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Radicals of fixed subrings under Jordan automorphisms

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ABSTRACT. Let R be an associative ring and let G be a finite group of Jordan automorphisms of R. Let R^G be the set of elements in R fixed by all $g \in G$.

In this paper we will study the relationship between the Levitzki radical of R^G and R as that a Jordan ring. We also show that if R is a P.I. algebra, then the algebraicity of R^G implies the algebraicity of R.

Let R be an associative ring. By an automorphism of R, we will mean an ordinary automorphism of R as an associative ring. We let $\text{Aut}(R)$ denote the group of all automorphisms of R. If A is an additive subgroup of R, A is a (quadratic) Jordan subring of R if A is closed under squares (that is, $x^2 \in A$ if $x \in A$) and under the quadratic operator $xU_y = yxy$. Any Jordan subring A necessarily satisfies

$$\text{(J)} \qquad xy + yx \in A \quad \text{whenever} \quad x, y \in A.$$

If R has no 2-torsion(i.e. $2a = 0$ implies $a = 0$ for every $a \in R$), then the additive subgroup A with the condition (J) is a Jordan subring.

A mapping $\phi : R \to R'$ of rings R and R' is a Jordan homomorphism if ϕ preserves the structure of R as a Jordan ring; that is, ϕ is additive, $\phi(x^2) = \phi(x)^2$ all $x \in R$, and $\phi(xyx) = \phi(x)\phi(y)\phi(x)$, all x, $y \in R$. A Jordan automorphism of R is simply a Jordan homomorphism which is also one to one and onto; we let $\text{Aut}_J(R)$ denote

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the group of all Jordan automorphisms of R. Let G be a subgroup of $\mathrm{Aut}_J(R)$. For $g \in G$ and $r \in R$, r^g means the image of r under g. The fixed ring of R under G is $R^G = \{r \in R \mid r^g = r \text{ for all } g \in G\}$. Clearly R^G is a Jordan subring of R.

Now say that G is finite with $|G| = n$. For $x \in R$, the trace of x is $tr_G(x) = \sum_{g \in G} x^g$. If there is no ambiguity about which group is involved, we simply write $tr_G(x) = tr(x)$. Note that $tr(x) \in R^G$.

A mapping $* : R \to R$ is called an involution if (1) $a^{**} = a$, (2) $(a+b)^* = a^* + b^*$, (3) $(ab)^* = b^* a^*$ for all $a, b \in R$.

When R has an involution $*$, and $G = \{e, *\}$ where e is the identity of G, we say that G is generated by involution $*$. In this situation, $R^G = \{x \in G \mid x^* = x\} = S_R$, the symmetric elements in R.

If I is an ideal of R, we say that I is G-invariant if $I^g \leq I$, for all $g \in G$. When I is G-invariant, $R = R/I$ has an induced group of automorphisms, given as follows: for $g \in G$, define $\bar{g}$ by $(x + I)^{\bar{g}} = x^g + I$.

Let K be the kernel of the mapping $g \to \bar{g}$, and let $\overline{G} = G/K$. Then $\overline{G}$ is a group of automorphisms of R. Clearly $R^{\bar{G}} \subseteq \overline{R}^{\bar{G}}$, where $R^{\bar{G}}$ denotes the image of R^G in $\overline{R}$.

THEOREM 1. *Let $\phi : R \to R'$ be a Jordan homomorphism of R onto a prime ring R'. Then ϕ is either a homomorphism or an anti-homomorphism* [5].

COROLLARY 2. *Let ϕ be a Jordan automorphism of R and let P be a prime ideal of R. Then P^ϕ is a prime ideal of R. Moreover, the prime ring R/P and R/P^ϕ are either isomorphic or anti-isomorphic* [8].

