JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 5, June 1992

Consequences of Lipschitz Stability*

SUNG KYU CHOI, KI SHIK KOO AND KEON-HEE LEE

ABSTRACT. In this note, we show that the ω-limit mapping is continuous and the Lipschitz constants vary continuously if the flow (x, π) is Lipschitz stable. Moreover we analyse the ω-limit sets under the generalized locally Lipschitz stable flows.

Throughout the paper we let (X, π) denote a flow on a locally compact metric space X with a metric d. The orbit, orbit closure, and prolongational limit sets on X are denoted, respectively, by O, $\overline{O}$ and J with unilateral versions carrying the appropriate $+$ or $-$ superscript.

A flow (X, π) is said to be *locally Lipschitz stable* if there exist $\delta > 0$ and $K \geq 1$ such that $d(xt, yt) \leq Kd(x, y)$ for all $t \in \mathbf{R}^+$ and all x, $y \in X$ with $d(x, y) < \delta$. A flow (X, π) is called Lipschitz stable if there exists $K \geq 1$ satisfying $d(xt, yt) \leq Kd(x, y)$ for all $t \in \mathbf{R}^+$ and all x, $y \in X$.

LEMMA 1. *The property of Lipschitz stability is independent of the metric.*

PROOF. The proof is straightforward.

*The present studies were supported by the Basic Science Research Institute Program, Ministry of Eduction, Korea, 1990, Project No. BSRI-90-110.
Received by the editors on April 23, 1992.
1980 *Mathematics subject classifications*: Primary 58F25.

LEMMA 2. *Let X be a bounded metric space. Then a flow (X,π) is Lipschitz stable if and only if it is locally Lipschitz stable.*

PROOF. The proof is straightforward.

Let X be a compact metric space, and let $K(X)$ be the set of all nonempty closed subsets of X with the *Hausdorff metric* :

$$\rho(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

A mapping $F : X \to K(X)$ is called upper (or lower) semicontinuous at $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$ then

$$F(y) \subset B_\varepsilon(F(x)) \qquad (\text{or } F(x) \subset B_\varepsilon(F(y))),$$

respectively, where $B_\varepsilon(A) = \{y \in X : d(x,y) < \varepsilon \text{ for some } x \in A\}$. Since the ω-limit set

$$\omega(x) = \{y \in X : xt_i \to y \text{ for some } t_i \to \infty\}$$

is a nonempty closed subset of X, we can consider the mapping

$$\omega : X \to K(X), \qquad x \rightsquigarrow \omega(x),$$

which is called the *ω-limit mapping.* The following theorem says that the ω-limit mapping is continuous if the flow (X,π) is Lipschitz stable.

THEOREM 3. *Let a flow (X,π) be Lipschitz stable. Then the ω-limit mapping is continuous.*

PROOF. Let $x \in X$ and $y \in \omega(x)$. If $\{x_n\}$ is a sequence in X converging to x, then we shall show that there exists a sequence $\{y_n\}$ in X such that $y_n \in \omega(x_n)$ and $y_n \to y$. Let $xs_i \to y$ for some $s_i \to \infty$. Then we have

$$\begin{aligned} d(x_n s_i, x_n s_j) &\le d(x_n s_i, x s_i) + d(x s_i, x s_j) + d(x s_j, x_n s_j) \\ &\le K d(x_n, x) + d(x s_i, x s_j) + K d(x, x_n). \end{aligned}$$

It follows that the sequence $\{x_n s_i\}_{i=0}^{\infty}$ is Cauchy and consequently it converges to a point, say y_n, in $\omega(x_n)$. Then the sequence $\{y_n\}$ converges to y. In fact, we get

$$\begin{aligned} d(y, y_n) &\leq d(y, xs_i) + d(xs_i, x_n s_i) + d(x_n s_i, y_n) \\ &\leq d(y, xs_i) + Kd(x, x_n) + d(x_n s_i, y_n) \to 0. \end{aligned}$$

