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Criteria of Sea Wave Breaking in Basins of Complex Topography
複雜한 海底地形에서의 碎破條件

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Abstract□ Empirical criteria for wave breaking on the coastal slope are substantiated theoretically for complex-shape basins. The theory developed here is a generalization of Carrier-Greenspan theory for a plane beach. The place and role of the linear theory for the description of run-up problem is discussed. The height of run-up on the beach of the basins with a "parabolic" profile is calculated for originally monochromatic wave.

要 旨：傾斜가 일정한 海岸에서 單調波의 碎破條件을 다룬 Carrier-Greenspan 理論을 一般化시켜 複雜한 실제 地形에까지 適用시킬 수 있게 하였으며 따라서 종래 주로 經驗에만 依存하던 碎破帶內의 波浪變形에 理論的 背景을 제공하였다. 拋物線形態의 地形을 對象으로 計算하였으며 이들 結果로부터 非線型性이 강한 처오름(run-up) 問題를 線型化시킴으로 나타나는 影響에 대하여도 檢討하였다.

1. INTRODUCTION

A number of empirical criteria of sea wave breaking on a plane beach were described by Galvin (1968), Mei (1983), Battjes (1988) and Massel (1990) who related the height and length (period) of breaking wave to the slope of the beach. The first theoretical model which enables researchers to determine the condition of monochromatic wave breaking was developed by Carrier and Greenspan (1958) back in 1958. Their model was based on the nonlinear theory of shallow water and on one-dimensional wave propagation over a plane slope. This model was adapted for calculation of tsunami waves in the coastal zone (Spielfogel, 1875; Synolakis, 1987; Voltzinger *et al.*, 1989; Kaistrenko *et al.*, 1991). It will be shown below that the ideas put forward by Carrier and Greenspan are applicable to basins that obey more complicated laws of depth variation. We describe also a simple way of run-up characteristic calculation (Pelinovsky, 1991) which can be applied to basin with complex geometry. The place and role of linear approximation in the solution of nonlinear problems of wave run-up description is discussed here.

2. CARRIER-GREENSPAN TRANSFORMATION

Consider a nonlinear system of equations for shallow water:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial \eta}{\partial x}=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial \eta}{\partial y}=0, \tag{2}$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\{(h+\eta)u\}+\frac{\partial}{\partial y}\{(h+\eta)v\}=0, \tag{3}$$

where h(x, y) is the unperturbed depth of the basin, u and v are the horizontal components averaged over depth, and η stands for free surface elevation. We will restrict ourselves to the analysis of wave processes in basins having bottom topography of the form

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$$h(x, y)=\alpha x-\nu|y|^m \tag{4}$$

with arbitrary parameters α, ν and *m*, which allows for the description of a broad class of bottom relief such as narrow bays, admission channels, etc. Assume that the wave propagates along the x-axis and its length is greater than the transverse size of the channel. This assumption enables us to write down the following one-dimensional equations for the flow velocity. U, averaged over cross-section (hydraulic approximation):

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+g\frac{\partial H}{\partial x}=\alpha, \tag{5}$$

$$\frac{\partial H}{\partial t}+\frac{m}{m+1}H\frac{\partial U}{\partial x}+U\frac{\partial H}{\partial x}=0, \tag{6}$$

where H is the total depth of the basin at y=0. Note that the cross-section in (5), (6) is described by a single parameter m. Being a hyperbolic system, (5)-(6) is effectively solved by Legendre transform, that was earlier used by Carrier and Greenspan (1958) for the calculation of waves on a sloping beach. In particular, substituting the variables (Golinko and Pelinovsky, 1990).

$$\eta=\frac{1}{2g}\left[\frac{m}{m+1}\frac{\partial\Phi}{\partial\lambda}-U^2\right], \tag{7}$$

$$U=\frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}, \tag{8}$$

$$x=\frac{1}{2g\alpha}\left[\frac{m}{m+1}\frac{\partial\Phi}{\partial\lambda}-U^2-\frac{m\sigma^2}{2m+2}\right], \tag{9}$$

$$t=\frac{1}{g\alpha}(\lambda-U) \tag{10}$$

reduces system (5)-(6) to a linear equation:

$$\frac{\partial^2\Phi}{\partial\lambda^2}-\frac{\partial\sigma^2}{\partial\sigma^2}-\frac{m+2}{m}\frac{2}{\sigma}\frac{\partial\Phi}{\partial\sigma}=0 \tag{11}$$

