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N-Dimensional sine and cosine functions

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ABSTRACT. We introduce n-dimensional sine and cosine functions which are generalization of the usual sine and cosine functions. We establish the property that n-dimensional sine and cosine functions have.

1. Introduction

S. T. Lin and Y. Lin [1] established the n-dimensional Pythagorian theorem. We shall establish generalized sine and cosine functions $\sin(n)x$ and $\cos(n)x$. If $n = 2$, $\sin(2)x = \sin x$ and $\cos(2)x = \cos x$, the usual sine and cosine functions. We list some properties of n-dimensional sine and cosine functions in the section 4. We have graphs of n-dimensional sine and cosine functions for $n = 2$, 3, 5, 10, 20, 30, 40, 50, 100, 200.

2. Definitions

In this section, we define simplexes, a k-simplex or a k-dimensional right triangle, the content of a simplex, and a sine (or a generalized sine) function of $(n-1)$ variables in an n-dimensional Euclidean space. Let R be the real line, and let $R^n = \{(x_1, \ldots, x_n) : x_i \in R\}$ be the n-dimensional Euclidean space.

DEFINITION 1. A set $S = \{A_i \in R^m : i = 0, 1, 2, \ldots, m\}$ is said to be in general position if the set of vectors $A_1 - A_0$, $A_2 - A_0, \ldots, A_m - A_0$

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is linearly independent. We define a simplex $\triangle[A_0, A_1, \ldots, A_m] = \{x = \sum a_i A_i : \sum a_i = 1, \text{ and } a_i \geq 0\}$, assuming that S is in general position. We call it the simplex spanned by the set S, and denoted it by often S^m instead of $\triangle[A_0, \ldots, A_m]$, A_i is called a vertex of the simplex S^m, and m is the dimension of it.

DEFINITION 2. Let S^m be a simplex of dimension m. A 1-simple $\triangle[A_i, A_j]$ is called an edge of the simplex. If there is a vertex $e(0)$ (in S) of S^m such that $\triangle[e(0), A_i]$ and $\triangle[e(0), A_j]$ are orthogonal ($i \neq j$, $A_i \neq e(0) \neq A_j$), then we say that S^m is a right simplex (or an m-dimensional right triangle).

DEFINITION 3. Let $S^m = \triangle[A_0, A_1, \ldots, A_m]$ be an m-simplex. An $(m-1)$-simplex $\triangle[A_0, A_1, \ldots, (A_i), \ldots, A_m]$ is a simplex obtained from S^m by deleting A_i. We define the *content* of $S^m (m = 3)$ as the volume of S^m and it will be denoted by $|S^m|$. The content $|\triangle[A_1, A_2]|$ of the 1-simplex $\triangle[A_1, A_2]$ is the length of the simplex $\triangle[A_1, A_2]$, $|\triangle[A_1, A_2, A_3]|$ is the area of the 2-simplex, and $|\triangle[A_1, A_2, A_3, A_4]|$ is defined as the volume of it. Similarly, we define the *content* of an $(m-1)$-simplex $\triangle[A_0, A_1, \ldots, (A_i), \ldots, A_m] = T$ as the volume of T and we denote the content of T by $|T|$ (see [2], for the contents).

3. N-dimensional sine function

In this section, we define a generalized sine function $\sin(n)x$, $(n \geq 3)$ in Definition 4.

DEFINITION 4. In R^n, we define $e(0) = (0, 0, \ldots, 0) \in R^n$, $e(1) = (1, 0, 0, \ldots, 0)$, $e(2) = (0, 1, 0, 0, \ldots, 0), \ldots, e(n) = (0, 0, \ldots, 0, 1)$. Let a_i be a positive number ($i = 1, 2, \ldots, n$). Define $A_i = a_i e(i)$. For instance $A_1 = (a_1, 0, 0, \ldots, 0)$. Let $e(0) = 0$. We define an angle $\alpha_{ij} = \angle O A_i A_j$ for the triangle $\triangle[O, A_i, A_j]$.

