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Nonlinear Transformation of Long Waves at a Bottom Step
海底段에서의 長波의 非線型 變形

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Abstract □ We consider the proparation of long finite amplitude nondispersive waves over a step bottom between two regions of finite different depths. Two dimensional motion is assumed, with the wave crests parallel to the step, and irrotational flow in the inviscid fluid is considered. To describe the transformation of finite amplitude waves we use the finite-amplitude shallow-water equations, the conditions of mass flow conservation and pressure continuity at the cut above the step in Riemann's variables. The equations define four families of curves-characteristics on which the values of the Riemann's invariants remain constant and a system of two nonlinear equations that relates the amplitudes of incident, reflected and transmitted waves. The system obtained is difficult to analyze in common form. Thus we consider some special cases having practical usage for tsunami waves. The results obtained are compared with the long wave theory and significant nonlinear effects are found even for quite small amplitude waves.

要　旨：서로 다른 有限水深을 갖는 두 領域을 연결하는 海底段위로 傳播하는 非分散 有限振幅長波를 考慮한다. 2次元 운동을 假定하고, 波峰線이 段과 平行하며, 非粘性流體에서의 非回轉運動으로 본다. 有限振幅波의 變形을 기술하기 위하여 有限振幅 淺海方程式과, 段위의 연결부에서 Riemann 變數로 나타낸 質量保存 및 壓力連續條件들을 사용한다. 式들에 의하면 Riemann 不變量이 一定한 네 組의 特性曲線과 入射, 反射 및 傳達波의 振幅을 관련지어 주는 2개의 非線型方程式이 定義된다. 언어진 方程式系는 통상의 형태로는 해석하기가 어려워 地震 海溢波에 실용적으로 사용할 수 있는 特殊한 境遇만 考慮한다. 얻어진 결과들을 長波理論과 비교하였고 아주 작은 振幅의 波인 경우에도 뚜렷한 非線型 效果가 提示되었다.

1. INTRODUCTION

The problem of long wave propagation over a bottom step is one of the most important problem arising while calculating the passing of the sea wave (in particular, the tsunami) over a bank or the shelf. The wave in the open ocean may have a different character due to the conditions of its generation and propagation. Thus if nonlinearity and dispersion have no time to develop, the wave form can be described by the linear problem solution and is similar to sinusoidal impulse (the wave form often used in tsunami calculations when the piston model of tsunami generation is taken). In cases when nonlinearity is of great importance, the wave becomes stepper while propagating until the dispersion becomes significant; this leads to the soliton formation when the nonlinearity and the dispersion are of the same order. Thus different cases occur when solving the problem of the wave passing over a bottom step. Usually, the relative wave amplitudes in the ocean are rather small, so it seems quite natural to use the linear problem solution obtained by Lamb (1932) from the conservation laws; this solution describes the reflected and transmitted waves at the distance from the bottom step exceeding the depths both to the left and to the right of the step. A more detailed analysis of the problem has

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been given by Bartholomeusz (1958), he developed the integral equation of the problem for the case of arbitrary depths. But these equations have been solved only in the long wave approximation and consequently just the same reflection and transmission coefficients have been found like those obtained by Lamb. The passage of the wave from the region of the infinite depth into the region of the finite depth and vice versa has been discussed by Newman (1965) in the linear approximation, calculations of the reflection and transmission coefficients and phase shifts have been carried out. The approximate solution of the problem of the passage of a small amplitude soliton over a bottom step is known as well; it has been given by Pelinovsky (1971) and Zabysky and Tappert (1971). The wave field far from the bottom step represents a decreasing amplitude soliton train. The solution has been obtained supposing that the boundary conditions at the step are linear. Sugimoto *et al.* (1987) have received an approximate solution of the wave transformation problem, they have introduced an 'edge-layer' theory and have obtained the 'reduced' boundary conditions at the step, the nonlinearity and dispersion appearing in different orders $o(\alpha)$ and $o(\beta^{1/2})$, respectively, where $\alpha = a/h$, $\beta = h^2/\lambda^2$, and a is the wave amplitude, λ is the wave length, h is the water depth ($\alpha \sim \beta$ for the soliton). Then the propagation of a soliton ($\alpha \sim \beta \ll 1$) over a bottom step has been considered. The reflection and transmission coefficients, that coincide with the Lamb's results to the first order as well as the corrections to them of the order $o(\beta^{1/2})$ leading to the appearance of linear phase shifts have been obtained. The results received by Sugimoto *et al.* (1987) may serve as a base to the approach given by Pelinovsky (1971) and Zabysky and Tappert (1971) for the solution of the problem of soliton transformation over a bottom step. Numerical and experimental study of these effects has been carried out by Seabra-Santos *et al.* (1987).