This means that the map ω is upper semicontinuous. To show that the map ω is lower semicontinuous, we let $x_n \to x$, $y_n \in \omega(x_n)$ and $y_n \to y$. Then it is enough to show that $y \in \omega(x)$. Since $y_n \in \omega(x_n)$, there exists a sequence $\{t_i^n\}$ in $\mathbf{R}^+$ such that $x_n t_i^n \to y_n$ as $t_i^n \to \infty$. We may assume that

$$|t_{n+1}^{n+1} - t_n^n| \geq n \quad \text{and} \quad d(y_n, x_n t_n^n) < \frac{1}{n}.$$

Then we have

$$d(y, x_n t_n^n) \leq d(y, y_n) + d(y_n, x_n t_n^n) < d(y, y_n) + \frac{1}{n}.$$

Hence we get $x_n t_n^n \to y$ and $t_n^n \to \infty$. This means that $y \in J^+(x) = \omega(x)$ and so completes the proof.

Let x, y be any two different points in X. Then the least positive number K satisfying

$$d(xt, yt) \leq Kd(x, y) \quad \text{for all} \quad t \in \mathbf{R}^+$$

will be called the *Lipschitz constant with respect to x and y*. In the following theorem, we show that the Lipschitz constants vary continuously if the flow (X, π) is *Lyapunov stable*. We say that a flow (X, π) is Lyapunov stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(xt, yt) < \varepsilon$ for all $t \in \mathbf{R}^+$ and all $x, y \in X$ with $d(x, y) < \delta$. To show this, we define the set $Y_K \subset X \times X$ for any $K > 0$ as follows.

$$Y_K = \{(x, y) \in X \times X/\Delta : d(xt, yt) \leq Kd(x, y) \text{ for all } t \in \mathbf{R}^+\},$$

where $\Delta = \{(x, x) : x \in X\}$.

THEOREM 4. *The function* $F : X \times X/\Delta \to [0,\infty]$ *given by* $F(x,y) = \inf\{K : (x,y) \in Y_K\}$ *is continuous if the flow* (X,π) *is Lyapunov stable.*

PROOF. Let $(x,y) \in X \times X/\Delta$, $F(x,y) = K$, and $0 < \varepsilon < K$ be arbitrary. First we assume $K < \infty$. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$d(x,z) < \delta_1 \text{ implies } d(xt,zt) < \frac{\varepsilon}{3}d(x,y), \quad \text{and}$$
$$d(y,w) < \delta_2 \text{ implies } d(yt,wt) < \frac{\varepsilon}{3}d(x,y) \quad \text{for all } t \in \mathbf{R}^+.$$

Choose $\delta > 0$ satisfying

$$\delta < \min\{\delta_1, \delta_2, \frac{\varepsilon}{6(K+\varepsilon)}\}d(x,y), \frac{\varepsilon}{12(K-\varepsilon)}d(x,y)\},$$

and select $(z,w) \in B(x,\delta) \times B(y,\delta)$. Then we have

$$\begin{aligned} d(zt,wt) &\le d(zt,xt) + d(xt,yt) + d(yt,wt) \\ &< d(xt,yt) + \frac{2}{3}\varepsilon d(x,y) \\ &< \frac{2}{3}\varepsilon d(x,y) + Kd(x,y) \\ &< Kd(z,w) + \frac{2}{3}\varepsilon d(x,y) + 2K\delta \\ &< Kd(z,w) + \varepsilon(d(x,y) - 2\delta) \\ &< Kd(z,w) + \varepsilon d(z,w) \\ &= (K+\varepsilon)d(z,w). \end{aligned}$$

Hence we get

$$(1) \qquad F(z,w) \le K + \varepsilon, \quad \text{and so} \quad F(z,w) - F(x,y) \le \varepsilon$$

