System Diagrams and Reliability Expression for Coherent Structure
- 結合構造에 관한 體系圖와 信賴度 方程式 -

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요 지

手作業 또는 컴퓨터에 의하여 信賴度 方程式을 얻을때 반드시 그것의 옳고 그름을 검정해 볼 必要가 있다. 本 論文에서는 結合構造에 있어서 二項變數를 使用한 二項 成分으로 二項 시스템을 設計하는 法을 提示하였고 檢定하기 위하여 檢定 方法 두가지를 信賴度 方程式의 正確性을 提案 하였다. 아울러 몇 가지 예를 들어 計算上의 效果가 倍加 되었음을 立證 하였다.

1. Introduction

Most reliability calculations are performed assuming that components and systems are either functioning or failed. This dichotomy is often a resonable assumption, but the assumption is sometimes made simply becasue there are no applicable results dealing with more complicated state spaces.

The design and reliability analysis of a large complex system is undertaken by decomposing the system into separate functional subsystems, decomposing each subsystem into components, and finally decomposing each component into its individual parts. System diagrams and event trees are used by reliability engineers to find the probability distribution for the state of the system. The reliability literature of the past 10 years contains many papers with system diagrams or event trees. Some work has also been done on computer programs for ternary components and systems.

First parts of this paper shows that binary state components and systems can be analyzed using system diagrams and event trees.

Main part of this paper shows that correctness of reliability expressions can be proved in a systematic way, though indeed for long expressions it is a formidable task. Two exhaustive tests are given.

Test 1 is the analogue of the method of perfect induction in switching theory. Test 2 handles the problem of checking an expression by breaking it down into disjoint subproblems which are more manageable and for which correctness can be verified separately.

2. System diagrams

The word "System diagram" as used here refers to the reliability block diagram, not to a schematic, or physical diagram. A system diagram is a logic diagram composed of series and parallel operators. However, a system diagram is not limited to systems which have only series and parallel combinations of components since it is well known that a systems can be represented in terms of its minimal path sets or minimal cut sets. Each component in a system diagram

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represents a binary random variable. The series operator replace n components in series with

$$\phi(x)=\prod_{i=1}^{n} x_i = \min (x_1,\ldots,x_n) \quad (1)$$

while the parallel operator replace n components in parallel with

$$\phi(x) = \coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n} (1 - x_i)$$

$$= \max (x_1,\ldots,x_n) \quad (2)$$

where $\phi(x)$ is system structure function and x_i is state of component i;
0 ≡ failed, 1 ≡ function

And a k-out-of-n structure functions if and only if at least k of the n components function. The structure function is given by

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{if } \sum_{i=1}^{n} x_i < k \end{cases} \quad (3)$$

or equivalently

$$\phi(x) = \prod_{i=1}^{n} x_i \qquad \text{for } k = n$$

while

$$\phi(x) = (x_1,\ldots,x_k) \amalg (x_1,\ldots,x_{k-1}\ x_{k+1}) \amalg \ldots \amalg (x_{n-k+1},\ldots,x_n)$$

$$= \max\{(x_1,\ldots,x_k),(x_1,\ldots,x_{k-1}\ x_{k+1}),\ldots,(x_{n-k+1},\ldots,x_n)\}$$

If components are statistically dependent or if the same components appears in multiple place in the system diagram, conditional probability expansions are used

$$\phi(x) = x_i\phi(1i,x) + (1-x_i)\ \phi(0_i,x) \qquad \text{for all } x \quad (4)$$

$$E\{ \phi(x) \} = p_i E\{ \phi(x) | x_i = 1 \} + (1 - p_i)\ E\{ \phi(x) | x_i = 0\} \quad (5)$$

where

$(1_i,\ x) = (x_1,\ldots,x_{i-1},\ 1,\ x_{i+1},\ldots,x_n)$
$(0_i,\ x) = (x_1,\ldots,x_{i-1},\ 0,\ x_{i+1},\ldots,x_n)$
$(\cdot_i,\ x) = (x_1,\ldots,x_{i-1},\ \cdot,\ x_{i+1},\ldots,x_n)$

Minimal cut sets are useful to reliability engineers since they provide a qualitative measure of the most important components in the system. Let K_j be the minimal cut set j; j=1,2,...,k. The value of $\phi(x)$ is calculated from

$$\phi(x) = \prod_{j=1}^{k} \{ 1 - \prod_{i\in k_j} (1 - x_i)\} \quad (6)$$

and system reliability given by

$$E\{ \phi(x) \} = \prod_{j=1}^{k} \{ 1 - \prod_{i\in k_j} (1 - p_i) \} \quad (7)$$

System diagrams are used to calculate either the probability that system functions or the probability that the system fails, depending on how the model is formulated.

