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Recurrence and the Shadowing Property

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ABSTRACT. In this paper we give a necessary condition for a homeomorphism restricted to its nonwandering set to have the shadowing property. Also, we consider a homeomorphism which cannot have the shadowing property

1. Introduction and Definitions

As an attempt to approach some problems of smooth dynamical systems theory from a nondifferential point of view, homeomorphisms of metric spaces with the shadowing property are studied together with the related concepts of various recurrence properties such as periodicity, recurrence, nonwanderingness and chain recurrence.

The shadowing property of P. Walters[8] plays an important role in the stability theory of dynamical systems. Related results, due to Hurley, Walters, Aoki, Morimoto, Ombach, can be founded in references. Recently, Aoki [2] established that a homeomorphism of a compact metric space has the shadowing property then so does its restriction to its nonwandering set. Also, Ombach [6] obtained some results for expansive homeomorphisms with the shadowing property. Moreover, we can find some examples of homeomorphisms without the shadowing property [2, 5].

In this paper we give a necessary condition for a homeomorphism restricted to its nonwandering set to have the shadowing property.

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Also, we show that if a homeomorphism f of a compact metric space has the shadowing property and its restriction to its nonwandering set is topologically transitive, then f is a nonwandering homeomorphism. Finally, we prove that a positively or negatively recurrent homeomorphism of a connected metric space which is not minimal cannot have the shadowing property.

Throughout this paper we assume that the space X is a metric space with a metric d, and let $f : X \to X$ be a homeomorphism.

For a homeomorphism $f : X \to X$, define the *recurrent set*

$$C(f) = \{x \in X : x \in \omega_f(x) \cap \alpha_f(x), \text{where } \omega_f(x) \text{ and } \alpha_f(x) \text{ denote the positive limit set and negative limit set of } x \text{ for } f, \text{ respectively}\},$$

the *nonwandering set*

$$\Omega(f) = \{x \in X : \text{for every neighborhood } U \text{ of } x \text{ and integer } n_0 > 0 \text{ there is } n \geq n_0 \text{ such that } f^n(U) \cap U \neq \phi\}.$$

A sequence of points $\{x_i\}_{i\in(a,b)}$ $(-\infty \leq a < b \leq \infty)$ is called a δ-*pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A finite δ-pseudo-orbit $\{x_0 = x, x_1, \cdots, x_n = y\}$ is called a δ-pseudo-orbit from x to y. A sequence $\{x_i\}_{i\in(a,b)}$ is called to be ϵ-*shadowed* by $x \in X$ if $d(f^i(x), x_i) < \epsilon$ holds for $i \in (a, b)$. We say that a homeomorphism f has the *shadowing property* also called *pseudo-orbit-tracing property* if for every $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo-orbit f can be ϵ-shadowed by some point x in X. Given $x, y \in X$ and $\alpha > 0, x$ is α-*related* to y (written $x \overset{\alpha}{\sim} y$) if there are α-pseudo-orbits of f from x to y and y to x. A point x in X is α-*chain recurrent* for f if for each $\beta > \alpha \geq 0$ there is a β-pseudo-orbit from x to x. The *chain recurrent set* of $f, CR(f)$, is the set of points in X that are α-chain recurrent for all $\alpha \geq 0$. There is a natural equivalence relation that is defined on $CR(f)$ by calling two points x, y be *related* (written $x \sim y$) if $x \overset{\alpha}{\sim} y$ holds for each $\alpha > 0$. Each equivalence class is called a *chain component.*

Let $B(x, \epsilon)$ denote $\{y \in X : d(x, y) < \epsilon\}$ and $\overline{M}$ denote the closure of $M \subset X$.

Basic terminologies are followed from [3].

2. Recurrent Sets and The Shadowing Property

Here we give a necessary condition for a homeomorphism restricted to its nonwandering set to have the shadowing property. To see this we need some lemmas.

Given $x, y \in X$, x is *α-related to y in $A \subset X$* means there are α-pseudo-orbits from x to y and y to x in A. $O_f(x), O_f^+(x)$ and $O_f^-(x)$ denote the *orbit, positive orbit* and *negative orbit* of a point x, respectively.

