norm $\|u\|$ and inner product $\langle u, v\rangle$.

Let A be a self adjoint operator with domain Dom(A) dense in $\mathcal{H}$ which is bounded below, and let the lower part of its spectrum consist of a finite or infinite number of isolated eigenvalues

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_\infty$$

each having finite multiplicity. Here λ_∞ denote the lowest limit point of the spectrum of A. We denote $a(u)$ by the closure of the quadratic form $\langle Au, u\rangle$ with a domain $\mathrm{Dom}(a) \supset \mathrm{Dom}(A)$. We assume that a self-adjoint operator A_0 is picked so that A_0 is bounded below and $A_0 \leq A$, and that the isolated eigenvalues of A_0

$$\lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_\infty^0$$

are computable to arbitrary precision. The closure of the quadratic form $\langle A_0 u, u\rangle$ is denoted by $a_0(u)$. Then $a_0(u) \leq a(u)$ for all $u \in \mathrm{Dom}(a) \subset \mathrm{Dom}(a_0)$.

We assume that the quadratic form $a(u)$ is decomposed as

$$a(u) = a_0(u) + \langle Tu, Tu\rangle_*$$

where T is a closed operator on $\mathcal{H}$ to another Hilbert space $\mathcal{H}_*$ with $A = A_0 + T^*T$. We take vectors $\{p_j\}_{j=1}^k \subset \mathrm{Dom}(T^*) \subseteq \mathcal{H}_*$ and $\{q_i\}_{i=1}^n \subset \mathrm{Dom}(A_0)$, and define the matrices as

$$F_1 = [\langle q_i, (A_0 - \mu)q_i\rangle], \quad F_2 = [\langle p_i, q_j\rangle_*], \quad H = [\langle (A_0 - \mu)q_i, T^* p_j\rangle],$$
$$G_1 = [\langle (A_0 - \mu)q_i, (A_0 - \mu)q_j\rangle] \quad \text{and} \quad G_2 = [\langle T^* p_i, T^* p_j\rangle],$$

for any i, $j = 1, \ldots, n$ or k. The following method then comes from [1].

THEOREM (Beattie and Goerisch). *Let μ and r be chosen so that $\lambda_{r-1}^0 < \mu \leq \lambda_r^0$. Suppose that $\{p_i\}_{i=1}^k \subset \mathrm{Dom}(T^*)$ and $\{q_i\}_{i=1}^n \subset \mathrm{Dom}(A_0)$ such that $\{(A_0 - \mu)q_i\}_{i=1}^n$ and $\{T^* p_i\}_{i=1}^k$ are jointly linearly independent. If the generalized matrix eigenvalue problem*

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \zeta \begin{bmatrix} G_1 & H \\ H^* & G_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.1}$$

has discrete finite eigenvalues ordered as

$$\zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_\ell < 0 \leq \zeta_{\ell+1} \leq \cdots$$

($\ell = 0$ if all discrete eigenvalues are either nonnegative or infinite), then for each eigenvalue ζ_p with $p \le \ell$ we have a corresponding lower bound for an eigenvalue of A,

$$\mu + \frac{1}{\zeta_p} \le \lambda_{r-p}. \tag{2.2}$$

Notice that the number of negative eigenvalues of (2.1) is less than r because the Gram matrix and F_2 are positive definite and because the number of eigenvalues of A_0 less than μ is at most $r-1$. So we can compute lower bounds to the first $r-1$ eigenvalues of A. Based on this theorem, the procedures of EVF method are obtained as following;

EIGENVECTOR-FREE METHOD([1]).

(1) *Take trial functions* $\{q_i\}_{i=1}^n \subset \mathrm{Dom}(A_0)$ *and* $\{p_j\}_{j=1}^k \subset \mathrm{Dom}(T^*)$.
(2) *Pick a value* $\mu \in (\lambda_{r-1}^0, \lambda_r^0]$ *for a selected* $r > 1$.
(3) *Solve the matrix eigenvalue problem defined by* (2.1).
(4) *The finite negative eigenvalues of the matrix problem* (2.1) *may each be associated with eigenvalue bounds as given in* (2.2).

We note that if $\{q_i\}_{i=1}^n$ and $\{p_j\}_{j=1}^k$ are chosen to have local support as with finite-element trial functions, then the resulting matrices will be sparse and the matrix eigenvalue problem may be efficiently handled using sparse techniques, even for quite large values of n and k.

