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Sets of Complete Continuity

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ABSTRACT. In this paper, we study some properties of sets of complete continuity.

Moreover, we prove that if the subsets C_1 and C_2 of a Banach space X are sets of complete continuity, then so is the set $C_1 \times C_2$ in the product space $X \times X$.

Throughout this paper, X is a real Banach space. A subset C of X is called a *set of complete continuity* if for every finite measure space (Ω, Σ, μ) and every bounded linear operator $S : L_1(\mu) \to X$ for which $S(\chi_E/\mu(E))$ belongs to C for each non-null measurable set E, the operator S is a Dunford-Pettis operator.

A bounded linear operator $S : L_1(\mu) \to X$ is a *Dunford-Pettis operator* if S sends weakly compact sets into norm compact sets.

The *average range* of a vector measure $F : \Sigma \to X$ is defined to be the set $\{F(E)/\mu(E) : E \in \Sigma, \mu(E) > 0\}$.

It is well-known that any one of the following statements about an operator $S : L_1(\mu) \to X$ implies all the others [4].

(1) The operator S is a Dunford-Pettis operator.

(2) The restriction of S to $L_1(\mu)$ defines a compact operator from $L_1(\mu)$ to X.

(3) The vector measure F defined for $E \in \Sigma$ by $F(E) = S(\chi_E)$ has a relatively norm compact range.

We get the following theorem from the above statements.

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Theorem 1. *Let C be a norm closed absolutely convex bounded subset of X.*

Then C is a set of complete continuity if and only if for each finite measure space (Ω, Σ, μ) and each μ-continuous vector measure $F : \Sigma \to X$ of bounded variation with average range contained in C, the measure F has a relatively norm compact range.

Proof. Suppose that for each finite measure space (Ω, Σ, μ) and each μ-continuous vector measure F of bounded variation with average range contained in C, F has a relatively norm compact range.

Let $S : L_1(\mu) \to X$ be a bounded linear operator with $S(\chi_E/\mu(E)) \in C$ for each $E \in \Sigma$. Define the vector measure F by $F(E) = S(\chi_E)$. Then it is easy to show that F is a μ-continuous vector measure of bounded variation with $F(E)/\mu(E) \in C$ for each E in Σ. By the assumption, F has a relatively norm compact range and hence S is a Dunford-Pettis operator. This implies that C is a set of complete continuity.

Conversely, suppose that C is a set of complete continuity. Let F be any μ-continuous vector measure of bounded variation with average range contained in C. Let $S : L_1(\mu) \to X$ be a bounded linear operator with $S(\chi_E) = F(E)$.

Since $S(\chi_E/\mu(E)) \in C$, S is a Dunford-Pettis operator. Hence F has a relatively norm compact range.

Theorem 2. *Let C_1 and C_2 be subsets of X and let $C_1 \subset C_2$. If C_2 is a set of complete continuity, then so is the set C_1.*

Proof. Let (Ω, Σ, μ) be a finite measure space and let $S : L_1(\mu) \to X$ be any bounded linear operator such that $S(\chi_E/\mu(E)) \in C_1$ for each non-null measurable set E. Since $S(\chi_E/\mu(E)) \in C_2$ and C_2 is a set of complete continuity, S is a Dunford-Pettis operator. Hence C_1 is a set of complete continuity.

THEOREM 3. *Let C_1 and C_2 be subsets of X. If C_1 and C_2 are sets of complete continuity, then so is the set $C_1 \times C_2$ in the product space $X \times X$.*

PROOF. Let (Ω, Σ, μ) be a finite measure space and let $S : L_1(\mu) \to X \times X$ be any bounded linear operator for which $S(\chi_E/\mu(E)) \in C_1 \times C_2$ for each non-null measurable set E. Then $S_1 = P_1 \circ S : L_1(\mu) \to X$ and $S_2 = P_2 \circ S : L_1(\mu) \to X$ are bounded linear operators such that $S_1(\chi_E/\mu(E)) \in C_1$ and $S_2(\chi_E/\mu(E)) \in C_2$, where P_1 and P_2 are projections from $X \times X$ to X defined by $P_1(x, y) = x$, $P_2(x, y) = y$, respectively.

Since C_1 and C_2 are sets of complete continuity, S_1 and S_2 are Dunford-Pettis operators. If W is a weakly compact subset of L_1, then $S(W)$ is a relatively norm compact set since $S_1(W)$ and $S_2(W)$ are relatively norm compact sets. Hence S is a Dunford-Pettis operator. This implies that $C_1 \times C_2$ is a set of complete continuity.

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