COROLLARY 3. *Let R be a prime ring, and G a group of Jordan automorphisms of R. Let H be the subgroup of G consisting of all* *automorphisms. If $H \neq G$, then $[G:H]=2$. Moreover, G/H induces an involution $*$ on the associative ring R^H as followings:*

Choose $g \in G$, $g \notin H$, and let $x^* = x^g$, for any $x \in R^H$. The involution is independent of the choice of g, and the set of symmetric elements S_{R^H} of R^H under $*$ *is precisely the set R^G.*[8]

The Levitzki radical of R, which we shall denote by $L(R)$ is defined as the sum of all locally nilpotent ideals of R and R is Levitzki-semi-simple (L-semisimple) if $L(R)=0$.

It is well-known that the Levitzki radical $L(R)$ of an associative ring R is the intersection of all the prime ideals P of R for which R/P is Levitzki-semi-simple [4].

The Levitzki radical of a Jordan ring A, which we shall denote by $L(A)$, is defined as the sum all locally solvable ideals of A and A is Levitzki-semi-simple if $L(A)=0$.

It is also known that the Levitzki radical $L(J)$ of a Jordan ring J is the intersection of all prime ideals P_α of J for which J/P_α is Levitzki-semi-simple [10].

LEMMA 4. (Bergman and Isaacs). *Let G be finite group of automorphisms acting on an associative ring R such that R has no $|G|$-torsion. Then if R^G* (or *more generally $tr(R)$) is nilpotent, then R is nilpotent.*

THEOREM 5. (Beidar). *Let G be a finite group of automorphisms acting on an associative ring R such that $|G|$ is a bijection on R.*

Then $L(R^G) = L(R) \cap R^G$.

We are now ready to prove one of our main theorems.

THEOREM 6. *Let R be an associative ring and let G be a group of Jordan automorphisms of R. Assume that $|G|$ is a bijection on R, and R^G is locally nilpotent. Then R is locally nilpotent.*

PROOF. We proceed by induction on $|G|$.

If $n = 1$, then $R = R^G$ and there is nothing to prove.

Thus, assume that for any group K of Jordan automorphisms with $|K| < |G|$, the theorem is true. To show that R is locally nilpotent, that is, $L(R) = R$, we show that there is no proper prime ideal P such that R/P is Levitzki-semi-simple. Now assume to the contrary that there is a proper prime ideal P such that R/P is Levitzki-semi-simple. We first assume that P is G-invariant, that is,

$$P^g \subseteq P \quad \text{for} \quad g \in G.$$

Let $\overline{R} = R/P$. $\overline{R}^{\bar{G}} \subseteq R^{\bar{G}} \cong R^G/P \cap R^G$. Thus $L(\overline{R}^{\bar{G}}) = L(R^{\overline{G}}) = R^{\overline{G}}$. We may therefore reduce to the case when R is Levitzki-semi-simple and R is prime. Now since R is prime, every Jordan automorphism is either an automorphism or an anti-automorphism by Theorem 1.

Let H be the subgroup of G consisting of automorphisms. If $H = G$, then by Theorem 5, we have $R^G = L(R^G) = R^G \cap L(R) = (0)$. If $R^G = (0)$, then by Lemma 4, R is nilpotent. Thus $R = L(R) = (0)$. If H is a proper subgroup of G, then H is a subgroup of index 2. By Theorem 5, $L(R^H) = L(R) \cap R^H$. The fixed ring R^H is equipped with the involution induced by the action of G/H by Corollary 3. Let S_{R^H} be the symmetric elements in R^H. Actually in this case the fixed subring R^G of G is just S_{R^H}. So by M. Rich [9], we have

$$L(R^G) = R^G = L(S_{R^H}) = S_{R^H} \cap L(R^H) = R^G \cap L(R) = (0).$$

But this implies that by Lemma 4, R is nilpotent. $L(R) = R = (0)$. We may therefore assume that P is not G-invariant. Let $I = \bigcap_{g \in G} P^g$. Then I is G-invariant.