In the paper presented an exact nonlinear solution of the problem of long nondispersive finite amplitude wave transformation at a bottom step is given. The solution obtained permits to find the relation between the amplitudes of the surface level displacements and velocities of the reflected and transmitted waves and those of the incident wave. The solution obtained gives us the possibility to define the linear approximation limits of usage.

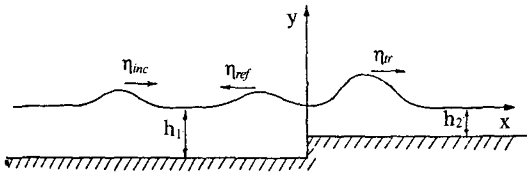

Fig. 1. The geometry of the problem: the long finite amplitude wave propagation over a stepped bottom.

2. BASIC FORMULATION

Let's consider the propagation of the plane (two dimensional) wave in the inviscid fluid which is under the action of gravity over the bottom step (see Fig. 1). We choose the coordinate system (x, y) with the free surface at rest and oy is positively upward. The fluid occupies the region $0<y<-h_1$ when $-\infty<x<0$ and $0<y<-h_2$ when $0<x<\infty$.

To describe the transformation of finite amplitude wave we'll use the finite-amplitude shallow-water equations that are accurate concerning the amplitude, the dispersive effects being not taken into account:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(h(x)+\eta)u] = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0 \quad (1)$$

where η is the water level displacement, u is the horizontal partical velocity, g is the accelaration of gravity, and h(x) is the water depth:

$$h(x) = \begin{cases} h_1 & \text{when } x<0 \\ h_2 & \text{when } x>0 \end{cases}$$

The following conditions at the step (Lamb, 1932; Lighthill, 1975) which represent the conditions of mass, flow conservation and pressure continuity have to be added to the equation (1):

$$(h_2+\eta_2)u_2 = (h_1+\eta_1)u_1$$
$$u_2^2/2 + g\eta_2 = u_1^2/2 + g\eta_1 \quad (2)$$

In the linear approximation the conditions take the form given by Lamb (1932):

$$h_2 u_2 = h_1 u_1, \quad \eta_2 = \eta_1 \tag{3}$$

In the calculations below we use the Riemann variables:

$$I_{\pm i} = u_i \pm 2[\sqrt{g(h_i+\eta_i)} - \sqrt{gh_i}] \tag{4}$$

where $i=1$ for $x<0$ and $i=2$ for $x>0$. Now we shall rewrite both the equations (1) and the conditions (2) at the step using the new variables (4). Then the equations (1) and the conditions (2) at the step take the following forms:

$$\frac{\partial I_{\pm i}}{\partial t} + \left(\pm\sqrt{gh_i} + \frac{3}{4} I_{\pm i} + \frac{1}{4} I_{\pm i}\right)\frac{\partial I_{\pm i}}{\partial x} = 0 \tag{5}$$

$$(I_{+1}+I_{-1})\left(\frac{I_{+1}+I_{-1}}{4} + \sqrt{gh_1}\right)^2 = I_{+2}\left(\frac{I_{+2}}{4} + \sqrt{gh_2}\right)^2$$

$$\frac{(I_{+1}+I_{-1})^2}{8} + \left(\frac{I_{+1}-I_{-1}}{4} + \sqrt{gh_1}\right)^2 - gh_1$$

$$= \frac{(I_{+2})^2}{8} + \left(\frac{I_{+2}}{4} + \sqrt{gh_2}\right)^2 - gh_2 \tag{6}$$

