論 文

A New Recursive Formula to Derive the Fourier Transforms of Cosine-Pulses Using Modified Class-I PRS Model

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수정된 제1종 부분응답 전송 시스템 모델을 이용한 여현 펄스 푸리에 변환의 새로운 순환 공식

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ABSTRACT

This paper presents a new and easy method to obtain the Fourier transforms of the n-th order cosine-pulses whose maximum amplitudes are uniform. The new method is focused on deriving a formula which is recursively related following their orders and can be well applied to some numerical solutions. On the other hand, this method also offers more compact procedures in view of analytical solutions than the conventional methods because the results are consist of the sum of two functions which are easily calculated. Especially, the formula can be represented as a complete recursion by the separation of coefficients originated by the authors and the resulting difference equation is given by the sum of the original 'sinc' functions shifted by some symmetrical factors and multiplied by some constants. The constants are easily decided from the binomial coefficients and the shifting factors from the corresponding exponential differences in the expansion of $(a+b)^n$.

要 約

본 논문에서는 일정한 구간 내에 한정되고 최대치가 균일하도록 설정된 임의 차수 여현펄스의 푸리에 변환을 유도하기 위한 새롭고 용이한 방법을 제안하였다. 제안된 방법은 수치적 해법에 원활하게 적용될 수 있도록 함수의 각 차수에 따라 순환적으로 유도되는 공식에 촛점을 두고 있다. 반면에, 유도된 관계식은 용이하게 계산될 수 있는 두 함수의 합에 의하여 나타나므로 해석적 해법의 관점에서도 기존의 방법보다 간결한 과정을 제공한다. 특히, 저자 등에 의하여 발견된 계수 분리법에 의하여 공식은 완전 순환적 알고리듬으로 표현되며, 그 결과로 나타나는 차동방정식은 초기 'Sinc' 함수가 차수에 따라 지연되어 상수가 곱해진 형태의 합으로 주어진다. 이 때, 곱해지는 각 상수는 이항계수로부터 용이하게 결정되며, 'Sinc' 함수의 지연요소도 이항식 $(a+b)^n$의 전개식에서 해당되는 항의 지수차에 의하여 쉽게 얻어진다.

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Ⅰ. INTRODUCTION

It is common to analyze the system functions and the signals used in communcication systems by means of the Fourier transform. Analytical developmants and some numerical solutions for the transform can be easily located in the literatures of communication theory [1-3]. The n-th order cosine-pulses are used frequently to represent the transfer functions or the time responses of some systems. Nevertheless, if the order of the function is increased, the procedure to derive the Fourier transform using conventional methods will be complex and difficult. Although the method of consecutive differentiations or by the convolution theorem can be used to derive the transforms of the n-th order cosine pulses, the higher the orders of the function, the more tedious procedures become [2]. In general, the method of consecutive differentiations become terribly complex since the function has more than 3-rd order, and the method by the convolution theorem since 4-th, or 5-th order. Moreover, these methods can not offer the forms of recursive formula.

On the other hand, the class-I PRS(partial response signaling) system was introduced in 1963 by the name of duobinary [4]. It is a transmission technique for digital data using the concept of controlled amount of interference between adjacent samples. In 1975, P.Kabal and S.Pasupathy generalized this concept and proposed a model separated into two parts, namely, the transversal filter and the bandlimiting filter [5]. We present an easy iterative process to derive the Fourier transforms of the n-th order cosine pulses by means of this model for class-I PRS system modified for our purpose.

Throughout this paper, we describe the problem and some prerequisites for the foundation of later developments. In section Ⅲ, the producedure of conventional methods using consecutive differentiations and the convolution theorem are represented briefly. And we attend the fact that the method by the convolution theorem gives us a clue for the separation of coefficients developed in section Ⅴ. Continuously, we derive an iterative formula for the transforms of the n-th order cosine-pulse using modified class-I PRS model, and we separate the coefficients from the formula to develop a completely recursive relationship in a more compact form. These formulas are expressed as propositions, which are proved by induction in Appendix.

Ⅱ. Problem Description and Some Prerequisites

The functions of our interests are the truncated cosine-pulses which have the same amplitude for any order n = 1, 2, 3, ⋯. We define the n-th order cosine-pulse as follows :

(*Definition 1* : The n-th order cosine-pulse)

$$p_n(t) \equiv A\left[\cos\frac{\pi t}{2\tau}\right]^n \Pi\left[\frac{t}{2\tau}\right],\quad \text{for } n = 1, 2, 3, \cdots \tag{1}$$

where

$$\Pi\left[\frac{t}{2\tau}\right] \equiv \begin{cases} 1, & \text{if } |t| \le \tau \\ 0, & \text{elsewhere} \end{cases}$$

Deriving the frequency-domain function $P_n(f)$ from (1) is just the problem we will solve throughout this paper.