Take and fix $\alpha > 0$. Then we can split $\overline{C(f)}$ into a union $\overline{C(f)} = \bigcup C_\lambda$ of equivalence classes C_λ under the α-relation in $\overline{C(f)}$.

LEMMA 2.1. *For $\alpha > 0$, every $x \in \overline{C(f)}$ is α- related to $f^k(x)$ in $\overline{C(f)}$ for all $k > 0$.*

PROOF. Let $x \in \overline{C(f)}$. Using the continuity of f we can take $\beta > 0$ with $\beta < \alpha$ such that $d(z, x) < \beta$ implies $\max\{d(f^i(x), f^i(z)) : 0 \leq i \leq k+1\} < \alpha$. Since $x \in \overline{C(f)}$ there is $z \in B(x, \beta) \cap C(f)$ and integer $\ell \geq k+2$ with $f^\ell(z) \in B(x, \beta)$. then the sequence $\{f^k(x), f^{k+1}(z), f^{k+2}(z), \cdots, f^{\ell-1}(z), x\}$ is an α-pseudo-orbit in $\overline{C(f)}$ from $f^k(x)$ to x. Obviously, $\{x, f(x), \cdots, f^k(x)\}$ is α-pseudo-orbit from x to $f^k(x)$.

This lemma shows that each equivalence calss C_λ is f-invariant.

LEMMA 2.2. *Each equivalence class C_λ is open and closed in $\overline{C(f)}$.*

PROOF. First, we show that each C_λ is closed in $\overline{C(f)}$. To see this, we choose a sequence $\{x_i\}$ in C_λ which converges to x'. Take $\beta > 0$ with $0 < \beta < \frac{1}{2}\alpha$ such that $f(B(x',\beta)) \subset B(f(x'),\frac{1}{2}\alpha)$. Let $s > 0$ be an integer with $d(x_s,x') < \beta$. Also, let $y \in B(x',\beta) \cap C(f)$ and $f^\ell(y) \in B(x',\beta)$ for some $\ell > 0$. Then the sequence $\{x', f(y), f^2(y), \cdots, f^{\ell-1}(y), x_s\}$ is an α-pseudo-orbit from x' to x_s in $\overline{C(f)}$. On the otherhand, take $\gamma > 0$ such that $B(x_s,\gamma) \subset B(x',\frac{1}{2}\alpha)$ and $f(B(x_s,\gamma)) \subset B(f(x_s),\frac{1}{2}\alpha)$. Let $w \in B(x_s,\gamma) \cap C(f)$ and $f^n(w) \in B(x_s,\gamma)$ for some $n > 0$. Then the sequence $\{x_s, f(w), f^2(w), \cdots, f^{n-1}(w), x'\}$ is an α-pseudo-orbit from x_s to x' in $\overline{C(f)}$. This implies that x_s is α-related to x' in $\overline{C(f)}$ and so $x' \in C_\lambda$. Hence C_λ is closed in $\overline{C(f)}$.

Next, we show that C_λ is open in $\overline{C(f)}$. Let $x \in C_\lambda$. for every $y \in C_\lambda$ there is an α-pseudo-orbit $\{x_0 = x, x_1, \cdots, x_p = y\}$ from x to y in $\overline{C(f)}$. Choose ξ with $0 < \xi < \frac{1}{3}\alpha$ such that $f(B(x_0,\xi)) \subset B(x_1,\alpha)$. Then for every $a \in B(x_0,\xi) \cap \overline{C(f)}$ the sequence $\{a, x_1, x_2, \cdots, x_p\}$ is an α-pseudo-orbit from a to y. On the ortherhand, let $\{y_0 = y, y_1, \cdots, y_q = x\}$ be an α-pseudo-orbit from y to x in $\overline{C(f)}$. Since $x \in B(f(y_{q-1}),\alpha)$, we can choose a point z in $B(f(y_{q-1}),\alpha) \cap B(x,\xi) \cap C(f)$. Since $z \in B(x,\xi) \cap C(f)$ there is an integer $m > 0$ with $f^m(z) \in B(x,\xi)$. Therefore, for each a in $B(x,\xi) \cap \overline{C(f)}$ the sequence $\{y_0, y_1, \cdots, y_{q-1}, z, f(z), \cdots, f^{m-1}(z), a\}$ is an α-pseudo-orbit from y to a. This means that $B(x,\xi) \cap \overline{C(f)} \subset C_\lambda$. Hence C_λ is open in $\overline{C(f)}$, and so completes the proof.