I is a Levitzki-semi-simple ideal of R, that is R/I is Levitzki-semi-simple. As in the previous case, after passing to $\overline{R} = R/I$, we may assume that R is Levitzki-semi-simple with $\bigcap_{g\in G} P^g = (0)$. Let $\operatorname{orb} P = \{P^g \mid g \in G\}$ and m be the smallest positive integer such that, for any choice of m distinct members of $\operatorname{orb} P$, say $P_1, P_2, \ldots, P_m$, we have $\bigcap_{i=1}^{m} P_i = (0)$. Clearly $m \leq n$. If $m = 1$, then $P = (0)$.

This says that P is G-invariant, a contradiction.

We may assume that $m > 1$. Now by the minimality of m, there exist $m-1$ distinct members $P_1, P_2, \ldots, P_{m-1}$ of $\operatorname{orb} P$ such that $V = \bigcap_{i=1}^{m} P_i \neq (0)$. Let $K = \{g \in G \mid$ permutes $P_1, P_2, \ldots, P_{m-1}\}$. If $K = G$, we have a contradiction since G is transitive on $\operatorname{orb} P$ and $m-1 < m$.

Thus K is a proper subgroup of G. Since $|K|$ divides $|G|$, $|K|$ is a bijection on R. In fact, $|K|$ is a bijection on V. For, clearly V has no $|K|$-torsion and R/V is semiprime. $|K|$ is a bijection on R/V.

Indeed $|K|R/V = R/V$ and $|K|r \in V$ implies $|K|rR \subseteq V$ and $r|K|Rr = rRr \subseteq V$.

Hence $r \in V$ and R/V is $|K|$-torsion free. Thus $|K|V = V$ and K is a bijection on V. Now V is a K-invariant ideal of R. Let $\operatorname{Ann}_R(V) = \{r \in R \mid Vr = (0)\}$. Since V is an ideal in R, $\operatorname{Ann}_R(V)$ is a two-sided ideal in R. Let $J = \bigcap_{g\in G} \operatorname{Ann}_R(V)^g$. Since V is a semiprime ideal in R, $V \cap \operatorname{Ann}_R(V) = (0)$ and so $V \cap J = (0)$. For, $V \cap \operatorname{Ann}_R(V)$ is a nilpotent ideal in R. For any $x \in V$, $tr_G(x) = tr_K(x) + c(x)$ where $c(x) = \sum_{g\notin K} x^g$.

Since V is K-invariant, $tr_K(x) \in V$ and $tr_K(x) \in V^K$. If $g \notin K$, then for some P_i, $P_i^g \notin \{P_1, \ldots, P_{m-1}\}$. Thus $x^g \in P_i^g$ and $x^g V = Vx^g = (0)$ since $x^g V \subseteq P_i^g \cap (P_1 \cap \cdots \cap P_{m-1}) = (0)$ by the minimality of m. Thus $c(x) \in \operatorname{Ann}_R(V)$. Since $c(x)^h = c(x)$ for any $h \in K$, $c(x) \in J$ and $c(x) \in J^K$.

Therefore we have $tr_K(y)$, $c(x) = 0$. Now we prove that the fixed subring V^K of V under the action of K is (Jordan) locally nilpotent. Denote $\langle tr_G(x_1), \ldots, tr_G(x_m)\rangle$ and $\langle tr_K(x_1), \ldots, tr_K(x_m)\rangle$ the (Jordan) subrings of R^G and V^K, respectively, generated by $\{tr_G(x_1), \ldots, tr_G(x_m)\}$ and $\{tr_K(x_1), \ldots, tr_K(x_m)\}$ for $x_1, \ldots, x_m \in V$, then, for any positive integer m and nonnegative integers $q_1, \ldots, q_m$, we have

$$\prod_{i=1}^{m} tr_G(x_i)^{q_i} = \prod_{i=1}^{m} tr_K(x_i)^{q_i} + \prod_{i=1}^{m} c(x_i)^{q_i}$$