The equation (5) defines four families of curves

$$c_{\pm i} = \pm\sqrt{gh_i} + \frac{3}{4} I_{\pm i} + \frac{1}{4} I_{\mp i}$$

on the (x, t) plane, the qualitative behaviour of which being shown by solid lines in Fig. 2 for the case of positive impulse propagation (the dotted lines indicate the characteristies of linear problem). The dashed region shows the area where the interaction between the incident and reflected wave takes place near the step. As it follows from (5), the values of the invariant remain constant on the characteristics, thus the wave interaction leads only to the appearance of the additional delays for the wave with definite value of Riemann invariant. Outside the interaction area the waves are separate and the values of Riemann invariants for the incident, transmitted and reflected waves are determined by the following formulae:

$$I_{+1} = 4\sqrt{gh_1}\left[\sqrt{1+\frac{\eta_{inc}}{h_1}} - 1\right].$$

$$I_{-1} = -4\sqrt{gh_1}\left[\sqrt{1+\frac{\eta_{ref}}{h_1}} - 1\right].$$

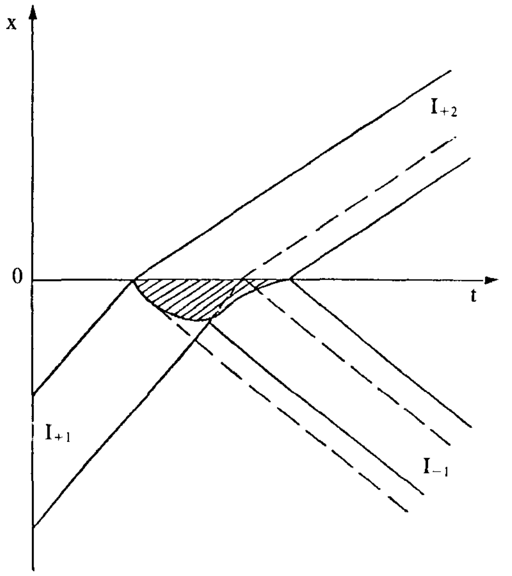

Fig. 2. Qualitative behaviour of characeristics for the problem of finite amplitude wave propagation over a bottom step.

$$I_{+2} = 4\sqrt{gh_2}\left[\sqrt{1+\frac{\eta_{tr}}{h_2}} - 1\right].$$

$$I_{-2} = 0 \tag{7}$$

where η_{inc}, η_{ref}, η_{tr} are the surface level displacements in the incident, reflected and transmitted waves correspondingly.

The system of nonlinear equations that determines the relation between surface level displacements in the incident, transmitted and reflected waves follows from the condition of invariant value conservation on the characteristics (5) and boundary condition (6):

$$q^{3/2}(M-N)(M+N+1)^2 = P(P+1)^2$$

$$q[3(M^2+N^2) - 2MN + 2(N+M)] = 3P^2 + 2P \tag{8}$$

where $q = h_1/h_2$ is the depth ratio and the following designations are introduced:

$$M = \sqrt{1+\frac{\eta_{inc}}{h_1}} - 1$$

$$N = \sqrt{1+\frac{\eta_{ref}}{h_1}} - 1$$

$$P = \sqrt{1+\frac{\eta_{tr}}{h_2}} - 1$$

The system (8) obtained is difficult to analyze in common form. Let's consider some special cases that have practical usage for tsunami waves.