LEMMA 2.3. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If $f : X \to X$ has the shadowing property, then for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-shadowed by some point in X. In particular, every periodic δ-*

pseudo-orbit can be ϵ-shadowed by some point in $C(f)$.

PROOF. Let $\epsilon > 0$ be given and $\delta = \delta(\frac{1}{2}\epsilon)$ be the number with the property of the shadowing property for f. Let $\{x_i\} = \{\cdots, x_{-1} = y_n, x_0 = y_0, \cdots, x_n = y_n, x_{n+1} = y_0, \cdots\}$ be a periodic δ-pseudo orbit. Since f has the shadowing property there is a point y with $d(f^i(y), x_i) < \frac{1}{2}\epsilon$ for all $i \in \mathbb{Z}$. So $d(f^{ni+j}(y), y_j) < \frac{1}{2}\epsilon$ for all $i \in \mathbb{Z}$ and $0 \leq j < n$. In particular, $f^{ni}(y) \in B(x_0, \frac{1}{2}\epsilon)$ for all $i \in \mathbb{Z}$ and hence we have $\overline{O_{f^n}(y)} \subset \overline{B(x_0, \frac{1}{2}\epsilon)}$. Since $\omega_{f^n}(y)$ contained in $\overline{O_{f^n}(y)}$ is compact and invariant for the homeomorphism $f^n : X \to X$ it contains a minimal set M for f^n. By the minimality of M we have $\overline{O_{f^n}(z)} = \omega_{f^n}(z) = \alpha_{f^n}(z) = M$ for each z in M. Thus $z \in \omega_{f^n}(z) \cap \alpha_{f^n}(z) \subset \omega_f(z) \cap \alpha_f(z)$ and so $z \in C(f)$. Now, we need to show that $\{x_i\}$ is ε-shadowed by z. Assume that $d(f^{ni_0+j_0}(z), y_{j_0}) \geq \varepsilon$ for some $i_0 \in \mathbb{Z}$ and $0 \leq j_0 < n$. Since $z \in \omega_{f^n}(y)$ we can choose a sequence $\{f^{n\ell_i}(y)\}$ converges to z for some $\ell_i \to +\infty$. Thus, by continuity of f it follows that there is sufficiently large L in $\{\ell_i\}$ satisfying $d(f^{n(L+i_0)+j_0}(y), y_{j_0}) > \frac{1}{2}\varepsilon$. But this contradicts the fact that $d(f^{ni+j}(y), y_j) < \frac{1}{2}\varepsilon$ for all $i \in \mathbb{Z}$ and $0 \leq j < n$. Therefore we conclude that $\{x_i\}$ is ε-shadowed by z, and the proof is complete.

Using these lemmas we can obtain the following result. For closed subsets A, B of X, we denote $d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$.

THEOREM 2.4. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If $f : \Omega(f) \to \Omega(f)$ has the shadowing property, then so does $f : \overline{C(f)} \to \overline{C(f)}$.*

PROOF. Let $\varepsilon > 0$ be given and select $\delta = \delta(\frac{1}{2}\varepsilon) > 0$ as in Lemma 2.3. Lemma 2.2 asserts that $\overline{C(f)}$ can be split into a finite union of equivalence classes C_λ under δ-relation in $\overline{C(f)}$, i.e. $\overline{C(f)} = \bigcup_{i=1}^{k} C_i$. Let $\alpha = \min\{d(C_i, C_j) : 1 \leq i, j \leq k\}$. Take β with $0 < \beta < \min\{\delta, \alpha\}$

and let $\{x_i\}$ be any β-pseudo orbit in $\overline{C(f)}$. We claim that there exists a point in $\overline{C(f)}$ ε-shadowing the β-pseudo-orbit $\{x_i\}$. Since each C_i is f-invariant the pseudo-orbit $\{x_i\}$ is contained in some C_j. Thus for each integer $n > 0$ there is δ-pseudo-orbit $\{z_0^n, z_1^n, \cdots, z_{n_k}^n\}$ from x_n to x_{-n}. So we can obtain a periodic δ-pseudo-orbit A_n by putting

$$A_n = \{\cdots, z_{n_k-1}^n, x_{-n}, x_{-n+1}, \cdots, x_{n-1}, x_n, z_1^n, \cdots, z_{n_k-1}^n, x_{-n}, \cdots\}.$$