In addition, we establish some propoerties of sinc function and new-defined 'cosinc' function for later developments.

(*Definition 2*: The sinc function and the cosinc function)

$$\text{sinc}(x) \equiv \frac{\sin \pi x}{\pi x} \tag{2}$$

$$\text{cosinc}(x) \equiv \frac{\cos \pi x}{\pi x} \tag{3}$$

It is well known that the sinc function is an even function and it can be easily shown that the cosinc function is odd. Other properties of them used in this paper are as follows :

Property 1 : $f_1(x) = \text{sinc}(ax+b) + \text{sinc}(ax-b)$ is also an even function, where a,b are real.

Property 2 : $f_1(x) = \text{sinc}(ax+b) + \text{sinc}(ax-b)$ can be simplified as

$$\frac{2(ax)^2}{b^2-(ax)^2}\ \text{sinc}(ax),\ \text{when b is an odd number}$$

$$\frac{2(ax)^2}{(ax)^2-b^2}\ \text{sinc}(ax),\ \text{when b is an even number},$$

where a is real and b is a positive integer.

Property 3 : $f_2(x) = \text{sinc}(ax+k/2) + \text{sinc}(ax-k/2)$ can be simplified as

$$\frac{4k(ax)}{k^2-(2ax)^2}\ \text{cosinc}(ax),\ \text{when } k=1, 5, 9, \cdots$$

$$\frac{4k(ax)}{(2ax)^2-k^2}\ \text{cosinc}(ax),\ \text{when } k=3, 7, 11, \cdots,$$

where a is real and k is an odd number.

It is straightforward that these properties are clearly true from the definitions (2) and (3). They are conveniently used to calculate the transforms resulting in one-term styles in the later sections.

Ⅲ. Brief Review of the Conventional Methods

A. Consecutive Differentiations

We can obtain the Fourier transform of the function (1) by the method of consecutive differentiations when the order of the function is relatively low.

For the 1-st order, we have first two derivatives of the function as follows :

$$\frac{dp_1(t)}{dt} = -A\left[\frac{\pi}{2\tau}\right]\sin\frac{\pi t}{2\tau}\ \Pi\left[\frac{t}{2\tau}\right] \tag{4}$$

$$\frac{d^2p_1(t)}{dt^2} = -A\left[\frac{\pi}{2\tau}\right]^2\cos\frac{\pi t}{2\tau}\ \Pi\left[\frac{t}{2\tau}\right] + A\left[\frac{\pi}{2\tau}\right]\left\{\delta(t+\tau)+\delta(t-\tau)\right\} \tag{5}$$

The first term of (5) can be rewritten with a constant multiple of the original function $p_1(t)$ and the second term contains a constant multiple of shifted impulses only. Therefore, we have

$$(j2\pi f)^2 P_1(f) = -\left[\frac{\pi}{2\tau}\right]^2 P_1(f) + 2A\left[\frac{\pi}{2\tau}\right]\cos 2\pi f\tau \tag{6}$$

and after a few steps, we obtain

$$P_1(f) = \frac{8Af\tau^2}{1-(4f\tau)^2}\ \text{cosinc}(2f\tau) \tag{7}$$

where the functipn cosinc(·) has been defined in (3).

For the 2-nd order, we need first three derivatives and they are given by

$$\frac{dp_2(t)}{dt} = -A\left[\frac{\pi}{2\tau}\right]\sin\frac{\pi t}{\tau}\ \Pi\left[\frac{t}{2\tau}\right] \tag{8}$$

$$\frac{d^2p_2(t)}{dt^2} = -2A\left[\frac{\pi}{2\tau}\right]^2\cos\frac{\pi t}{\tau}\ \Pi\left[\frac{t}{2\tau}\right] \tag{9}$$

$$\frac{d^3p_2(t)}{dt^3} = 4A\left[\frac{\pi}{2\tau}\right]^3\sin\frac{\pi t}{\tau}\ \Pi\left[\frac{t}{2\tau}\right] + 2A\left[\frac{\pi}{2\tau}\right]^2\left\{\delta(t+\tau)-\delta(t-\tau)\right\} \tag{10}$$