3. The analysis of the solutions

3.1 The case of small amplitude waves

First of all we shall investigate the case of small surface level displacement in the transmitted wave, when $\eta_{tr}/h_2 \ll 1$ and consequently $\eta_{inc}/h_1 \ll 1$, $\eta_{ref}/h_1 \ll 1$ as well. In the first approximation Lamb's formulae for the reflection R and transmission T coefficients follow from the system (8):

$$R=\frac{\sqrt{q}-1}{\sqrt{g}+1} \qquad T=\frac{2\sqrt{q}}{\sqrt{g}+1} \tag{9}$$

In the second approximation the additions μ and ν to the linear formulae are obtained:

$$\frac{\eta_{ref}}{h_1}=R\frac{\eta_{inc}}{h_1}\left(1+\frac{\mu}{R}\frac{\eta_{inc}}{h_1}\right)$$

$$\frac{\eta_{tr}}{h_1}=T\frac{\eta_{inc}}{h_1}\left(1+\frac{\nu}{T}\frac{\eta_{inc}}{h_1}\right) \tag{10}$$

where

$$\frac{\mu}{R}=\frac{1}{2(\sqrt{q}-1)}\left[\frac{T(3-q)}{2}-R^2-\frac{T}{\sqrt{q}}\right]$$

$$\frac{\nu}{T}=\frac{3}{8}T^2(1-q)\left(1+\frac{1}{\sqrt{q}}\right)-\frac{\mu}{2}(1+\sqrt{q})$$

To estimate how large the corrections obtained are the dependences of the relative additions to the 'linear' formulae

$$\frac{\Delta R}{R}\left(\frac{\eta_{inc}}{h_1}\right)^{-1}=\frac{\mu}{R} \quad \text{and} \quad \frac{\Delta T}{T}\left(\frac{\eta_{inc}}{h_1}\right)^{-1}=\frac{\nu}{T}$$

according to (10) on the relative depth ratio $\Delta h/h=(h_1-h_2)/h_1=1-1/q$ are given in Fig. 3. As seen from the Fig, the nonlinearity can bring to some increase of the reflected wave amplitude when the depth ratio $\Delta h/h<0.4$, but when $\Delta h/h>0.4$, the addition $\Delta R/R$ is negative and leads to the decrease of the reflected wave amplitude. Naturally, when $\Delta h/h=0$ and the step vanishes, ΔR and R vanishes as well. The addition to the transmission coefficient is always negative. When $\Delta h/h \to 1$ (i.e. $h_2 \to 0$). ΔR and

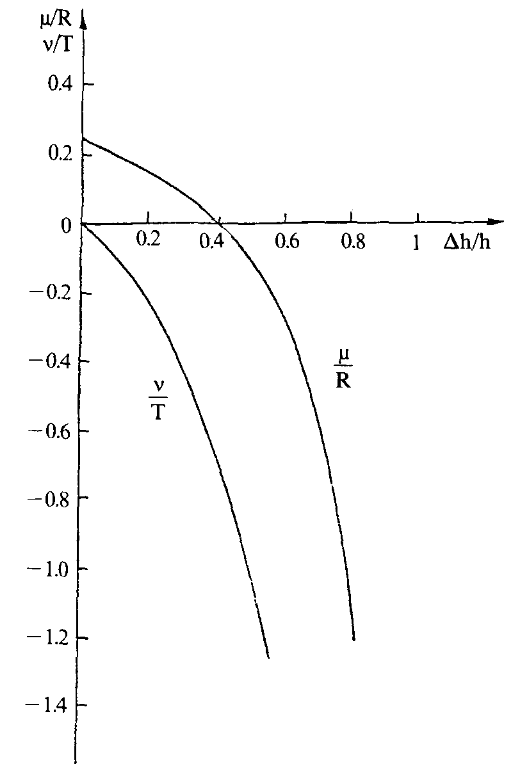

Fig. 3. Relative corrections to the reflection $\Delta R/R(\eta_{inc}/h_1)^{-1}=\mu/R$ and transmission $\Delta T/T(\eta_{inc}/h_1)^{-1}=\nu/T$ coefficients versus the depth ratio $\Delta h/h$.

