JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 5, June 1992

Stability of Closed Sets in Semiflows: The Negative Aspects

KYUNG BOK LEE

1. Introduction

The main motivation behind the study of semiflows is that they describe autonomous and nonautonomous ordinary differential equations. Semiflows considered as transformation semigroups were studied by Gottschalk in 1946. Later in 1960, Hajek considered continuous semiflows. But the first systematic study of continuous semiflows had to wait until 1969 the year in which the book of Bhatia and Hajek [1] appeared. The theory of continuous semiflows has been further developed by Ura, McCann, Das and Lakshmikanthan. A substantial contribution in the theory of discrete semiflows was made by Furstenburg and Brown[3]. A semiflows as in many theorems have associated with flows. The relation between these two systems and many of their properties are already developed jointly by Kaul and Elaydi in [3]. The following example(Fig. 1) illustrates the semiflows(b) and flow(c) associated with a given control system(a) [4].

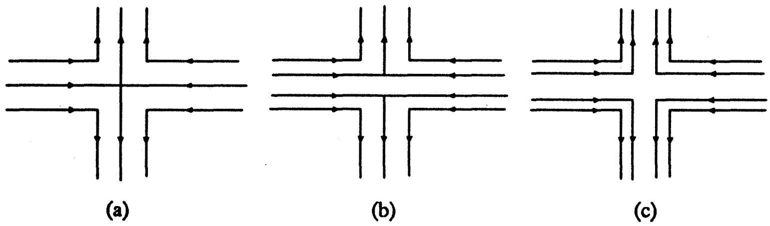

Fig. 1.

Received by the editors on April 11, 1992.
1980 *Mathematics subject classifications*: Primary 58F.

The concepts of Liapunov functions, stability and equistability in flows were studied. An elegant necessary and sufficient conditions for a flow to be stable, uniformly stable and equistable was appeared in [2]. In this paper, we introduce the concepts of a negative stability, uniform stability and equistability in semiflow (X, π) whose phase space X is a second countable meric space. Furthermore some relations between the above concepts and the negative Liapunov functions are *studied.* For this purpose, we introduce backward semicontinuous semiflows.

2. Definitions

A *semiflow* is a pair (X, π) which consists of a topological space X and a continuous map $\pi : X \times \mathbf{R}^+ \to X$ such that

i) $\pi(x, 0) = x$
ii) $\pi(\pi(x, s), t) = \pi(x, s + t)$

for all $x \in X$ and s, $t \in \mathbf{R}^+$, where $\mathbf{R}^+$ is the set of nonnegative real numbers. For simplicity, $\pi(x, t)$ will be denoted by xt. For $A \subset X$ and $S \subset \mathbf{R}^+$, let $F(A, S) = \{x \in X \mid xt \in A \text{ for some } t \in S\}$ and $F(A) = F(A, \mathbf{R}^+)$. For $x \in X$, $F(x) = F(\{x\}, \mathbf{R}^+)$ is called the *negative funnel* through x. A subset M of X is said to be *negatively invariant* if $F(M) \subset M$.

3. Stability of Closed Sets

DEFINITION 3.1. A subset M of X is *negatively stable* if for each $\varepsilon > 0$ and $x \in M$, there is $\delta > 0$ such that $F(B(x, \delta)) \subset B(M, \varepsilon)$.

THEOREM 3.2. *A closed subset M of X is negatively stable if and only if there exists a function $f : X \to \mathbf{R}^+$ with the following properties*

i) $f(x) = 0$ *if and only if* $x \in M$.

ii) *For every* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $f(x) \geq \delta$ *whenever* $d(M, x) \geq \varepsilon$; *also for any sequence* $\{x_n\}$, $f(x_n) \to 0$ *if* $x_n \to x \in M$.

iii) $f(y) \leq f(x)$ *for* $y \in F(x)$.