Example 1. Consider the system shown in fig. 1. If component 1 functions, then the system functions. If not, then 2-out-of-3 of components 2-4 must function and one of components 5 and 6 must functions in order for the system to function. A conditional probability expansion must be used to calculate system reliability, because components 2-4 appear in two place in the system diagram.

$$\begin{aligned}\phi(x) &= x_1 \amalg (x_2x_3 \amalg x_2x_4 \amalg x_3x_4)(x_5 \amalg x_6)\\ &= 1 - (1 - x_1)\{1 - (x_2x_3 \amalg x_2x_4 \amalg x_3x_4)(x_5 \amalg x_6)\}\\ &= 1 - (1 - x_1)[1 - \{1 - (1-x_2x_3)(1 - x_2x_4)(1 - x_3x_4)\}\\ &\quad \{1 - (1 - x_5)(1 - x_6)\}]\\ &= 1 - (1 - x_1)[1 - \{(x_3 + x_4 - x_3x_4)x_2 + x_3x_4(1 - x_2)\}\\ &\quad (x_5x_6 - x_5x_6)]\end{aligned}$$

$$\begin{aligned}E\{\phi(x)\} &= p_2 E\{\phi(x)| p_2 = 1\} + (1 - p_2) E\{\phi(x)| p_2 = 0\}\\ &= 1 - (1 - p_1)[1 - \{(p_3 + p_4 - p_3p_4) p_2 + p_3p_4(1 - p_2)\}\\ &\quad (p_5 + p_6 - p_5p_6)]\end{aligned}$$

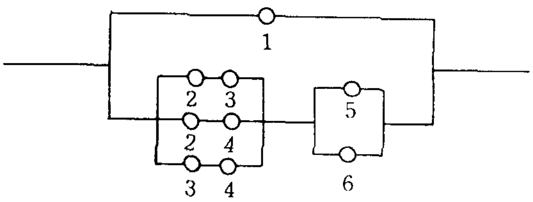

Fig. 1 System diagram

Example 2. Suppose that an airplane engine will operate, when in flight, with probability p independently from engine to engine; suppose that the airplane will make a successful flight if at least 50 percent of its engines remain operative. For what values of *p is a 4-engine plane* preferable to a 2-engine plane?

Solution. Let $\phi_1(x)$ be system structure function of 2-engine plane and $\phi_2(x)$ be system structure function of 4-engine plane. Then $\phi_1(x)$ is a 1-out-2 structure function and $\phi_2(x)$ is a 2-out-of-4 structure function given by

$$\begin{aligned}\phi_1(x) &= \coprod_{i=1}^{2} x_i\\ &= 1 - \prod_{i=1}^{2} (1 - x_i)\\ &= 1 - (1 - x_1)(1 - x_2)\\ &= x_1 x_2 + x_1(1 - x_2) + (1 - x_1) x_2\end{aligned}$$

$$\phi_2(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{4} x_i \geq 2 \\ 0 & \text{if } \sum_{i=1}^{4} x_i < 2 \end{cases}$$

or equivalently

$$\phi_2(x) = x_1x_2 \amalg x_1x_3 \amalg x_1x_4 \amalg x_2x_3 \amalg x_2x_4 \amalg x_3x_4$$

$$= x_1x_2x_3x_4 + x_1x_2x_3\,(1 - x_4) + x_1x_2\,(1 - x_3)x_4 + x_1(1- x_2)x_3x_4 + (1 - x_1)x_2x_3x_4 + x_1x_2(1 - x_3)(1 - x_4) + x_1(1 - x_2)x_3 (1 - x_4) + x_1(1 - x_2)(1 - x_3)x_4 + (1- x_1)(1 - x_2)\,x_3\,x_4 + (1 - x_1)x_2(1 - x_3)x_4 + (1 - x_1)x_2x_3(1 - x_4)$$

Now, we calculate system reliability

$$E\{\ \phi_1(x)\ \} = p^2 + 2\,p\,(\,1 - p\,)$$

$$E\{\ \phi_2(x)\ \} = p^4 + 4\,p^3\,(1-p) = 6\,p^2\,(\,1 - p\,)^2$$

Hence the 4-engine plane is safer if

$$E\{\ \phi_2(x)\ \} \geq E\{\ \phi_1(x)\ \}$$

By above inequality we get

$$p \geq 2/3$$

Hence the 4-engine plane is safer when the engine success probability is at least as large as 2/3.