Since $F(x,y) = K$, for $K - \frac{\varepsilon}{6} > 0$, there exists $s \in \mathbf{R}^+$ such that

$$d(xs,ys) > \left(K - \frac{\varepsilon}{6}\right) d(x,y).$$

since $(z,w) \in B(x,\delta) \times B(y,\delta)$,

$$\begin{aligned} d(xs,ys) &< d(xs,zs) + d(zs,ws) + d(ws,ys) \\ &< d(zs,ws) + \frac{2}{3}\varepsilon d(x,y), \quad \text{and so} \\ d(zs,ws) &> \left(K - \frac{\varepsilon}{6}\right) d(x,y) - \frac{2}{3}\varepsilon d(x,y) \\ &= Kd(x,y) - \frac{5}{6}\varepsilon d(x,y). \end{aligned}$$

Since $d(z,w) < d(x,y) + 2\delta$ and $d < \frac{\varepsilon}{12(K-\varepsilon)} d(x,y)$,

$$\begin{aligned} d(zs,ws) &> \left(Kd(x,y) - \frac{5}{6}\varepsilon d(x,y)\right)(K-\varepsilon) \\ &> \left(d(x,y) + \frac{2\varepsilon}{12(K-\varepsilon)} d(x,y)\right)(K-\varepsilon) \\ &> (d(x,y) + 2\delta)(K-\varepsilon) \\ &> (K-\varepsilon)d(z,w). \end{aligned}$$

This means that

$$F(x,y) - F(z,w) < \varepsilon \tag{2}$$

By (1) and (2), we obtain

$$|F(x,y) - F(z,w)| < \varepsilon.$$

Next we assume $F(x,y) = K = \infty$. Let $N > 0$ be sufficiently large number. For any $\varepsilon > 0$, there exists $s \in \mathbf{R}^+$ such that

$$d(xs,ys) > (N + 2\varepsilon)d(x,y).$$

Given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} d(x,z) < \delta_1 &\text{ implies } d(xt,zt) < \frac{\varepsilon}{3} d(x,y), \quad \text{and} \\ d(y,w) < \delta_2 &\text{ implies } d(yt,wt) < \frac{\varepsilon}{3} d(x,y). \end{aligned}$$

Choose $\delta > 0$ satisfying

$$\delta < \min\{\delta_1, \delta_2, \frac{\varepsilon d(x,y)}{3(N+\varepsilon)}\},$$

and select $(z,w) \in B(x,\delta) \times B(y,\delta)$. Then we get

$$\begin{aligned} d(zs, ws) &> d(xs, ys) - \frac{2}{3}\varepsilon d(x,y) \\ &> (N+2\varepsilon)d(x,y) - \frac{2}{3}\varepsilon d(x,y) \\ &> \left(d(x,y) + \frac{\varepsilon d(x,y)}{3(N+\varepsilon)}\right)(N+\varepsilon) \\ &> (N+\varepsilon)(d(x,y)+2\delta) \\ &> (N+\varepsilon)d(z,w). \end{aligned}$$

This shows that $(z,w) \in B(x,\delta) \times B(y,\delta)$ implies $F(z,w) > N$. Consequently the map F is continuous, and so completes the proof.

Finally we introduce the concept of generalized local Lipschitz stability which is a generalization of that of local Lipschitz stability and analyse the ω-limit sets under the generalized locally Lipschitz stable flows.

DEFINITION 5. A flow (X,π) is called *generalized locally Lipschitz stable* if there exists $\delta > 0$, $K \geq 1$ and a continuous function $\alpha : X \times \mathbf{R}^+ \to \mathbf{R}^+$ such that $d(xt, y\alpha(x,t)) \leq Kd(x,y)$ for all $t \in \mathbf{R}^+$ and all x, $y \in X$ with $d(x,y) < \delta$.

THEOREM 6. *Let (X,π) be a generalized locally Lipschitz stable flow on a complete and connected space X. Then we have either*

i) *$\omega(x) = \phi$ for all $x \in X$, or*

ii) *$\omega(x) \neq \phi$ is compact and minimal for all $x \in M$.*

PROOF. Let $Y = \{x \in X : \omega(x) \neq \phi\}$. Then it is enough to show that the set Y is both open and closed in X.