ΔT increase unlimitedly. This situation will be discussed below (in 3.2.). Fig. 4 shows the dependence of nonlinear additions to the transmission $\Delta T/T$(a) and reflection $\Delta R/R$(b) coefficients on the depth ratio $\Delta h/h$ and the incident wave amplitude. From the Figs one can see that for small depth ratios the nonlinear additions are small in wide limits of the relative incident wave amplitudes, at the same time for large depth ratios the nonlinear addition is sufficient even for small relative amplitudes of the incident wave. Hence in the last case the situstion of 'weak' nonlinearity discussed above becomes insufficient as the condition $\eta_{tr}/h_2 \ll 1$ breaks.

3.2 The case of considerable depth ratios

In the case of considerable depth ratios an appropriate model for describing the wave transformation is the other limit of (6) when the incident and reflected waves are linear (i.e. the surface level disp-

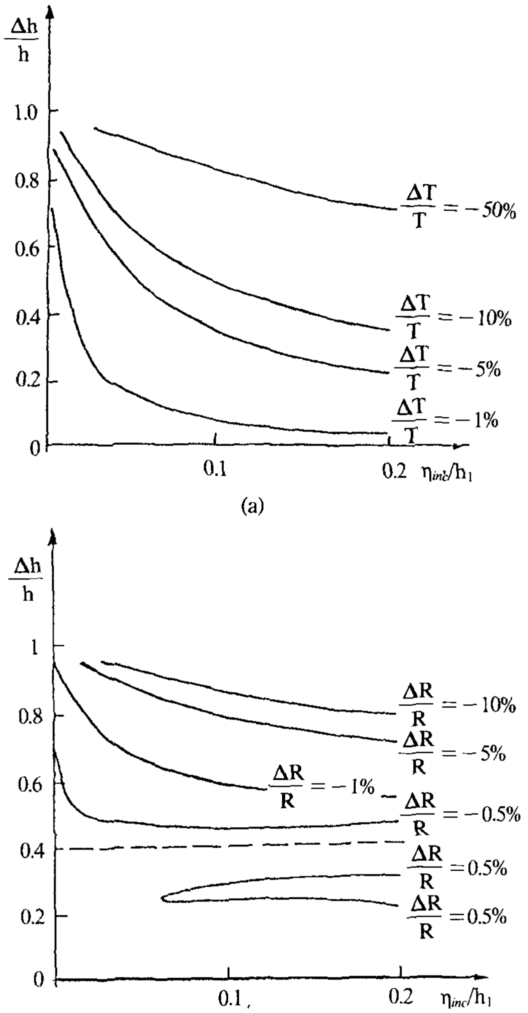

Fig. 4. The dependences of nonlinear corrections to the transmission coefficient ΔT/T (a) and reflection coefficient ΔR/R (b) on the depth ratio Δh/h and incident wave amplitude η_{inc}/h_1.

lacements are small enough) but the relative amplitude of the transmitted wave η_{tr}/h_2 can't be considered to be small. In that case we obtain from (8):

$$\frac{\eta_{ref}}{h_1}=\frac{\eta_{inc}}{h_1}-\frac{2P(P+1)}{q^{3/2}}$$
$$\frac{\eta_{inc}}{h_1}=\frac{P(P+1)^2}{q^{3/2}}+\frac{P(3P+2)}{2q} \tag{11}$$

where $P=\sqrt{1+\eta_{tr}/h_2}-1$. Fig. 5 shows the solution of the system (11), i.e. the dependence of reflected and transmitted wave amplitudes on the incident

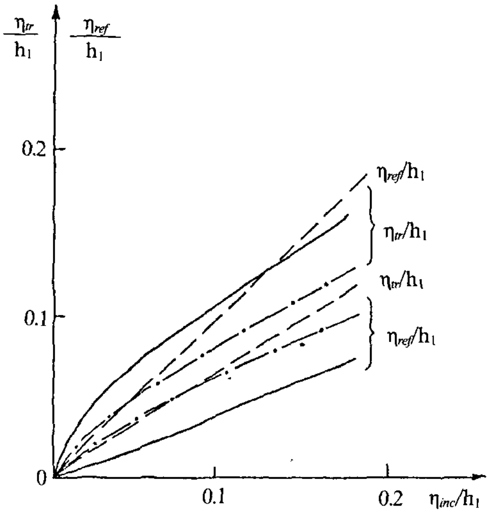

Fig. 5. Relative amplitudes of reflected η_{ref}/h_1 and transmitted η_{tr}/h_1 waves versus the relative amplitude of incident wave corresponding to formula (11).