Hence $\prod_{i=1}^{m} tr_G(x_i)^{q_i} = 0$ implies $\prod_{i=1}^{m} tr_K(x_i)^{q_i} = 0$ since $V \cap J = 0$. Now since R^G is locally nilpotent, $\langle tr_G(x_1), \ldots, tr_G(x_m)\rangle$ is (Jordan) nilpotent for every $x_1, \ldots, x_m \in V$ and hence $\langle tr_K(x_1), \ldots, tr_K(x_m)\rangle$ is (Jordan) nilpotent. Since $|K|$ is a bijection on V, $tr_K(V) = V^K$. Thus V^K is locally nilpotent. By induction hypothesis, we have $L(V) = V \neq (0)$. Hence R is not Levitzki semi-simple. This is a contradiction.

Therefore there is no proper prime ideal P such that R/P is Levitzki semi-simple. Hence R is locally nilpotent.

LEMMA 7. *If bRb is locally nilpotent, then b^2R is locally nilpotent.*

PROOF. Let $\langle b^2r_1, b^2r_2, \ldots, b^2r_n\rangle$ be the subring of R generated by $b^2r_1, \ldots,$
b^2r_n. Then for any positive integers m and $q_1, \ldots, q_m$, $\prod_{i=1}^{m}(b^2r_1)^{q_i} = b(br_1b)^{q_1}, \ldots, (br_mb)^{q_m-1}br_m$. Hence the local nilpotency of bRb implies that of b^2R.

With the help of Lemma 7 we get one of our main results.

LEMMA 8. *Let R be an associative ring. Let G be a group of Jordan automorphisms of R and $|G|$ be a bijection on R.*

Then (1) if R is Levitzki semi-simple, then R^G is Levitzki semi-simple.

(2) $L(R^G) = R^G \cap L(R)$.

PROOF. (1) Let $b \in L(R^G)$. Then $bR^Gb = (bRb)^G \subseteq L(R^G)$. So bR^Gb is locally nilpotent and hence bRb is locally nilpotent by Theorem 6.

Now by Lemma 7, b^2R is locally nilpotent. So $b^2R \cap L(R) = (0)$. Since $b^2 = 0$ for all $b \in L(R^G)$ and R is semiprime, by Lemma 3.18, $L(R^G) = (0)$.

(2) Since $L(R) \cap R^G$ is a locally nilpotent ideal of R^G, the inclusion $L(R) \cap R^G \subseteq L(R^G)$ is obvious. Now for other inclusion we need to show that $L(R^G) \subseteq L(R)$.

For any prime ideal P of R for which R/P is Levitzki semi-simple, $\bigcap_{g \in G} P^g$ is a G-invariant semiprime ideal in R. Let $\overline{R} = R/\bigcap_{g \in G} P^g$. Then $\overline{R}$ is Levitzki semi-simple and $\overline{R}^{\bar{G}} = R^{\bar{G}} = R^G/R^G \cap \bigcap_{g \in G} P^g \cong R^G/P \cap R^G$. Since $\overline{R}$ is Levitzki semi-simple $R^{\bar{G}} \cong R^G/P \cap R^G$ is Levitzki semi-simple. Hence $L(R^G) \subseteq P \cap R^G$ for all prime ideal P for which R/P is Levitzki semi-simple. Therefore $L(R^G) \subseteq L(R) \cap R^G$ and so $L(R^G) = R^G \cap L(R)$.

We consider the transferring of the algebraicity from R^G to R.

Let A be an associative algebra over a field Φ. N. Jacobson [6] defined the algebraic kernel as the maximal algebraic ideal which contains every algebraic ideal in A.

THEOREM 9. (Kharchenko). *Let R be an associative algebra which is P.I. R is a ring with involution $*$. Let S be the set of all symmetric elements of R. If S is algebraic over Φ. Then R is algebraic over Φ.* [7].