3. Procedures for bounds to frequencies of beams

We first consider the free vibration of a uniform rotating beam clamped at one end and free at the other. This may be modeled by the differential equation (cf. [4])

$$EI\frac{d^4u}{dz^4} - \frac{mC\Omega^2}{2}\frac{d}{dz}(\ell^2 - z^2)\frac{du}{dz} - 4\pi^2 f^2 mCu = 0, \qquad 0 < z < \ell$$

with boundary conditions

$$u(0) = \frac{du}{dz}(0) = \frac{d^2u}{dz^2}(\ell) = \frac{d^3u}{dz^3}(\ell) = 0,$$

where u is the transverse displacement of the beam, I is the moment of inertia of the cross section about the principal axis in the plane of rotation, E is the modulus of elasticity, m is mass per unit volume, ℓ is the length of the beam, C is the cross-sectional area, Ω is the angular velocity of rotation, and f is the natural frequency. This problem has been treated previously with other methods in [2, 4].

For convenience, we introduce the nondimensional variable $x = \frac{z}{\ell}$ and write the above differential equation as an eigenvalue problem,

$$\frac{d^4u}{dx^4} - \frac{a^2}{2}\frac{d}{dx}(1-x^2)\frac{du}{dx} = \lambda u, \qquad 0 < x < 1 \tag{3.1}$$

with boundary conditions

$$u(0) = \frac{du}{dx}(0) = \frac{d^2u}{dx^2}(1) = \frac{d^3u}{dx^3}(1) = 0$$

Here the parameter a^2 is proportional to the angular velocity of rotation, $a^2 = \frac{mC\ell^4\Omega^2}{EI}$, and the eigenvalue λ is related to the natural frequency f by $\frac{4\pi^2 mC\ell^4 f^2}{EI}$. We denote by A the differential operator of the equation (3.1) and by $\mathrm{Dom}(A)$ its domain, i.e.,

$$Au = \frac{d^4u}{dx^4} - \frac{a^2}{2}\frac{d}{dx}(1-x^2)\frac{du}{dx}$$

and

$$\mathrm{Dom}(A) = \{u \in H^4(0,1) \mid u(0) = \frac{du}{dx}(0) = \frac{d^2u}{dx^2}(1) = \frac{d^3u}{dx^3}(1) = 0\}.$$

The quadratic form associated with the operator A is given by

$$a(u) = \int_0^1 \left(\left|\frac{d^2u}{dx^2}\right|^2 + \frac{a^2}{2}(1-x^2)\left|\frac{du}{dx}\right|^2 \right) dx$$

with boundary conditions $u(0) = \frac{du}{dx}(0) = 0$. If we take the base operator A_0 as

$$A_0 u = -\frac{a^2}{2}\frac{d}{dx}(1-x^2)\frac{du}{dx}$$

with boundary conditions $u(0) = \lim_{x\to\Gamma}(1-x)\frac{du}{dx} = 0$, and the perturbation operator T as

$$Tu = -\frac{d^2u}{dx^2}$$

with boundary conditions $u(0) = \frac{du}{dx}(0) = 0$, then the quadratic forms associated with the operators A and A_0 are

$$a(u) = a_0(u) + \langle Tu, Tu\rangle, \quad a_0(u) = \frac{a^2}{2}\int_0^1 (1-x^2)\left|\frac{du}{dx}\right|^2 dx$$

with $\mathrm{Dom}(a_0) = \{u \in H^1(0,1) \mid u(0) = 0\}$, and the adjoint operator of T is obtained as

$$T^* u = -\frac{d^2u}{dx^2}$$

with $\mathrm{Dom}(T^*) = \{u \in H^2(0,1) \mid u(1) = \frac{du}{dx}(1) = 0\}$. The eigenvalues of A_0 with the boundary conditions are easily found by

$$\lambda_\ell^0 = a^2\ell(2\ell-1), \qquad \text{for} \quad \ell = 1,2,3,\dots .$$

It follows that for a given function $u \in \mathrm{Dom}(a)$, the quadratic forms a and a_0 satisfy the inequality

$$a_0(u) \le a(u).$$

The eigenvalues associated with these quadratic forms thus satisfy the inequality

$$\lambda_\nu^0 \le \lambda_\nu, \qquad \text{for} \quad \nu = 1,2,3,\dots .$$

Define a uniform mesh on $[0,1]$ with a mesh size $h = \frac{1}{N}$. Furthermore, define cubic spline functions on this mesh, $B_i(x)$, centered at $x_i = ih$ for $i = -1, 0, 1, \dots, N+1$ so that

$$B_i(x_i) = 4, \quad B_i(x_{i\pm1}) = 1 \quad \text{and} \quad B_i(x_{i\pm2}) = 0.$$