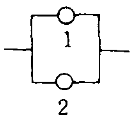

Fig. 2 2-engine plane

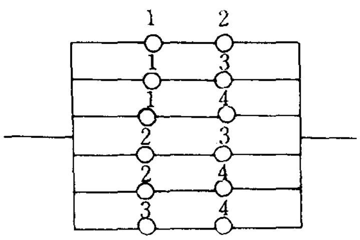

Fig. 3 4-engine plane

3. Event Trees

Constructing the event tree is often itself a useful exercise in understanding a system. When system failure, rather than success, is stressed, the event tree is commonly called a fault tree. Fault trees and event trees are logic diagrams consisting of a top event and a structure delineating the ways in which top event can occur. The tree structure consists of AND GATES and OR GATES which perform the same functions in the event tree as the series and parallel operators in the system diagram.

A system event of major importance will be represented by a rectangle called the top event, appearing at the top of the event tree.

Top event

Immediately below each rectangle will be either an AND GATE represented by Fig. 4.

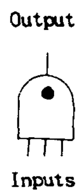

Fig. 4 AND GATE

or an OR GATE represented by Fig. 5.

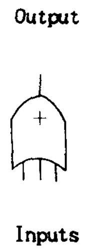

Fig. 5 OR GATE

The output event to an AND GATE occurs if and only if all input events occur. More information on event trees and a different binary formulation can be found in 2.

Example 3. The event tree for the structure function contained in example 1 is shown in fig. 6.

Example 4. The event trees for the structure function contained in example 2 are shown in fig. 7.

The event tree aids in determining the possible causes of an accident, and serves as a display of results. If the system design is not adequate, the event tree can be used to show what the weak points are and how they lead to undesirable events. If the design is adequate, the event tree can be used to show that all conceivable causes have been considered. Also, the event tree provides a convenient and efficient format helpful in the computation of the probability of system success.

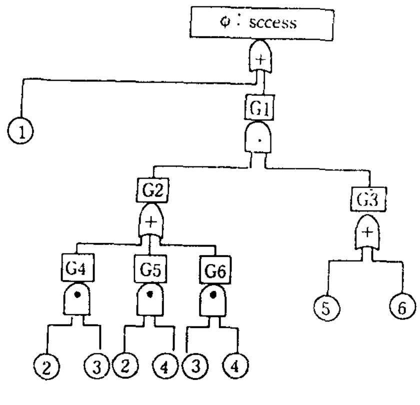

Fig. 6 Event tree of example 1

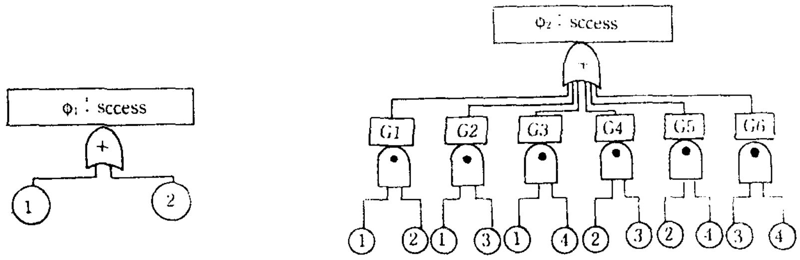

Fig. 7 Event tree of example 2

4. Reliability expression

For a coherent system, the success function S(x) is monotonically increasing, and its complement the failure function S(x) is monotonically decreasing. Hence S is expressible as a sum of products of the uncomplemented literals x_i alone while $\bar{S}$ is expressible as a sum of products of the complemented literals $\bar{x}_i$ alone.

The resulting sum of products expression is unique and lead to a minimal coverage of the function, since each of products appearing in it represents an essential or core prime implicant of the function.

Test 1. A reliability expression R(P) is correct if and only if it is a multiaffine function that yields the correct results of 0 or 1 for the value of the input vector P implied by all the states of the system.

Example 5. The following unreliability expression is given for the small bridge system shown in fig. 8.