By Lemma 2.3, there is z_n in $C(f)$ ε-shadowing the pseudo-orbit A_n with $d(f^i(z_n), x_i) < \frac{1}{2}\varepsilon$ for $-n \leq i \leq n$. Let a subsequence of $\{z_n\}$ converge to z as $n \to +\infty$. Then $z \in \overline{C(f)}$. Using the continuity of f we can easily prove that $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. It follows that $f : \overline{C(f)} \to \overline{C(f)}$ has the shadowing property, and so completes the proof.

REMARK. In general, the converse of the above theorem does not hold. For example, $X = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Define a continuous real flow $g : X \times \mathbb{R} \to X$ on X using an autonomous system of differential equation satisfying : the points $(0,1)$ and $(0,-1)$ are fixed for g. For $p = (x,y) \in S^1$, let $\omega_g(p) = (0,-1)$ and $\alpha_g(p) = (0,1)$ if $x < 0$ and $\omega_g(p) = (0,1)$ and $\alpha_g(p) = (0,-1)$ if $x > 0$. The phase portrait of $\{(x,y) \in X : x^2 + y^2 < 1\}$ is the same as that of the flow defined by the differential equations (polar coordinate)

$$\frac{dr}{d\theta} = r(1-r), \quad \frac{d\theta}{dt} = 1.$$

Consider a homeomorphism $f : X \to X$ defined by $f(x) = g(x,1)$. For the homeomorphism f, we have $\Omega(f) = S^1 \cup \{0,0\}$ and $\overline{C(f)} = \{(0,0),(0,1),(0,-1)\}$. Obviously, $f : \Omega(f) \to \Omega(f)$ does not have the shadowing property though $f : \overline{C(f)} \to \overline{C(f)}$ has the shadowing property.

3. Recurrent Sets and Chain Component Sets

In this section we obtain some properties of chain component sets and relations between the various recurrent sets.

Let X_1 and X_2 be metric spaces with X_2 compact. Let FX_2 be the set of all closed nonempty subsets of X_2 with the Hausdorff metric

$$\rho(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

Recall that a subset S of a topological space X_1 is residual if S can be realized as a countable intersection of open dense subsets of X_1.

LEMMA 3.1. *Let $h : X_1 \to FX_2$ be either upper or lower semicontinuous. Then the set of continuity points of h is a residual subset of X_1.*

This is [3 , Lemma 1]. The map h is upper semicontinuous at x in X_1 if for any sequence $x_n \to x$, if y_n is in $h(x_n)$ and $y_n \to y$, then y is in $h(x)$.

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers. Note that for any $\alpha \geq 0$ the set of all α-chain recurrent points in X is closed.

THEOREM 3.2. *Let $f : X \to X$ be a homeomorphism of a comapct metric space X. Then there is a residual subset $\mathbb{R}_1$ of $\mathbb{R}^+$ such that the map $H : \mathbb{R}^+ \to FX$ defined by*

$$H(\alpha) = \text{the set of } \alpha\text{-chain recurrent points in } X$$

is continuous at each numbers in $\mathbb{R}_1$.