In order to take the projecting vectors $\{q_i\}$ and $\{p_j\}$ within $\mathrm{Dom}(A_0)$ and $\mathrm{Dom}(T^*)$ respectively, we define them by

$$q_1 = B_0 - 4B_{-1}, \quad q_2 = B_0 - 4B_1, \quad q_i = B_{i-1}, \quad \text{for } i = 3, \dots, N+2$$

and

$$p_j = B_{j-2}, \quad \text{for } j = 1, \dots, N, \quad p_{N+1} = B_{N-1} - \frac{1}{2}B_N + B_{N+1}.$$

This provides an $(2N+3)$-th order problem. Since the order is dependent only on the mesh size, the eigenvalue estimates will be denoted by $\lambda_\nu^{(N)}$. The elements of matrices F_1, F_2, G_1, G_2 and H of (2.3) are given by the inner products:

$$\begin{aligned}
F_1^{ij} &= \frac{a^2}{2}\left(\int_0^1 q_i' \cdot q_j'\, dx - \int_0^1 q_i' \cdot x^2 q_j'\, dx\right) - \mu \int_0^1 q_i \cdot q_j\, dx, \\
F_2^{ij} &= \int_0^1 p_i \cdot p_j\, dx, \\
G_1^{ij} &= \frac{a^4}{4}\left(\int_0^1 q_i'' \cdot q_j''\, dx - 2\int_0^1 q_i'' \cdot x^2 q_j''\, dx + \int_0^1 q_i'' \cdot x^4 q_j''\, dx\right. \\
&\quad - 2\int_0^1 q_i'' \cdot x q_j'\, dx + 2\int_0^1 q_i'' \cdot x^3 q_j'\, dx \\
&\quad \left. -2\int_0^1 q_i' \cdot x q_j''\, dx + 2\int_0^1 q_i' \cdot x^3 q_j''\, dx + 4\int_0^1 q_i' \cdot x^2 q_j'\, dx\right) \\
&\quad - a^2\mu\left(\int_0^1 q_i' \cdot q_j'\, dx - \int_0^1 q_i' \cdot x^2 q_j'\, dx\right) + \mu^2 \int_0^1 q_i \cdot q_j\, dx, \\
G_2^{ij} &= \int_0^1 p_i'' \cdot p_j''\, dx, \\
H^{ij} &= \frac{a^2}{2}\left(\int_0^1 q_i'' \cdot p_j''\, dx - 2\int_0^1 x q_i' \cdot p_j''\, dx - \int_0^1 x^2 q_i'' \cdot p_j''\, dx\right) \\
&\quad - \mu \int_0^1 q_i' \cdot p_j'\, dx.
\end{aligned}$$

For the upper bounds the basis functions are chosen as

$$\phi_1 = B_{-1} - \frac{1}{2}B_0 + B_1 \quad \text{and} \quad \phi_i = B_i, \quad \text{for } i = 2, \ldots, N+1$$

to satisfy the boundary conditions. Upper bounds to the eigenvalues of the rotating beam are obtained as the eigenvalues λ of $(N+1)$-st order symmetric generalized algebraic eigenvalue problem,

$$(\langle A_0\phi_i, \phi_j\rangle + \langle T\phi_i, T\phi_j\rangle)x = \lambda(\langle \phi_i, \phi_j\rangle)x$$

for i, $j = 1, \ldots, N+1$. Here

$$\langle A_0\phi_i, \phi_j\rangle = \frac{a^2}{2}\left(\int_0^1 \phi_i' \cdot \phi_j'\, dx - \int_0^1 \phi_i' \cdot x^2\phi_j'\, dx\right),$$

$$\langle T\phi_i, T\phi_j\rangle = \int_0^1 \phi_i'' \cdot \phi_j''\, dx \quad \text{and} \quad \langle \phi_i, \phi_j\rangle = \int_0^1 \phi_i \cdot \phi_j\, dx.$$

The result is contained in Table 1. Here the upper bounds come from Rayleigh-Ritz problem of $N = 200$.

Table 1. Clamped Beam Problem

$a^2 = 200$ $\mu = 65000$ $r = 13$

N	λ_1	λ_2	λ_3	λ_4	λ_5
Base	200.000000	1200.00000	3000.00000	5600.00000	9000.000000
40	233.793442	1771.61203	7305.16111	21716.2501	51978.7670
70	233.793442	1771.61204	7305.16292	21716.3288	51978.8794
100	233.793442	1771.61205	7305.16308	21716.3359	51978.2470
130	233.793442	1771.61205	7305.16311	21716.3373	51978.3221
Ritz	233.793442	1771.61205	7305.16315	21716.3383	51978.3639