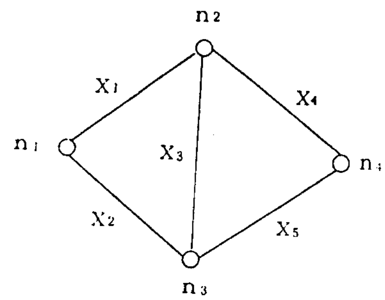

Fig. 8 A bridge system having five branches and four nodes

The success function S_{14} of the system is the union of its minimal s-t paths;

$$S_{14} = x_1x_4 \cup x_2x_5 \cup x_1x_3x_5 \cup x_2x_3x_4 \quad (8)$$

By DeMorgan's law, its complement is

$$\bar{S}_{14} = (\overline{x_1x_4}) \cap (\overline{x_2x_5}) \cap (\overline{x_1x_3x_5}) \cap (\overline{x_2x_3x_4}) \quad (9)$$

Based on (9), the following unreliability expression was proposed

$$\begin{aligned} Q_{14} &= (1 - p_1p_4)(1 - p_2p_5)(1 - p_1p_3p_5)(1 - p_2p_3p_4) \\ &= 1 - p_1p_4 - p_2p_5 - p_1p_3p_5 - p_2p_3p_4 + p_1p_2p_4p_5 + p_1^2p_3p_4p_5 \\ &\quad + p_1p_2p_3p_4^2 + p_1p_2p_3p_5^2 + p_2^2p_3p_4p_5 + p_1p_2p_3^2p_4p_5 - p_1^2p_2p_3p_4p_5^2 \\ &\quad - p_1p_2^2p_3p_4^2p_5 - p_1^2p_2p_3^2p_4^{2p}{}_5 - p_1p_2^2p_3^2p_4p_5^2 + p_1^2p_2^2p_3^2p_4^2p_5^2 \end{aligned} \quad (10)$$

This unreliability expression yields correct results for all the 2^n values of P mentioned in test 1. However, it yields a wrong result for almost any other valid value of P. The pitfall in going from (9) to (10) is that minimal paths are assumed statistically independent while they are not. Test 1 detects that (10) is incorrect by simply finding that it is not a multiaffine function which is defined

$$Q_{14} = q_5 \{q_4 + q_1q_4(q_2 + p_2q_3)\} + q_2q_5(q_1 + p_1q_3q_4) \quad (11)$$

Expression (10) can be corrected by suppressing all the exponents occurring in it, i.e. by applying the $[\cdot]^*$ operator defined by

$$[\prod_j P_j^{ij}]^* = \prod_j P_j \quad (12)$$

So that Q_{14} takes the correct multiaffine form.

$$Q_{14} = 1 - p_1p_4 - p_2p_5 - p_1p_3p_5 - p_2p_3p_4 + p_1p_2p_4p_5 + p_1p_3p_4p_5 + p_1p_2p_3p_4 + p_1p_2p_3p_5 + p_2p_3p_4p_5 - 2p_1p_2p_3p_4p_5 \quad (13)$$

Test 1 is tedious and impractical even for expressions of moderate size. Certain patterns of repetitions take place in the checks of test 1. To save some work, these repetitions can be exploited by replacing each group of similar checks by a single one. Reliability expression for the network in fig. 8 is

$$R = ((p_1+q_1p_2)p_3+p_1p_2p_3)(p_5+p_4q_5)+(p_1q_2+q_1p_2)q_3p_4p_5 \quad (14)$$

It is in multiaffine form. It yields the correct result of 1 for the 8 spanning trees $x_1x_2x_4$, $x_1x_4x_5$, $x_2x_4x_5$, $x_1x_2x_5$, $x_1x_3x_4$, $x_2x_3x_5$, $x_1x_3x_5$, and $x_2x_3x_4$.

It also yields the correct result of 0 for the 6 network cutsets $\bar{x}_1\bar{x}_2$, $\bar{x}_4\bar{x}_5$, $\bar{x}_1\bar{x}_3\bar{x}_4$, $\bar{x}_2\bar{x}_3\bar{x}_5$, $\bar{x}_2\bar{x}_3\bar{x}_4$, and $\bar{x}_1\bar{x}_3\bar{x}_5$.

For example, for the spanning tree $x_1x_2x_4$

$$R\{p_1 = p_2 = p_4 = 1\} = ((1+0)p_3+q_3)(p_5+q_5)+(0+0) = 1$$

and for the network cut set $x_2\bar{x}_3\bar{x}_5$

$$R\{p_2 = p_3 = p_5 = 0\} = ((p_1+0)0+0)(0+p_4)+(p_1+0)(p_4)0 = 0$$

Hence, (14) is correct.

Test 2. A reliability expression is correct if it is a multiaffine function that reduces to the correct reliability expressions of the subsystems derived from the original system through a Bayesian decomposition with respect to k admissible keystone, elements. Test 2 is useful when the subsystems obtained are simple. If k=n, test 1 are the same.