3. DYNAMICS OF RUN-UP ZONE

The wave field may be calculated rather easily in this case: first, the wave equation (11) is solved using standard procedures, and then space-time distribution of the wave field elements is calculated by means of (7)-(10). In practice, however, one has to employ numerical methods because of implicit dependencies of coordinates in (7)-(10), consequently, results are not so transparent. Nevertheless, many characteristics of the process can be found explicilty. This refers, first of all, to the run-up zone and to the determination of the criteria of wave breaking. Consider this problem that is significant for applied purposes, in more detail. The boundary of the run-up zone corresponds to σ=0, because physcially σ is a squared total depth (see (7)-(10)) and in the language of run-up, the total depth turns to zero. As a result, the formula (10) can be written, for σ=0, in the form

$$U(\lambda)=F(t+\frac{1}{g\alpha}U) \tag{12}$$

The physical meaning of the function F may be clarified analyzing a linear system of equations for shallow water:

$$\frac{\partial U}{\partial t}+g\frac{\partial\eta}{\partial x}=0 \tag{13}$$

$$\frac{\partial\eta}{\partial t}+\frac{m}{m+1}\alpha x\frac{\partial U}{\partial x}=0, \tag{14}$$

for which a linear equivalent of (7)-(10) has a form

$$\eta=\frac{1}{2g}\left[\frac{m}{m+1}\frac{\partial\Phi}{\partial\lambda}\right], \tag{15}$$

$$U=\frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}, \tag{16}$$

$$x=\frac{1}{g\alpha}\left[\frac{m\sigma^2}{2m+2}\right], \tag{17}$$

$$t=\frac{1}{g\alpha}\lambda. \tag{18}$$

As a result of these substitutions the physical variables, actually, become dimensionless. The relations (15)-(18) reduce the linear system of equations (13)-(14) to the wave equation (11) again. Now the motionless shore line (x=0) corresponds to the point α=0. By virtue of their simplicity linear problems are widely used in the literature, especially in the processing of experimental data. Our analysis demonstrates that they have a fundamental meaning too. Indeed, the flow velocity along the shore line considered within a linear theory is the function F(t) in (12). Consequently, the solution of a linear problem may be used for the construction of the solution of a nonlinear problem. Knowing the flow velocity along the moving shore line, one can readily find the displacement of water level by simple integration of (12):

$$\eta=\Psi(t+\frac{1}{g\alpha}U)-\frac{1}{2g}U^2, \tag{19}$$

where Ψ is the corresponding linear solution for the elevation of water level along the shore line. From (19) we arrive at a significant conclusion: the maximal heights of run-up (or run-down) are equal within linear and nonlinear theories because the flow velocity turns to zero at the extreme points corresponding to the maxima of run up and run-down. Thus, the solution of the linear problem may be used for correct calculation of maximal height of run-up and of maximal depth of rundown. This explains that the expressions of linear theory are verified, to a good accuracy, in experiments, particularly so in the region of very long waves when the effects of breaking are insignificant.

4. CRITERIA OF BREAKING

In contrast to extreme, the time evolution of the moving shoreline is different within linear and nonlinear theories, in particular, "ture" record of flow velocity contains a sharper leading front. This conclusion is verified by direct calculation from Eq. (12) of the steepness (time derivative) of velocity field:

$$\frac{dU}{dt}=\frac{dF/dt}{1-(1-g\alpha)\ dF/dt} \tag{20}$$

In particular, for a regular (monochromatic) wave with the run up amplitude $R=\Psi_{max}$ and frequency ω, Eq. (20) takes on a simple form

$$\frac{dU}{dt}=\frac{2\omega R \sin \omega t}{\alpha(1-Br \sin \omega t)} \tag{21}$$