PROOF. By the compactness of X, for each $\alpha \geq 0$, the set of α-chain recurrent points is not empty. From Lemma 3.1 it is enough to show that the map H is upper semicontinuous. Let a sequence $\{\alpha_n\}$ in $\mathbb{R}^+$ converge to α and $\{x_n\}$ in X converge to x with $x_n \in H(\alpha_n)$. We will show that $x \in H(\alpha)$. Let $\varepsilon > 0$ be arbitrary. Choose a number s satisfying that

$$\alpha_s < \alpha + \frac{1}{2}\varepsilon, \quad d(x_s, x) < \frac{1}{2}\varepsilon \quad \text{and} \quad d(f(x), f(x_s)) < \frac{1}{4}\varepsilon.$$

For $\beta = \alpha_s + \frac{1}{4}\varepsilon$, there is β-pseudo-orbit $\{z_0 = x_s, z_1, \cdots, z_i, z_{i+1} = x_s\}$ from x_s to x_s. Then the sequence $\{x, z_1, \cdots, z_i, x\}$ is $(\alpha + \varepsilon)$-pseudo orbit from x to x since

$$d(f(x), z_1) \le d(f(x), f(x_s)) + d(f(x_s), z_1) < \frac{1}{4}\varepsilon + \beta < \alpha + \varepsilon,$$
$$d(f(z_k), z_{k+1}) < \beta < \alpha + \varepsilon, \quad 1 \le k < i, \quad \text{and}$$
$$d(f(z_i), x) \le d(f(z_i), x_s) + d(x_s, x) < \beta + \frac{1}{4}\varepsilon < \alpha + \varepsilon.$$

This implies $x \in H(\alpha)$ and thus H is upper semicontinuous.

In general, the following inclusions hold :

$$\overline{\mathrm{Per}(f)} \subset \overline{C(f)} \subset \Omega(f) \subset CR(f),$$

where $\overline{\mathrm{Per}(f)}$ denote the set of periodic points of f. It is a good problem to find various f's where the inclusions are strict or equalities. For this problem we can see many known results [2, 6, 7].

THEOREM 3.3. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If $f : X \to X$ has the shadowing property, then we have $\overline{C(f)} = \Omega(f)$.*

PROOF. Let $x \in \Omega(f)$ and $\varepsilon > 0$ be given. Let $\delta > 0$ be the number with the property of the shadowing property of f. Choose $\alpha > 0$ small enough that $d(x, y) < \alpha$ implies that $d(f(x), f(y)) < \delta$. Since $x \in \Omega(f)$ we can select y in $B(x, \alpha)$ with $f^{\ell}(y) \in B(x, \alpha)$ for some integer $\ell > 1$. Then $\{x_i\} = \{\cdots, f^{\ell-1}(y), x, f(y), f^2(y), \cdots, f^{\ell-1}(y), x, \cdots\}$ is a periodic δ-pseudo orbit. Hence there is $z \in C(f)$ ε-shadowing the pseudo-orbit $\{x_i\}$ with $d(x,z) < \varepsilon$. Since ε was arbitrary this show that $x \in \overline{C(f)}$, and so completes the proof.

In [4], Hurley claimed that f is topologically stable, X is connected and $\Omega(f)$ has interior then $\Omega(f) = X$. Moreover, it was shown that if an expansive homeomorphism f of a compact metric X has the shadowing property and $f|_{\Omega(f)}$ is topologically transitive, then $X = \Omega(f)$ [2]. We extend this result to arbitrary homeomorphisms on compact spaces. Note that expansiveness are missing.

THEOREM 3.4. *Let $f : X \to X$ be a homeomorphism of a compact metric space X which has the shadowing property and $f|_{\Omega(f)}$ is topologically transitive, then $X = \Omega(f)$.*

PROOF. Suppsoe that $X \neq \Omega(f)$. Since $\omega_f(x_0)$ and $\alpha_f(x_0)$ are contained in $\Omega(f)$ for $x_0 \in X\backslash\Omega(f)$, we have $\overline{O_f^+(x_0)} \cap \Omega(f) \neq \emptyset$ and $\overline{O_f^-(x_0)} \cap \Omega(f) \neq \emptyset$. Let $\varepsilon = d(x_0, \Omega(f))$ and $\delta > 0$ be the number with the property of the shadowing property for f. Then for sufficiently large n, $d(\Omega(f), f^n(x_0)) < \delta$ and $d(\Omega(f), f^{-n}(x_0)) < \delta$ and hence there are points $x_{n+1}, x_{-n-1} \in \Omega(f)$ with $d(f^n(x_0), x_{n+1}) < \delta$ and $d(f^{-n}(x_0), f(x_{-n-1})) < \delta$. Since $f|_{\Omega(f)}$ is topologically transitive there is δ-pseudo-orbit
$\{x_{n+1}, z_1, \cdots, z_k, x_{-n-1}\}$ from x_{n+1} to x_{-n-1} in $\Omega(f)$. So we can obtain a periodic δ-pseudo-orbit