We define a generalized sine function of $(n-1)$ variables as follows:

$$\sin(\alpha_{i1},\alpha_{i2},\ldots,(\alpha_{ii}),\ldots,\alpha_{in}) = (\alpha_i) = \sin(\alpha_i)$$
$$= \frac{|\triangle[O,A_1,\ldots,(A_i),\ldots,A_n]|}{|\triangle[A_1,A_2,\ldots,A_n]|}.$$

If $\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{in} = x$, then we define $\sin(\alpha_i)$ as $\sin(n)x = \sin(x,x,\ldots,x) = \sin(\alpha_i)$. We shall show that $\sin(n)x = \sin x/[1 + (n-2)\cos^2 x]^{\frac{1}{2}}$ in Proposition 1. We need a symmetric matrix U_n of order $(n+1)$.

DEFINITION 5. We define a symmetric matrix $U_n = (u_{ij})$ as follows:

$$\begin{aligned} u_{ij} &= 0, && \text{if } i = j;\\ &= 1, && \text{if } i = 1, j = 2,3,\ldots,n+1;\\ &= 1, && \text{if } j = 1, i = 2,3,\ldots,n+1;\\ &= \csc^2\alpha, && \text{if } i = 2, j = 3,4,\ldots,n+1;\\ &= \csc^2\alpha, && \text{if } j = 2, i = 3,4,\ldots,n = 1;\\ &= 2, && \text{otherwise} \end{aligned}$$

We prove the following lemma.

LEMMA 1. *The determinant* $\det(U)$ *of the matrix* $U = U_n$ *is equal to* $\det(U) = [(-1)^n 2^{n-1}][(1+(n-2)\cos^2\alpha)/\sin^2\alpha]$.

PROOF. We can see that

$$\det(U) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \csc^2\alpha & \csc^2\alpha & \csc^2\alpha & \cdots & \csc^2\alpha \\ 1 & \csc^2\alpha & 0 & 2 & 2 & \cdots & 2 \\ 1 & \csc^2\alpha & 2 & 0 & 2 & \cdots & 2 \\ 1 & \csc^2\alpha & 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \csc^2\alpha & 2 & 2 & 2 & \cdots & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -2\csc^2\alpha & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 2 & 2 & \cdots & 2 \\ 1 & 0 & 2 & 0 & 2 & \cdots & 2 \\ 1 & 0 & 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 2 & 2 & 2 & \cdots & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2-2\csc^2\alpha & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -2 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & -2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & -2 \end{vmatrix}$$

$$= \begin{vmatrix} (n-1)/2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 2(1-\csc^2\alpha) & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -2 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & -2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & -2 \end{vmatrix}$$

$$= \begin{vmatrix} (n-1)/2 & 1 \\ 1 & 2(1-\csc^2\alpha) \end{vmatrix} \begin{vmatrix} -2 & 0 & 0 & \dots & 0 \\ 0 & -2 & 0 & \dots & 0 \\ 0 & 0 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{vmatrix}$$
$$= [(-1)^n 2^{n-1}][(1+(n-2)\cos^2\alpha)/\sin^2\alpha].$$

This proves the lemma.

LEMMA 2. *The content* $|\triangle[A_1, A_2, \dots, A_n]|$ *of an* $(n-1)$*-simplex* $\triangle[A_1, A_2, \dots, A_n]$ *is obtained by the following* [1]*:* $\det(V) = (-1)^n 2^{n-1}[(n-1)!]^2 |\triangle[A_1, A_2, \dots, A_n]|^2$, *where* $V = (v_{ij})$ *is a matrix of order* $(n+1)$ *defined as follows:*

$$\begin{aligned} v_{ij} &= 0, && \text{if } i = j; \\ &= 1, && \text{if } i = 1, j = 2, 3, \dots, n+1; \\ &= 1, && \text{if } j = 1, i = 2, 3, \dots, n+1; \\ &= |\triangle[A_i, A_j]|^2, && \text{if } i \neq j, i \neq 1, \text{ and } j \neq 1 \end{aligned}$$