where

$$Br=\frac{\omega^2}{g\alpha} \tag{22}$$

stands for the parameter of breaking. Its physical meaning is clear from (21): for Br=1, the derivative dU/dt vanishes to infinity, which corresponds, within the theory of shallow water, to wave breaking. This definition of the parameter Br agrees with the known criteria of Iribarren, Battjes and others, that were found empirically for a plane slope (Mei, 1983; Berkhoff, 1976; Battjes, 1988). Thus, the criterion of breaking is substantiated theoretically for arbitrary relief of coastal zone. Theory, however, enables us to advance further and obtain the criterion of breaking for waves of arbitrary shape, which is significant for the interpretation of data of observations of irregular waves or tsunami. Indeed, solving a linear problem we can find oscillations of the shoreline, $\Psi(t)$. The horizontal velocity F(t) is readily calculated in an explicit form: $F(t)=(1/\alpha)d\Psi/dt$. Substituting this expression into (20) yields an unbounded derivative dU/dt (which corresponds to breaking), provided that

$$Br=\frac{d^2\Psi/dt^2}{g\alpha^2}=1. \tag{23}$$

To be more exact, Eq. (13) contains the maximum of $d^2\Psi/dt^2$. The parameter Br represented in this form allows one to evaluate the probability of breaking of waves of arbitrary shape.

The parameter of breaking has a clear physical interpretation. The value Ψ/α corresponds to the horizontal run-up of the wave or, to be more exact, to the run up that is counted along the slope when y=0. Then

$$a=(1/\alpha)\ d^2\Psi/dt^2 \tag{24}$$

is the acceleration in the wave that is positive towards the beach. With this value taken into account, the condition of breaking takes on a form

$$a=g\alpha \tag{25}$$

i.e. the wave breaks when the "run-up" acceleration in the wave a exceeds the "run-down" component of acceleration of gravity $\gamma\alpha$. There is an alternative interpretation: from the linear equations (13)-(14)

follows a simple relation between time and space derivatives of the level along the shoreline (x=0):

$$\frac{\partial^2\Psi}{\partial t^2}+g\alpha\frac{\partial\Psi}{\partial x}=0 \qquad (26)$$

Consequently, the condition of breaking (23) takes on the form

$$\mathrm{Max}\ \partial\Psi/\partial x=\alpha, \qquad (27)$$

i.e. the wave breaks if its slope along the shoreline is comparable to the slope of the bottom. This criterion is known as the empirical Miche criterion and the theory presented here substantiates it.

Thus it has been ascertained that the known empirical criteria of wave breaking in the surf zone are substantiated within the nonlinear theory of shallow water.

There is another, extremely important conclusion. The expressions presented above depend only on the mean slope of the bottom, α, and are not affected by details of the bottom topography (the coefficients ν and μ in (4)). Therefore empirical criteria of breaking include only the mean slope of the bottom and theory confirms the predominant role of this parameter. At the same time, the bottom topography significantly affects the wave field itself and its structure, because the parameter m enters Eqs. (5)-(6). In particular, in the case of monochromatic wave run up, the solution of Eq. (11) that determines the wave dynamics in linear and nonlinear theories is

$$\Psi(\sigma,\ \lambda)=A\frac{I_{1/m}(l\sigma)}{\sigma^{1/m}}\mathrm{Sin}\ (l\lambda), \qquad (28)$$

where A and l are arbitrary constants determined through initial conditions and I_l is Bessel's function. Consider also an expression that follows from (7)-(10) and (18) for the height of monochromatic wave run up (Golinko and Pelinovsky, 1990):

$$\frac{R}{\xi}=\frac{2\sqrt{\pi}}{\Gamma(1+1/m)}(2\pi L/\lambda)^{0.5+1/m}, \qquad (29)$$

where ξ is the original wave height at a distance L from the shoreline, λ is its length, and Γ is gamma-function. As m decreases, the dependence of the wave height on L/λ becomes sharper. Thus we arrive at a conclusion that the range of run-up in narrow bays is greater than on plane slopes. The expression (29) is valid only in the region of nonbreaking waves.

5. CONCLUSION

Therefore, we have shown that empirical criteria for wave breaking on the coastal slope can be substantiated theoretically for basins of complex-shape. The theory developed here is a generalization of Carrier-Greenspan theory for a plane beach. We have shown also that the linear theory describes correctly extreme characteristics of wave run-up. The height of run-up on the beach of the basins with a "parabolic" profile is calcualted for the originally monochromatic wave.

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