PROOF. Necessity. Let M be negatively stable. Assume that the metric function d on X is bounded. Define $f : X \to \mathbf{R}^+$ by $f(x) = \sup_{y \in F(x)} d(M, x)$. Let $f(x) = 0$. Then we have $d(M, x) \leq \sup d(M, y) = f(x) = 0$ with $x \in F(x)$. Since M is a closed set, we obtain $x \in M$. Conversely, if $x \in M$, there exists $\delta > 0$ such that $F(B(x, \delta)) \subset B(M, \varepsilon)$ for each $\varepsilon > 0$, since M is negatively stable. For any $y \in F(x)$, we have $y \in B(M, \varepsilon)$, and so $d(M, y) \leq \varepsilon$. Therefore, we obtain $f(x) = \sup_{y \in F(x)} d(M, y) \leq \varepsilon$. This means that $f(x) = 0$. For given $\varepsilon > 0$, take $\delta = \varepsilon$. Since $d(M, x) \geq \varepsilon$ and $x \in F(x)$, we have

$$f(x) = \sup_{y \in F(x)} d(M, y) \geq d(M, x) \geq \varepsilon = \delta.$$

Now, let $x_n \to x \in M$. By the fact that M is negatively stable, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $F(B(x, \delta)) \subset B(M, \varepsilon)$. Since $x_n \to x$, there exists N such that for all $n \geq N$, $x_n \in B(x, \delta)$. Since $F(x) \subset F(B(x, \delta)) \subset B(M, \varepsilon)$, we obtain $f(x) = \sup_{y \in F(x)} d(M, y)$, and so $f(x_n) \to 0$. Let $y \in F(x)$. Then we have

$$f(y) = \sup_{z \in F(y)} d(M, z) \leq \sup_{z \in F(x)} d(M, z) = f(x).$$

Sufficiency. By the above facts, for every $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \geq \delta$ whenever $d(M, x) \geq \varepsilon$. Given $\varepsilon > 0$, we let $m = \inf\{f(x) : d(M, x) \geq \varepsilon\}$. Then we obtain $m \geq \delta$. Let $x \in M$, we will show that there is $\delta > 0$ such that $f(y) < m$ whenever $d(x, y) < \delta$. Suppose not. Then there exists $y_n \in X$ such that $d(x, y_n) < 1/n$ and

$f(y_n) \geq m$ for each $n \in N$. The sequence $\{y_n\}$ converges to $x \in M$, but the sequence $\{f(y_n)\}$ does not converges to 0. This contradicts to the above facts. Also, we assert that $F(B(x,\delta)) \subset B(M,\varepsilon)$. Assume that $F(B(x,\delta)) \not\subset B(M,\varepsilon)$. Then there is $y \in B(M,\delta)$ such that $F(y) \not\subset B(M,\varepsilon)$. Let $z \in F(y) - B(M,\varepsilon)$. By the above results, we get $f(z) \leq f(y) < m$. But $d(M,z) \geq \varepsilon$ implies $f(z) \geq m$. This contradiction proves the theorem.

DEFINITION 3.3. A subset M of X is *negatively uniformly stable* if for each $\varepsilon > 0$, there is $\delta > 0$ such that $F(B(M,\delta)) \subset B(M,\varepsilon)$.

THEOREM 3.4. *A closed subset M of X is negatively uniformly stable if and only if there is a function $f : X \to \mathbf{R}^+$ such that*

i) *for every $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \geq \delta$ whenever $d(M,x) \geq \varepsilon$.*
ii) *for every $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \leq \delta$ whenever $d(M,x) \leq \varepsilon$.*
iii) *$f(y) \leq f(x)$ for $y \in F(x)$.*

PROOF. Necessity. Assume that the metric function d on X is bounded. Define a map $f : X \to \mathbf{R}^+$ by $f(x) = \sup_{y \in F(x)} d(M,y)$.

For each $\delta > 0$, there is $\varepsilon > 0$ and $x \in X$ such that $d(M,x) \leq \varepsilon$ and $f(x) < \delta$. Take $\delta = \varepsilon$. Then there is $x \in X$ such that $d(M,x) \leq \varepsilon$ and $f(x) < \varepsilon$. Since $x \in F(x)$, we have $\varepsilon \leq d(M,x) \leq \sup_{y \in F(x)} d(M,y) = f(x) < \varepsilon$. This contradiction proves i).