THEOREM 10. (Armendariz). *Let R be an associative P.I. algebra over Φ and G be a finite group of automorphisms of R such that $|G|$*

induces a bijection on R. If R^G is an algebraic algebra over Φ, then R is an algebraic algebra over Φ. [1]

THEOREM 11. *Let R be an associative P.I. algebra over a field Φ and G a group of Jordan automorphisms of R such that $|G|$ is a bijection on R. Then if R^G is algebraic over Φ, then R is algebraic over Φ.*

PROOF. We proceed by induction on $|G|$.

If $|G| = 1$, then $R = R^G$ and we are done. Suppose that the theorem is true for any groups $K \leq \mathrm{Aut}_J(R)$ with $|K| < |G|$. Now assume to the contrary that R is not algebraic over Φ. Then there exists $x \in R$ which is transcendental over Φ. Let P be a maximal ideal with respect to the property $P \cap \Phi[x] = (0)$. Let $S = \{I \mid I \text{ is a two-sided ideal in } R, I \cap \Phi[x] = (0)\}$. By Zorn's lemma, such a maximal ideal P exists.

We note that $\Phi[x]$ is an integral domain. We first prove that P is a prime ideal in R. Let A and B be two-sided ideals in R such that $A \subseteq P$ and $B \subseteq P$. Then $A + P \neq P$ and $B + P \neq P$: Since $(A+P) \cap \Phi[x] \neq (0)$ and $(B+P) \cap \Phi[x] \neq (0)$, there exists $f(x) \neq 0$, $g(x) \neq 0$ in $\Phi[x]$ such that $f(x) \in A + P$ and $g(x) \in B + P$. Thus $0 \neq f(x)g(x) \in (A+P)(B+P) = AB + P$.

Hence $AB \subseteq P$. Therefore P is a prime ideal in R. We claim that R/P has zero algebraic kernel.

If not, we have a nonzero algebraic ideal I/P of R/P. Thus $I \cap \Phi[x] \neq (0)$ by the maximality of P. Take up $0 \neq f(x) \in I \cap \Phi[x]$. Then $f(x) + P$ is an algebraic element in R/P. There exists not all zero elements $a_0, a-1, \ldots, a_t$ in Φ such that $a_0 f(x)^t + a_1 f(x)^{t-1} + \cdots + a_{t-1} f(x) + a_t \in P \cap \phi[x]$ and so $a_0 f(x)^t + a_1 f(x)^{t-1} + \cdots + a_{t-1} f(x) + a_t = 0$. We claim that $a_0 = a_1 = \cdots = a_{t-1} = a_t = 0$. We proceed this by induction on t. When $t = 1$, let $f(x) = \sum_{i=1}^{n} b_i x^i$ with $b_i \in \Phi$ and $b^n \neq 0$.

Then $a_0\left(\sum_{i=0}^{n} b_i x^i\right) + a_1 = 0$. Thus $a_0 b_n = 0$. Hence $a_0 = 0$ and $a_1 = 0$. Suppose that this true for $t-1$. From

$$\begin{aligned} 0 &= a_0 f(x)^t + \cdots + a_{t-1} f(x) + a_t \\ &= a_0\left(\sum_{i=0}^{n} b_i x^i\right)^t + \cdots + a_{t-1}\left(\sum_{i=0}^{n} b_i x^i\right) + a^t a_0 b_n^t = 0 \end{aligned}$$

and hence $a_0 = 0$. Thus we have

$$0 = a_1\left(\sum_{i=1}^{n} b_i x^i\right)^{t-1} + \cdots + a_{t-1}\left(\sum_{i=0}^{n} b_i x^i\right) + a_t$$

By induction hypothesis, $a_1 = \cdots = a_{t-1} = a_t = 0$ and so $a_0 = a_1 = \cdots = a_t = 0$.

This is a contradiction to the fact that $f(x) \neq 0$.