Similarly, if we check (8) and (10), we can see that the first term of (10) can be rewritten with a constant multiple of (8) and the remainings contain a constant multiple of shifted impulses only. Terefore, (10) gives

$$(j2\pi f)^3 P_2(f) = -4\left[\frac{\pi}{2\tau}\right]^2 (j2\pi f)\, P_2(f) + j\,4A\left[\frac{\pi}{2\tau}\right]^2 \sin 2\pi f\tau \tag{11}$$

After somewhat tedious steps of calculations, we can find the Fourier transform of $p_2(t)$ in a compact form as

$$P_2(f) = \frac{A\tau}{1-(2\tau f)^2}\ \text{sinc}(2\tau f) \tag{12}$$

where the function sinc(·) has been defined in (2).

As the order of the function is increased, the situations are quite different. In other words, the processes like above and the forms of equations corresponding to (6) or (11) will be more and more complicated to perform analytically.

B. Convolution theorem

As another method to derive the Fourier transform of the n-th order cosine-pulse, the convolution integral has been introduced in many books [1,2]. In this subsection, we derive the transforms of the functions given above using the method by the convolution theorem and show that the procedure presented here is much simpler than that of consecutive differentiations and so we can have the transforms of higher order functions. In addition, we can find a clue for the separation of coefficients described in section V during the process of derivations.

When the order of the function is relatively low (n=1, 2, 3, or 4), we can easily obtain the Fourier transform considering (1) as a product of two functions so that they are convolved in the frequency-domain.

$$p_n(t) = x_n(t)\, y(t)\ , \quad \text{for } n=1,2,3,\cdots \tag{13}$$

where

$$x_n(t) = A\left[\cos\frac{\pi t}{2\tau}\right]^n, \quad \text{for } n=1,2,3,\cdots \tag{14}$$

and

$$y(t) = \Pi\left[\frac{t}{2\tau}\right], \quad \text{for all } n \tag{15}$$

With this manipulation, we can see that only the form of $x_n(t)$ is varied but $y(t)$ is fixed for its order. In addition, the Fourier transform of $y(t)$ is easily obtained as

$$Y(f) = 2\tau\,\text{sinc}\,(2\tau f) \tag{16}$$

For n=1, 2 and 3, it is relatively simple to derive the transforms of $x_n(t)$. We have

$$X_1(f) = \frac{A}{2}\left\{\delta(f-1/4\tau)+\delta(f+1/4\tau)\right\} \tag{17}$$

$$X_2(f) = \frac{A}{4}\left\{\delta(f-1/2\tau)+2\delta(f)+\delta(f+1/2\tau)\right\} \tag{18}$$

$$X_3(f) = \frac{A}{8}\left\{\delta(f-3/4\tau)+\delta(f-3/4\tau)\right\} + \frac{3A}{8}\left\{\delta(f-1/4\tau)+\delta(f+1/4\tau)\right\} \tag{19}$$

By the convolution theorem, we can obtain the Fourier transforms of $p_n(t)$, for n=1, 2 and 3 as follows :

$$P_1(f) = A\tau\{\text{sinc}(2\tau f+1/2)+\text{sinc}(2\tau f-1/2)\} \tag{20}$$

$$P_2(f) = \frac{A\tau}{2}\left\{\text{sinc}(2\tau f-1)+2\,\text{sinc}(2\tau f) + \text{sinc}(2\tau f+1)\right\} \tag{21}$$

$$P_3(f) = \frac{A\tau}{4}\left\{\text{sinc}(2\tau f-3/2)+\text{sinc}(2\tau f+3/2)\right\} + \frac{3A\tau}{4}\left\{\text{sinc}(2\tau f-1/2)+\text{sinc}(2\tau f+1/2)\right\} \tag{22}$$

Applying the *Property 3* and *2* in section II to (20) and (21), we can see that (20) and (21) are equivalent to (7) and (12) respectively. We can also apply the *Property 3* to (22) so that the form of $P_3(f)$ will be more compact as

$$P_3(f) = \frac{48\,A\tau^2 f}{\{1-(4\tau f)^2\}\{9-(4\tau f)^2\}}\ \text{cosinc}\,(2\tau f) \tag{23}$$

In general, as the order of the function (1) is increased, the form of $X_n(f)$ will be more and more complicated and the process for convolving and calculation will be tedious and difficult. Moreover, we can not find an iterative property from the trends of (20), (21), and (22). But if we ignore the coefficients, $P_n(f)$ contains shifted sinc functions only for any n. It will be a clue for the separation of coefficients presented in section V.