For each $\varepsilon > 0$, there is $\delta > 0$ such that $F(B(M,2\delta)) \subset B(M,\varepsilon)$. Assume that $d(M,x) \leq \delta$. Since $x \in B(M,2\delta)$, we have $F(x) \subset F(B(M,2\delta)) \subset B(M,\varepsilon)$. This shows that $f(x) < \delta$, and so proves the condition ii).

The condition iii) is clear by the fact that $F(y) \subset F(x)$ for any $y \in F(x)$.

Sufficiency. Let $\varepsilon > 0$ be arbitrary. By the condition i), we can choose $\delta > 0$ such that $f(x) \geq 2\delta$ whenever $d(M,x) \geq \varepsilon$. also, by condition ii), there $\eta > 0$ such that $f(x) \leq \delta$ whenever $d(M,x) \leq \eta$.

Let $y \in F(B(M,\eta))$. Then there exists $x \in X$ such that $y \in F(x)$ and $d(M,x) \leq \eta$. By condition iii), $y \in F(x)$ implies $f(y) \leq f(x) < \delta < 2\delta$. Hence, we have $y \in B(M,\varepsilon)$. Assume that $y \notin B(M,\varepsilon)$. Since $d(M,y) \geq \varepsilon$, we obtain $f(y) \geq 2\delta$. This contradiction implies $F(B(M,\eta)) \subset B(M,\varepsilon)$. Consequently, M is negatively uniformly stable, and so completes the proof.

DEFINITION 3.5. A semiflow (x,π) is *negatively backward semicontinuous* if for each $x \in X$, $t \in \mathbf{R}^+$ and $\varepsilon > 0$, there is $\delta > 0$ such that $B(x,\varepsilon) \cap F(y) \neq \emptyset$ whenever $y \in B(xt,\delta)$.

DEFINITION 3.6. A subset M of X is said to be *negatively equistable* if for each $x \notin M$, there is $\delta > 0$ such that $x \notin \overline{F(B(M,\delta))}$.

LEMMA 3.7. *If a closed subset M of X is negatively equistable, then it is negatively invariant.*

PROOF. Assume that $F(x) \not\subset M$ for some $x \in M$. Then there is $y \in F(x) - M$, and so $yt = x$ for some $t \in \mathbf{R}^+$. Since M is negatively equistable, there is $\delta > 0$ such that $y \notin F(B(M,\delta))$. Since $x \in M \subset B(M,\delta)$, we have

$$F(x) \subset F(B(M,\delta))$$

and

$$y \in F(x) \subset \overline{F(x)} \subset \overline{F(B(M,\delta))}.$$

This contradiction proves the Lemma.

THEOREM 3.8. *Let a semiflow (x, π) be negatively backward semicontinuous. A closed subset M of X equistable if and only if there is a function $f : X \to \mathbf{R}^+$ with the following properties.*

i) $f(x) = 0$ *if and only if* $x \in M$.
ii) *For every* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $f(x) < \varepsilon$ *if* $d(M, x) < \delta$.
iii) $f(y) \leq f(x)$ *for* $y \in F(x)$.
iv) *For* $x \in X$ *and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $f(x) - \varepsilon < f(y)$ *if* $d(x, y) < \delta$.

PROOF. Necessity. For $x \notin M$, we let

$$I(x) = \{r > 0 : x \notin \overline{F(B(M,r))}\}.$$

In view of negative equistability of M, we have $I(x) \neq \emptyset$. Define a map $f : X \to \mathbf{R}^+$ by

$$f(x) = \begin{cases} \sup I(x), & x \notin M. \\ 0, & x \in M. \end{cases}$$

Now we will show that $f(x) \leq d(M, x)$ for any $x \in X$. If $x \in M$, the result is clear. So we suppose that $x \notin M$. Then we have that $r > d(M, x)$ implies $x \in B(M, r)$. Since $x \in B(M, r) \subset \overline{B(M, r)}$, we obtain $r \notin I(x)$. Hence $r \in I(x)$ implies $r \leq d(M, x)$. This means that $f(x) \leq d(M, x)$.