Example 6. The following multiaffine expression is (6) for the system in fig. 9.

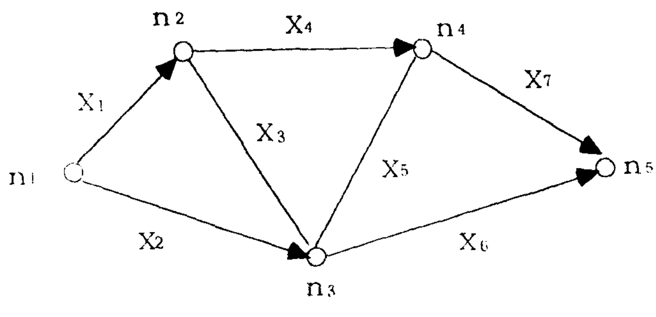

Fig. 9 A moderate system having seven s-independent branches

Branches 3 and 5 are bidirectional.

$$R_{15} = p_6(p_2+p_1q_2(p_3+p_4q_3(p_5+q_5p_7)))+q_6p_7(p_2(p_5+p_4q_5(p_3+p_1q_3))+p_1q_2(p_4q_5+p_5(p_3+p_4q_3))) \quad (15)$$

An application of test 2 with branches 2, 3, 5, 6 taken as keystone elements, results in table 1 which consists of 16 lines. The results in the 16 lines of this table are correct, as can be easily seen by considering the corresponding subsystems derived from the original system. The table could have been shortened by combining some of its lines, for example the entries in lines 1001, 1011, 1101, 1111 are the same, and these lines can be combined as 1..1.

table 1

$p_2p_3p_5p_6$	R_{15}
0000	$0+p_7(0+p_1(p_4+0)) = p_1p_4p_7$
0001	$1(0+p_1(0+p_4(0+p_7)))+0 = p_1p_4p_7$
0010	$0+p_7(0+p_1(0+1(0+p_4))) = p_1p_4p_7$
0011	$1(0+p_1(0+p_4(1+0)))+0 = p_1p_4$
0100	$0+p_7(0+p_1(p_4+0)) = p_1p_4p_7$
0101	$1(0+p_1(1+0))+0 = p_1$
0110	$0+p_7(0+p_1(0+1(1+0))) = p_1p_7$
0111	$1(0+p_1(1+0))+0 = p_1$
1000	$0+p_7(1(0+p_4(0+p_1))) = p_1p_4p_7$
1001	$1(1+0)+0 = 1$
1010	$0+p_7(1(1+0)) = p_7$
1011	$1(1+0)+0 = 1$
1100	$0+p_7(1(0+p_4(1+0))) = p_4p_7$
1101	$1(1+0)+0 = 1$
1110	$0+p_7(1(1+0)) = p_7$
1111	$1(1+0)+0 = 1$

Example 7. This example illustrates an alternative way of applying test 2 to (15). Initially branch 2 alone is taken as a keystone elements. When $p_2=1$, the system reduces to a series parallel subsystem, while when $p_2=0$ the system is still complex and can be decomposed further with respect to branch 5 to give 2 simple subsystems. Expression (15) is now reduced under the mutually exclusive and exhaustive conditions $\{p_2=1\}$, $\{p_2=0, p_5=0\}$, $\{p_2=0, p_5=1\}$.
The results are;

$$R_{15}\{p_2=1\} = p_6+q_6p_7(p_5+p_4q_5(p_3+p_1q_3))$$

$$\begin{aligned} R_{15}\{p_2=0, p_5=0\} &= p_6p_1(p_3+q_3p_4p_7)+q_6p_7(p_1p_4) \\ &= p_1(p_3p_6+p_4p_7(q_6+p_3p_6)) \end{aligned} \tag{16}$$

$$\begin{aligned} R_{15}\{p_2=0, p_5=1\} &= p_6p_1(p_3+q_3p_4)+q_6p_7p_1(p_3+q_3p_4) \\ &= p_1(p_3+q_3p_4)(p_6+q_6p_7) \end{aligned}$$

These are correct for the subsystems under consideration.

5. Conclusion

The checking methods be proposed above are very useful for detecting faults in hand derivations and debugging computer programs. They are initially developed for the case of a system with perfectly reliable nodes, and then modified to handle node unreliability. The test 1, 2 also apply to a flow network having a capacity constraint. In example 6, 7 computational efficiency can be enhanced.

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