First we show that the set Y is open in X. Let $x \in Y$, and let $z(\neq x) \in \omega(x)$. Then there exists a sequence $\{s_i\}$ in $\mathbf{R}^+$ such that $xs_i \to z$ and $s_i \to \infty$. Choose $\varepsilon > 0$ satisfying $x \notin \overline{B(z,\varepsilon)}$ and $\overline{B(z,\varepsilon)}$ is compact. Select $0 < \delta < \varepsilon$ satisfying $xs_i \in B(z,\delta)$. Since (X,π) is generalized locally Lipschitz stable, there exists $\alpha > 0$ such that if $d(x,y) < \alpha$ then $B(x,\alpha) \cap B(z,\varepsilon) = \phi$, $yt \in B(O^+(x), \varepsilon - \delta)$ and $xt \in B(O^+(y), \varepsilon - \delta)$ for all $t \in \mathbf{R}^+$. Hence we can choose $t_i > 0$ satisfying $d(xs_i, yt_i) < \varepsilon - \delta$. Then we get $d(yt_i, z) < \varepsilon - \delta + \delta = \varepsilon$. Since $\{yt_i\} \subset \overline{B(z,\varepsilon)}$ and $t_i \not\to 0$, we obtain $y \in Y$. This means that Y is open.

Next we show that the set Y is closed in X. For each positive integer K, we can choose $\delta_K > 0$, $\gamma_K > 0$, $s_K > 0$ such that

$$d(x,y) < \delta_K \text{ implies } xt \in B\left(O^+(y), \frac{1}{K^2}\right) \text{ and } yt \in B\left(O^+(x), \frac{1}{K^2}\right),$$

$$d(x,y) < \gamma_K \text{ implies } xt \in B\left(O^+(y), \frac{1}{2}\delta_K\right) \text{ and } yt \in B\left(O^+(x), \frac{1}{2}\delta_K\right),$$

$$d(x,y) < s_K \text{ implies } xt \in B\left(O^+(y), \frac{1}{4}\gamma_K\right) \text{ and } yt \in B\left(O^+(x), \frac{1}{4}\gamma_K\right)$$

for all $t \in \mathbf{R}^+$. We may assume that

$$s_K < \gamma_K < \delta_K < \frac{1}{K^2}, \quad \delta_{K+1} < \delta_K, \quad \gamma_{K+1} < \gamma_K \quad \text{and} \quad s_K < s_{K+1}.$$

Let $\{x_n\}$ be a sequence in Y converging to $x \in X$. By choosing a subsequence, we may assume that $d(x_n, x) < \frac{1}{2}s_n$ for each n. Then we obtain

$$d(\omega(x_n), \omega(x_{n+1})) < \delta_n \quad \text{for each} \quad n.$$

In fact, we assume $d(\omega(x_n), \omega(x_{n+1})) \geq \delta_n$ for each n. Then there exist

$\alpha > 0$ and t_0, $s_0 \in \mathbf{R}^+$ such that

$$d(z,w) > \frac{1}{2}\delta_n \quad \text{for} \quad z \in B(\omega(x_n),\alpha) \quad \text{and} \quad w \in B(\omega(x_{n+1}),\alpha),$$
$$d(x_nt,\omega(x_n)) < \alpha \quad \text{for} \quad t \geq t_0 > 0, \quad \text{and}$$
$$d(x_{n+1}t,\omega(x_{n+1})) < \alpha \quad \text{for} \quad t \geq s_0 > 0.$$

Since $d(x_n, x_{n+1}) < d(x_n, x) + d(x, x_{n+1}) < \frac{1}{2}(s_n + s_{n+1}) < s_n$, we have

$$x_{n+1}t \in B(O^+(x_n),\gamma_n) \quad \text{for} \quad t \in \mathbf{R}^+, \quad \text{and so}$$
$$d(x_{n+1}t_0, x_ns) < \gamma_n \quad \text{for some} \quad 0 \leq s < t_0.$$