If $x \notin M$, then $I(x) \neq \emptyset$. If $r \in I(x)$, then $f(x) \geq r > 0$. Hence $f(x) = 0$ implies $x \in M$. This proves conditon i).

For $\varepsilon > 0$, if we take $\delta = \varepsilon$, then the condition ii) is clear.

Let $y \in F(x)$ and $x \in M$. By Lemma 3.7, we have $F(x) \subset M$, and so $f(y) = 0 = f(x)$. Choose $x \notin M$. Suppose that $I(y) \not\subset I(x)$. Then there is $r \in I(y) - I(x)$. Hence we can select $\delta > 0$ such that $B(y, \delta) \cap F(B(M, r)) = \emptyset$ by $y \in \overline{F(B(M, r))}$. Since a semiflow (x, π) is negatively backward semicontinuous, there is $\varepsilon > 0$ such that $B(y,\delta)\cap F(z) \neq \emptyset$ for any $z \in B(x,\varepsilon)$. Since $x \in F(B(M,r))$, we get $B(x,\varepsilon)\cap F(B(M,r)) \neq \emptyset$. Choose $w \in B(x,\varepsilon)\cap F(B(M,r))$. Since $\emptyset \neq B(y,\delta)\cap F(w) \subset B(y,\varepsilon)\cap F(B(M,r))$, we obtain a contradiction, and so proves the condition iii).

If $f(x) - \varepsilon < f(x)$ for $x \notin M$, $\varepsilon > 0$, there is $r \in I(x)$ such that $f(x) - \varepsilon < r$. Since $x \notin \overline{F(B(M,r))}$, there is $\delta > 0$ satisfying $B(x,\delta) \cap F(B(M,r)) = \emptyset$. Let $y \in B(x,\delta)$. Since $y \notin \overline{F(B(M,r))}$, we have $r \in I(y)$. Hence we obtain the condition iv).

Sufficiency. Let $x \notin M$ and $\varepsilon = f(x)$. Then by condition i) we have $\varepsilon > 0$. Also, by condition ii), there exists $\delta > 0$ such that $f(y) < \varepsilon/2$ for $d(M,y) < \delta$. Now we show that $x \notin \overline{F(B(M,r))}$. Assume that $x \in \overline{F(B(M,r))}$. Then we have $B(x,1/n) \cap F(B(M,\delta)) \neq \emptyset$ for each $n \in N$. Choose $y_n \in B(x,1/n) \cap F(B(M,\delta))$. There is $z_n \in B(M,\delta)$ such that $y_n \in F(z_n)$. By condition iv), there is $\eta > 0$ such that $f(x) - \varepsilon/2 = \varepsilon/2 < f(y)$ whenever $d(x,y) < \eta$. Since the sequence (y_n) converges to x, there is $m > 0$ if $d(x,y_m) < \eta$. By condition iii), we get $\varepsilon/2 < f(y_m) = f(z_m) < \varepsilon/2$. Hence we obtain a contradiction. This completes the proof.

The author wishes to thank Prof. Chin-Ku Chu and Prof. Jong Suh Park for their help.

References

[1] P. N. Bhatia and O. Hajek, *Local Semidynamical systems*, Lecture Notes in Mathematics, Vol. 90, Springer Verlag, Berline, 1968.

[2] P. N. Bhatia and G. P. Szegö, *Stability theory of Dynamical systems*, Berline, 1970.

[3] S. Elaydi and S. K. Kaul, *Semiflows with global extensions*, Nonlinear Analysis, Theory, Methods and Applications **10** (1986), 713–726.

[4] S. K. Kaul, *A semidynamical system associated with a general control system*, Nonlinear Analysis, Theory, Methods and Applications **13** (1989), 1–5.

Department of Mathematics
Hoseo University
Asan, 377-850, Korea