$$\{y_i\} = \{\cdots, x_{-n-1}, f^{-n}(x_0), \cdots, x_0, f(x_0), \cdots, f^{n-1}(x_0), x_{n+1}, z_1, \cdots, z_k, x_{-n-1}, \cdots\}.$$

Since f has the shadowing property there is y in $C(f)$ ε-shadowing the pseudo-orbit $\{y_i\}$ with $d(y, x_0) < \varepsilon$. This means $d(x_0, \Omega(f)) < \varepsilon$ and this contradicts the fact that $d(x, \Omega(f)) = \varepsilon$.

THEOREM 3.5. *Let $f : X \to X$ be a homeomorphism of a compact metric space X and f have the shadowing property. Let $x_0 \notin \Omega(f)$ and $\omega_f(x_0) \subset C_\lambda$ for some chain component C_λ of f. Then there is an open set U containing x_0 such that each negative limit set of a point x in U does not intersect C_λ.*

PROOF. It is known that the positive and negative limit sets of any point are contained in a single chain component, respectively. Assume that there is no such open set. Let $d(x_0, \Omega(f)) = \varepsilon$ and $\delta = \delta(\varepsilon)$ be the number with the property of the shadowing property for f. Since each neighborhood of x_0 contains points whose negative limit set intersect C_λ, using the same method as the proof of Theorem 3.4, we can construct a periodic δ-pseudo orbit $\{x_i\}$ containing x_0. So there is y in $C(f)$ ε- shadowing the $\{x_i\}$ with $d(x_0, y) < \varepsilon$. Thus contradicting.

4. Homeomorphisms Without The Shadowing Property

In this section we consider homeomorphisms which cannot have the shadowing property. For example, every isometry of a compact Riemannian manifold of positive dimension has not the shadowing property [5] and every distal homeomorphism of a compact connected metric space has not the shadowing property [1].

A homeomorphism $f : X \to X$ is called *positively(negatively) recurrent* if $x \in \omega_f(x)(x \in \alpha_f(x))$ for each x in X. A homeomorphism $f : X \to X$ is called *minimal* if orbit closure of any point in X is dense in X. It is known that a minimal homeomorphism of compact connected metric space which is not one point does not have the shadowing property [2].

An ε-chain from x to y is a finite sequence $\{z_0 = x, z_1, \cdots, z_n = y\}$ such that $d(z_i, z_{i+1}) < \varepsilon$ for each $0 \leq i < n$.

THEOREM 4.1. *Let $f : X \to X$ be a homeomorphism of a con-*

nected metric space X which is not minimal. If f is positively or negatively recurrent, then f does not have the shadowing property.

PROOF. Suppose that f has the shadowing property. Since f is not minimal there is $y_0 \in X$ with $\overline{O_f(y_0)} \neq X$. Let $x_0 \in X \setminus \overline{O_f(y_0)}$ and $d(x_0, \overline{O_f(y_0)}) = 3\varepsilon$. Let $\delta = \delta(\varepsilon)$ with $0 < \delta < \varepsilon$ be a number with the property of the shadowing property for f. By connectedness of X there is $\frac{\delta}{2}$-chain $\{z_0 = x_0, z_1, z_2, \cdots, z_{n+1} = y_0\}$ from x_0 to y_0.