For $t_0 - s > 0$, we have

$$(x_ns)(t_0 - s) \in B(O^+(x_{n+1}t_0), \frac{1}{2}\delta_n) \quad \text{and so}$$
$$d(x_nt_0, x_{n+1}t') < \frac{1}{2}\delta_n \quad \text{for some} \quad t' > 0.$$

But $x_nt_0 \in B(\omega(x_n),\alpha)$ and $x_{n+1}t' \in B(\omega(x_{n+1}),\alpha)$, and thus

$$d(x_nt_0, x_{n+1}t') > \frac{1}{2}\delta_n.$$

This contradiction shows that $d(\omega(x_n), \omega(x_{n+1})) < \delta_n$ for each n. For each n, choose $z_n \in \omega(x_n)$ inductively as follows. Let $z_1 \in \omega(x_1)$ and $z_n \in \omega(x_n)$. Since $d(\omega(x_n), \omega(x_{n+1})) < \delta_n$, there exist $b_n \in \omega(x_n)$ and $b_{n+1} \in \omega(x_{n+1})$ such that $d(b_n, b_{n+1}) < \delta_n$. Since $\omega(x_n)$ is minimal, by the generalized local Lipschitz stability, we have $\omega(x_n) = \omega(b_n)$. Choose $T >$ such that

$$d(b_nT, z_n) < \frac{1}{n^2} \quad \text{and} \quad b_nT \in B\left(O^+(b_{n+1}), \frac{1}{(n+1)^2}\right).$$

Then we have

$$d(b_nT, b_{n+1}T') < \frac{1}{(n+1)^2} \quad \text{for some} \quad T' > 0.$$

By the invariance of $\omega(x_{n+1})$, we have $b_{n+1}T' \in \omega(x_{n+1})$. Let $z_{n+1} = b_{n+1}T'$. Then we get

$$\begin{aligned} d(z_n, z_{n+1}) &\le d(z_n, b_nT) + d(b_nT, z_{n+1}) \\ &< \frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{2}{n^2}. \end{aligned}$$

Since $\{z_n\}$ is a Cauchy sequence, we may assume $z_n \to z$. By choosing a subsequence if necessary, we may assume

$$d(z, z_n) < \frac{1}{n^2}.$$

Since $z_n \in \omega(x_n)$, there exists a sequence $\{t_i^n\}$ in $\mathbf{R}^+$ such that

$$x_n t_i^n \to z_n \quad \text{and} \quad t_i^n \to \infty.$$

For each $K > 0$, we choose $t_{i_K}^K \in \{t_i^K\}$ such that

$$d(x_K t_{i_K}^K, z_K) < \frac{1}{K^2}.$$

We assume $t_{i_{K+1}}^{K+1} > t_{i_K}^K$, and we let $a_n = x_n t_{i_n}^n$. Then we have

$$\begin{aligned} d(z, a_n) &\le d(z, x_n) + d(x_n, a_n) \\ &< \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}. \end{aligned}$$

This means that $a_n \to z \in \omega(x)$ and so $x \in Y$. Consequently the set Y is closed and thus complete the proof.

References

[1] S. Elaydi and H.R. Farran, *On weak isometries and their embeddings in flows*, Nonlinear Analysis, TMA **8** no. 12 (1984), 1437–1441.

[2] ———, *Lipschitz stable dynamical systems*, Nonlinear Analysis, TMA **9** no. 7 (1985), 729–738.

[3] ———, *On variations of equicontinuity in dynamical systems*, Bull. Austral. Math. Soc. **42** (1990), 391–397.

Department of Mathematics
Chungnam National University
Taejon, 305-765, Korea

Department of Mathematics
Taejon University
Taejon, 300-716, Korea