wave amplitude for three values of depth ratios q = 10, 100, 10000. In the limit case, when $\eta_{tr}/h_2 \to \infty$, (but so that $\eta_{tr}/h_1 \ll 1$, i.e. $q \gg (\eta_{tr}/h_1)^{-1}$) it is easy to obtain from (11):

$$\frac{\eta_{inc}}{h_1}=\frac{3}{2}\frac{\eta_{inc}}{h_1}+0\left[\left(\frac{\eta_{tr}}{h_1}\right)^{3/2}\right]$$
$$\frac{\eta_{ref}}{h_1}=\frac{\eta_{inc}}{h_1}+0\left[\left(\frac{\eta_{tr}}{h_1}\right)^{3/2}\right] \tag{12}$$

As it follows from (12), the amplitude of the transmitted wave is 2/3 of the incident wave amplitude (in contrast to the linear approximation solution when $\eta_{tr}/\eta_{inc}=T=2$ as $h_2 \to 0$). This limit coincides with the situation q = 10000 shown in Fig. 5. But it should be taken into account that in the limit $h_2 \to 0$ the wave breaks immediately at the step and so the case discussed gives the upper limit estimation of the solution of the problem.

3.3 Applicability condition of the results obtained

From the theory developed one manages to find the condition of its applicability. While the wave amplitude increases, the distance at which the wave becomes stepper decreases (in the frames of our nondispersive theory). That is why when the wave amplitude is high and the distance the wave travels from the source is large, the shock wave will approach the step. This effect, generally speaking, is not connected with the bottom step and is determined only by the wave evolution during its propagation. There is, however, an other effect, that is completely determined by the presence of the bottom step. This effect is concerned with the interaction of the incident and reflected waves. Indeed, the reflected wave travels near the step in the field of the incident wave and if the last one is quite intensive, then the reflected wave will not be able to move out of the step and will break. Taking into account the definition of the characteristics we get the condition of the reflected wave existence:

$$C_{-1} = -\sqrt{gh_1} + \frac{1}{4} I_{+1} \le 0 \qquad (13)$$

(it is enough to consider the characteristic $I_{-1}=0$ that corresponds to the beginning of the reflection). Thus using (4) the condition of the solution smoothness is obtained in the following form:

$$\eta_{inc} \le 3h_1 \qquad (14)$$

In contrast, if $\eta_{inc} > 3h_1$, the wave is surely to break. When the inequality (14) takes place and a smooth wave approaches the step, then it will not break during its transformation over the step (but it may break at some distance away from the step as soon as the nonlinear effects accumulate).

3.4 Particle flux velocity determination

The particle velocities in the incident, reflected and transmitted fluxes are easily obtained from (7) and the definition of the corresponding invariants.

$$u_{inc} = 2\sqrt{gh_1}\left[\sqrt{1+\frac{\eta_{inc}}{h_1}} - 1\right]$$

$$u_{ref} = -2\sqrt{gh_1}\left[\sqrt{1+\frac{\eta_{ref}}{h_1}} - 1\right]$$

$$u_{tr} = -2\sqrt{gh_2}\left[\sqrt{1+\frac{\eta_{tr}}{h_1}} - 1\right]$$

In the case of weak nonlinearity from these relations we have:

$$\frac{u_{inc}}{\sqrt{gh_1}} = \frac{\eta_{inc}}{h_1} - \frac{1}{4}\left(\frac{\eta_{inc}}{h_1}\right)^2,$$