$a^2 = 10000$ $\mu = 910000$ $r = 7$

N	λ_1	λ_2	λ_3
Base	100.000000	60000.0000	150000.000
40	10215.0332	61582.3109	159783.051
70	10215.0885	61582.6118	159783.980
100	10215.0905	61582.6228	159784.017
130	10215.0907	61582.6242	159784.022
Ritz	10215.0930	61582.6373	159784.059

N	λ_4	λ_5	λ_6
Base	280000.000	450000.000	660000.000
40	319654.833	557153.539	884797.692
70	319658.096	557170.293	885609.127
100	319658.267	557171.431	885677.091
130	319658.298	557171.643	885690.672
Ritz	319658.387	557171.897	885697.824

Next we consider the free vibration of a uniform rotating beam simply supported at one end and free at the other. The differential equation governing this problem is the same as the clamped case but with a different boundary condition

$$u(0) = \frac{d^2u}{dx^2}(0) = \frac{d^2u}{dx^2}(1) = \frac{d^3u}{dx^3}(1) = 0.$$

Then the base operator A_0 and its domain are the same as in the clamped case, but

$$\mathrm{Dom}(T^*) = \{u \in H^2(0,1) \mid u(0) = u(1) = \frac{du}{dx}(1) = 0\}.$$

The projecting vectors $\{q_i\}$ are the same as those of the previous case, but the vectors p_j are defined as

$$p_1 = B_0 - 4B_{-1}, \qquad p_2 = B_1 - 4B_{-1}$$

and

$$p_i = B_{i-1}, \quad \text{for } i = 3, \ldots, N-1, \quad p_N = B_{N-1} - \frac{1}{2}B_N + B_{N+1}.$$

This yields $(2N+2)$-order eigenvalue problem.

For the upper bounds the trial functions are chosen to be

$$\phi_1 = B_0 - 4B_{-1}, \quad \phi_2 = B_0 - 4B_1, \quad \phi_i = B_{i-1}, \quad \text{for } i = 3, \ldots, N+2$$

in order to satisfy boundary conditions. Table 2 contains the result. Here the upper bounds come from Rayleigh-Ritz problem of $N = 200$.

Table 2. Simply Supported Beam Problem

$a^2 = 5$ $\mu = 2805$ $r = 17$

N	λ_1	λ_2	λ_3
Base	5.00000000	30.00000000	75.0000000
40	5.00000000	269.67028572	1853.3382756
70	5.00000000	269.67028886	2585.9300079
100	5.00000000	269.67028915	2585.9333458
130	5.00000000	269.67028921	2585.9340304
Ritz	5.00000013	269.67028923	2585.9344046

$a^2 = 500$ $\mu = 138000$ $r = 12$

N	λ_1	λ_2	λ_3
Base	500.0000	3000.000000	7500.0000
40	500.0000	3316.362162	10958.2998
70	500.0000	3316.362184	10958.3014
100	500.0000	3316.362185	10958.3015
130	500.0000	3316.362186	10958.3016
Ritz	500.0000	3316.362186	10958.3016

N	λ_4	λ_5	λ_6
Base	14000.0000	22500.0000	33000.000
40	28159.2102	61332.9650	119134.256
70	28159.2528	61333.7617	119162.440
100	28159.2567	61333.8332	119164.951
130	28159.2575	61333.8478	119165.464
Ritz	28159.2580	61333.8567	119165.745

ACKNOWLEDGEMENT. *I would like to thank Professor Christopher A. Beattie for his guidance.*

REFERENCES

[1] C. Beattie and F. Goerisch, *Extensions of Weinstein-Aronszajn and Lehmann-Maehly Methods for Computing Lower Bounds to Eigenvalues of Self-adjoint operator*, ICAM Technical Report 90-09-01, (September, 1990), Virginia Polytechnic Institute and State University, Blacksburg.

[2] D. H. Hodges, *An Approximate Formular for the Fundamental Frequency of a Uniform Rotating Beam Clamped off the Axis of Rotation*, J. Sound and Vibration **77**(1) (1981), 1–18.

[3] G. Lee, *A Study of the Computation and Convergence Behavior of Eigenvalue Bounds for Self-Adjoint Operator*, Thesis, VPI & SU, 1991.

[4] N. Rubinstein and J. T. Stader, *Bounds to bending frequencies of a rotating beam*, J. Franklin Institute v. **294** No. 4 (1972).

[5] J. Weidmann, *Linear Operators in Hilbert Space*, Springer-Verlag, New York, 1980.

[6] A. Weinstein and W. Stenger, *Methods of Intermediate Problems for Eigenvalues, Theory and Ramificationss*, Academic Press, New York, 1972.

Department of Mathematics
Keonyang University
Nonsan, 320-800, Korea