First, assume that f is positively recurrent. Then, for each $z_i, z_i \in \omega_f(z_i)$. So there is positive integer $k(i)$ such that $d(f^{k(i)+1}(z_i), z_i) < \frac{1}{2}\delta$ for each $0 \leq i \leq n+1$. Thus we can construct δ-pseudo-orbit by putting

$$\{a_i\}_{i=0}^{\infty} = \{z_0, f(z_0), \cdots, f^{k(0)}(z_0), z_1, f(z_1), \cdots, f^{k(1)}(z_1), z_2, \\ \cdots, z_n, f(z_n), \cdots, f^{k(n)}(z_n), z_{n+1}, f(y_0), f^2(y_0), f^3(y_0), \cdots\}.$$

Since f has the shadowing property there is a in X such that $d(f^i(a), a_i) < \varepsilon$ for all $i \geq 0$. Let $K = k(0) + k(1) + \cdots + k(n) + n$. Then

$$d(a, a_0) = d(a, x_0) < \varepsilon$$
$$d(f^{K+i}(a), f^i(y_0)) < \varepsilon, \quad i \geq 0.$$

So we have

$$O_f^+(f^K(a)) \subset B(O_f^+(y_0), \varepsilon) \subset B(\overline{O_f^+(y_0)}, \varepsilon) \subset B(\overline{O_f(y_0)}, \varepsilon)$$

and hence

$$\omega_f(a) = \omega_f(f^K(a)) \subset \overline{O_f^+(f^K(a))} \subset \overline{B(\overline{O_f(y_0)}, \varepsilon)}.$$

However, by the positive recurrence of $f, a \in \omega_f(a)$ and therefore $d(\overline{O_f(y_0)}, a) \leq \varepsilon$. So we have

$$d(x_0, \overline{O_f(y_0)}) \leq d(x_0, a) + d(a, \overline{O_f^+(y_0)}) \leq 2\varepsilon.$$

Thus contradicting that $3\varepsilon = d(x_0, \overline{O_f(y_0)})$.

Next, assume that f is negatively recurrent. Then for each point z_i in the $\frac{1}{2}\delta$-chain $\{z_0, z_1, \cdots, z_{n+1}\}$ from x_0 to y_0, there is negative integer $c(i)$ such that $d(f^{c(i)}(z_i), z_i) < \frac{1}{2}\delta$. Let define a sequence $\{b_i\}$ by putting

$$\begin{aligned}\{b_i\}_{i=-\infty}^{0} = \{&\cdots, f^{-2}(y_0), f^{-1}(y_0), f^{c(n+1)}(z_{n+1}), f^{c(n+1)+1}(z_{n+1}), \cdots,\\ &f^{-1}(z_{n+1}), f^{c(n)}(z_n), f^{c(n)+1}(z_n), \cdots, f^{-1}(z_n), f^{c(n-1)}(z_{n-1}),\\ &f^{c(n-1)+1}(z_{n-1}), \cdots, f^{-1}(z_2), f^{c(1)}(z_1), f^{c(1)+1}(z_1), \cdots,\\ &f^{-1}(z_1), z_0\}.\end{aligned}$$

Obviously $\{b_i\}$ is δ-pseudo-orbit. So there is a point b in X with $d(f^i(b), b_i) < \varepsilon$ for each $i \leq 0$. In particular, we have

$$\begin{aligned}&d(b, b_0) = d(b, x_0) < \varepsilon \quad \text{and}\\ &d(f^{L+i}(b), f^i(y_0)) < \varepsilon, \quad i \leq -1,\end{aligned}$$

where $L = c(1) + c(2) + \cdots + c(n+1)$. Similarly, we can obtain the following fact that

$$b \in \alpha_f(b) \subset \overline{B(\overline{O_f(y_0)}, \varepsilon)}.$$

That is $d(\overline{O_f(y_0)}, b) \leq \varepsilon$, so that

$$d(x_0, \overline{O_f(y_0)} \leq d(x_0, b) \ + \ d(b, \overline{O_f(y_0)}) \leq 2\varepsilon.$$

Hence we can derive the same contradiction, and so completes the proof.

COROLLARY 4.2. *Let $f : X \to X$ be a homeomorphism of a compact metric space which is not one point. If all orbit closures are minimal sets, then f does not have the shadowing property.*

PROOF. If f is minimal then it is known that f does not have the shadowing property. Also, if f is not minimal, then f is positively and negatively recurrent since all orbit closures are minimal. From this fact the conclusion holds by the above theorem.

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