Therefore R/P has zero algebraic kernel. Now to proceed the induction process, we divide the following two cases. First we consider the case that P is G-invariant. Let $\overline{R} = R/P$. Then $\overline{R}^{\bar{G}} = R^{\bar{G}}$ and $\overline{R}$ is a prime P.I. algebra. By Theorem 1. $\bar{g}$ is an automorphism or anti-automorphism for all $g \in G$. By Theorem 9 and 10 R is algebraic over Φ. But since R/P has zero algebraic kernel, $R = P$ and so $\overline{R} = (0)$. This is a contradiction. We may therefore assume that P is not G-invariant. Let $\overline{R} = R/\bigcap_{t\in G} P^g$. Then since R/P has zero algebraic kernel, P contains the algebraic kernel. Since $R/P \cong R/{}^gP$ or R/P^g is anti-isomorphic to R/P^g for each $g \in G$, P^g contains the algebraic kernel of R for each $g \in G$. Hence $\bigcap_{g\in G} P^g$ contains the algebraic kernel of R. Thus $R/\bigcap_{g\in G} P^g$ has zero algebraic kernel. We may assume that R has zero algebraic kernel and $\bigcap_{g\in G} P^g = (0)$. Let $\operatorname{orb} P = \{P^g \mid g \in g\}$, and let m be the smallest positive integer such that for any choice of m distinct members of $\operatorname{orb} P$, say

$P_1, P_2, \ldots, P_m$, we have $\bigcap_{i=1}^{m} P_i = (0)$. Clearly $m \leq n$. If $m = 1$, then $P = (0)$. This says that P is G-invariant, a contradiction. We may therefore assume that $m > 1$. Let $V = P_1 \cap \cdots \cap P_{m-1} \neq (0)$. Let $K = \{g \in G \mid g \text{ permutes } P_1, P_2, \ldots, P_{m-1}\}$, then $K \neq G$.

Thus V is K-invariant and $|K|$ induces a bijection on V. We claim that V^K is algebraic. Let $x \in V$ and $tr_G(x) = tr_K(x) + c(x)$ where $c(x) = \sum_{g \notin K} x^g$. As the proof of Theorem 6 $V_c(x) = c(x)V = (0)$ for all $x \in V$. Since R^G is algebraic over Φ and $tr_G(x) \in R^G$, there exists $a_0, a_1, \ldots, a_n \in \Phi$, not all zero such that

$$a_0(tr_G(x))^n + a_1(tr_G(x))^{n-1} + \cdots + a_{n-1}(tr_G(x)) + a_n = 0$$

$$a_0[tr_K(x)^n + c(x)^n] + a_1[(tr_K(x)^{n-1} + c(x)^{n-1}] + \cdots + a_{n-1}[tr_K(x) + c(x)] + a_0 = 0$$

$$a_0 tr_K(x)^n + a_1 tr_K(x)^{n-1} + \cdots + a_{n-1} tr_K(x) + a_n + a_{0c}(x)^n + a_{1c}(x)^{n-1} + \cdots + a_{n-1}c(x) = 0$$

But as in the proof of Theorem 6

$$a_0 tr_K(x)^n + a_1 tr_K(x)^{n-1} + \cdots + a_{n-1} tr_K(x) + a_n \in V$$

and

$$a_0(c(x))^n + a_1 c(x)^{n-1} + \cdots + a_{n-1}c(x) \in J = \bigcap_{g \in G} \operatorname{Ann}(V)^g.$$

Since $V \cap J = (0)$,

$$a_0 tr_K(x)^n + a_1 tr_K(x)^{n-1} + \cdots + a_{n-1} tr_K(x) + a_n = 0.$$

Thus $V^K = tr_K V$ is algebraic over Φ. By induction on $|G|$, V is an algebraic ideal in R, $V \neq (0)$. This is a contradiction. This proves the theorem.

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