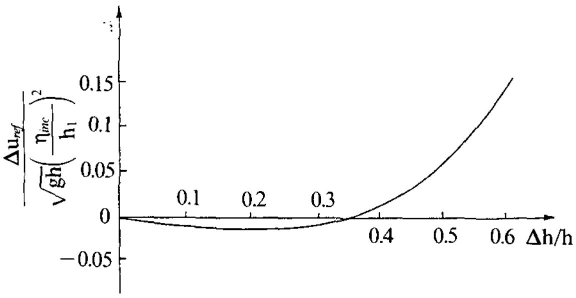

Fig. 6. The nonlinear reflected flow velocity correction versus depth ratio.

$$\frac{u_{ref}}{\sqrt{gh_1}} = -R\frac{\eta_{inc}}{h_1} + \left(\frac{\eta_{inc}}{h_1}\right)^2\left(\frac{R^2}{4} - \mu\right),$$

$$\frac{u_{tr}}{\sqrt{gh_2}} = Tq\frac{\eta_{inc}}{h_1} + \left(\frac{\eta_{inc}}{h_1}\right)^2\left(\nu q - \frac{T^2q^2}{4}\right) \qquad (16)$$

As it follows from (16) the linear effects taken into account lead to the decrease of the transmitted flux velocity (as ν is always negative) in comparison with the results of the linear theory. The nonlinear addition to the particle velocity of the reflected wave $\Delta u_{ref}/gh_1(\eta_{inc}/h_1)^2 = R^2/4 - \mu$ is shown in Fig. 6. As seen, the nonlinearity leads to a small increase in the reflected wave flux velocity, when $\Delta h/h < 0.35$, but for $\Delta h/h > 0.35$ the reflected flux velocity in nonlinear approximation is smaller than that of the linear approximation.

3.5 The application of results

Now let's discuss to which extent the results obtained as well as the existing solutions for long waves and small amplitude solitons can be applied to the tsunami waves. The relative wave amplitudes of the tsunami generated by the underwater earthquakes in the open ocean are small enough ($\eta/h <$ 0.02 see Mirchina and Pelinovsky, 1984; Mirchins and Pelinovsky, 1982; Mirchina, Selinovsky and Shavratsky, 1982). If the distance the wave travels before shoaling L is shorter than the characteristic length, the wave has no time to change its form before it runs up the step (see 3.3). The wave form remains similar to the sinusoidal impulse form. The results of the investigation carried out above allow to answer the question whether the finiteness of the wave amplitude influences the value of the reflected wave and, what is much more important, the transmitted wave amplitudes. As seen from Fig. 4, for the step heights $\Delta h/h<0.85$ the estimation of the transmitted wave by the help of linear formula leads to an error that doesn't exceed 10%. That is why the linear Lamb's theory may be considered to be of good accuracy when describing such situations. Hence there are some cases when the linear estimations give significant errors. For example, for 1964 tsunami generated by underwater earthquake in the vicinity of Alaska, $\eta_{inc}/h_1\sim 0.1$, and the difference from the linear formulae becomes significant (see Fig. 4), when $\Delta h/h>0.6$. But there exists an other possibility when the tsunami wave reaches the USA coast where $\Delta h/h$ approaches unity (i.e. $q\gg 1$). The second case considered above (formula (11), Fig. 5) is a proper model for describing such situation. Hence it should be emphasized that the wave must break in this case.

4. CONCLUDING REMARKS

We have obtained theoretical values of the reflection and transmission coefficients associated with the propagation of finite amplitude long waves over a step-shaped bottom. In the limit of small amplitude waves those results are consistent with the long wave theory of Lamb (1932) and hence, the linear nondispersive limit of the Bartholomeusz's (1958) theory. However, it is clear from our results that there are some cases when the long-wave limit is insufficient for describing the propagation of ocean waves in the shelf zone, because of the fact that the nonlinear corrections to the reflection and transmission coefficients may become significant for abrupt bottom changes and they should be taken into account when estimating the amplitudes of reflected and transmitted waves. Thus we may expect that even for relatively small amplitude ocean waves, the nonlinear effects will be important.

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