LEMMA 3. *If* $\triangle[O, A_1, A_2, \dots, A_s]$ *is a right* s*-simplex* ($\triangle[O, A_i]$ *and* $\triangle[O, A_j]$ *are orthogonal), then, the content of the* s*-simplex is given by:* $s!|\triangle[O, A_1, A_2, \dots, A_s]| = \prod_{i=1}^{s} |\triangle[O, A_i]|$. *We prove the following proposition.*

PROPOSITION 1. *In* R^n, $\sin(n)\alpha = \sin(\alpha, \alpha, \dots, \alpha) = \sin(\alpha_i) = (\sin\alpha)/[1+(n-2)\cos^2\alpha]^{1/2}$, *where* α *is an angle such that* $0 < \alpha < \pi/2$.

PROOF. Without loss of generality, we assume that $\alpha = \alpha_{12} = \alpha_{13} = \cdots = \alpha_{1n}$. We define $A = \cot\alpha e(1) = (\cot\alpha, 0, 0, \dots, 0)$.

We let $e(0) = 0$. We can see that $\angle OAe(k) = \alpha$ for the triangle $\Delta[O, A, e(k)]$, $k = 2, 3, \ldots, n$. Thus

$$\sin(\alpha_{12}, \alpha_{13}, \ldots, \alpha_{1n}) = \sin(\alpha, \alpha, \ldots, \alpha) = \sin(n)\alpha$$
$$= \frac{|\Delta\,[O, e(2), e(3), \ldots, e(n)]|}{|\Delta\,[A, e(2), e(3), \ldots, e(n)]|}.$$

We see that

$$|\Delta\,[O, e(2), e(3), \ldots, e(n)]| = (n-1)!, \text{ and } |\Delta\,[A, e(2), \ldots, e(n)]|$$
$$= (\det(U_n)/[(-1)^n 2^{n-1}((n-1)!)^2])^{1/2}.$$

Now we apply Lemma 1 and we obtain that $\sin(x, x, \ldots, x) = \sin x/(1 + (n-2)\cos^2 x)^{1/2}$. This proves Proposition 1.

DEFINITION 6. From Proposition 1, we define the n-dimensional sine function $\sin(n)(x) = \sin(x, x, \ldots, x) = \sin x/(1+(n-2)\cos^2 x)^{1/2}$, for $x \in R$. We define the n-dimensional cosine function $\cos(n)x$ as $\sin(n)(\pi/2 - x)$, and a tangent function $\tan(n)x$ as $\tan(n)x = \sin(n)x/\cos(n)x$. (We may call $\sin(n)x$ the n-dimensional pure sine function.)

Referring to Definition 4, we have the following:

PROPOSITION 2. *In R^n, $[\sin(\alpha_1)]^2 + [\sin(\alpha_2)]^2 + \cdots + [\sin(\alpha_n)]^2 = 1$.*

The proof of Proposition follows from the n-dimensional Pythagorian theorem in [1].

4. Note

In this section, we list some elementary properties from Proposition 1. We could have graphs of $\sin(n)x$ and $\cos(n)x$, for $n = 2, 3, 5, 10, 20, 30, 40, 50, 100$, and 200.

NOTE.

(1) $\sin(n)(\pi/4) = (n)^{1/2}$, $n \geq 2$.

(2) $\sin(n)(0) = 0$, $\sin(n)(\pi/2) = 1$.

(3) $\int_0^{\pi/2} \sin(n)x\, dx = \ln[(n-2)^{1/2} + (n-1)^{1/2}]/(n-2)^{1/2}$.

(4) $\lim_{n\to\infty} \int_0^{\pi/2} \sin(n)x\, dx = 0$.

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