Ⅳ. Deriving the Recursive Formula

It is well known that the transfer function of the form of a truncated cosine function [4,5]. We present a modified model of class-I PRS to obtain an iterative formula for deriving the Fourier transforms of the n-th order cosine-pulses as shown in Fig.1.

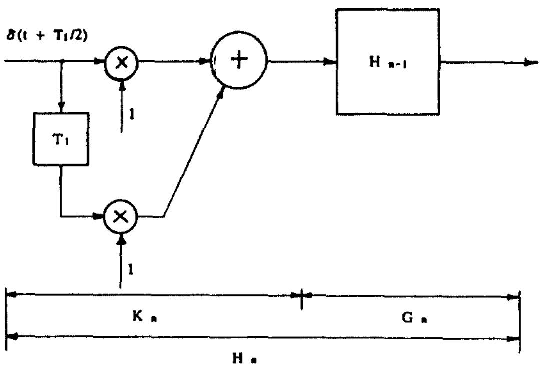

Fig. 1 Modified Model of Class-I PRS for the n-th Order Function

The model consists of a tapped delay line corresponding to the polynomial $K_n(D)=(1+D)$ in cascade with a filter with frequency response $H_{n-1}(f)$ which is equivalent to the overall transfer function of the system for the $(n-1)$-st order. We call the first half marked K_n the transversal filter and the second half marked G_n the bandlimiting filter. We alter the form of the bandlimiting filter along with the order n systematically. On the other hand, the transversal filter is fixed for all n.

In addition, we assume that the frequency response of the bandlimiting filter is the same as that of minimum-bandwidth PRS system[5] for the 1-st order function. In other words, we set

$$G_1(f)=H_0(f)=T_2\,\Pi(T_1 f) \tag{24}$$

which shapes a Fourier transform pair with

$$g_1(t)=h_0(t)=\frac{T_2}{T_1}\,\mathrm{sinc}\left[\frac{t}{T_1}\right] \tag{25}$$

With the assumptions above and the model in Fig.1, we can have the following proposition.

(*Proposition 1*)

$$P(n): H_n(f)=2^n T_2\,(\cos\pi fT_1)^n\,\Pi(T_1 f) \tag{26}$$

$$\Leftrightarrow h_n(t)=h_{n-1}(t+T_1/2)+h_{n-1}(t-T_1/2) \tag{27}$$

for all $n=1, 2, 3, \cdots$

with $H_0(f)$, $h_0(t)$ in (24), (25) respectively.

The symbol '⇔' denotes that the two functions on both sides form a Fourier transform pair. The proof of the (*Proposition 1*) is discussed in Appendix A.

Clearing the fact that the functions $H_0(f)$ and $h_0(t)$ are even, it can be easily shown that $H_n(f)$ and $h_n(t)$ are also even for all $n=1, 2, 3, \cdots$ using the mathematical induction[6] and the *Property 1* in section Ⅱ. We are now ready to derive an iterative relationship for the Fourier transform of the funtion (1).

From the dual property of the Fourier transform pair[1-3] and the fact that (26) and (27) are even functions, we have another transform pair as follows :

$$H_n(t)=2^n T_2\,(\cos\pi T_1 t)^n\,\Pi(T_1 t) \tag{28}$$

$$\Leftrightarrow h_n(f)=h_{n-1}(f+T_1/2)+h_{n-1}(f-T_1/2) \tag{29}$$

with $H_0(t)=T_2\,\Pi(T_1 t)$

$$\Leftrightarrow h_0(f)=\frac{T_2}{T_1}\,\mathrm{sinc}\left[\frac{f}{T_1}\right] \tag{30}$$

If we set the parameters as $T_1=1/2\tau$ and $T_2=A/2^n$ in (28), (29) and (30), for $n=1, 2, 3, \cdots$, another recursive formula we want appears

$$p_n(t)=A\left[\cos\frac{\pi t}{2\tau}\right]^n\,\Pi\left[\frac{t}{2\tau}\right]=H_n(t) \tag{31}$$

$$\Leftrightarrow P_n(f)=h_n(f)$$

$$=h_{n-1}\left[f+\frac{1}{4\tau}\right]+h_{n-1}\left[f-\frac{1}{4\tau}\right] \tag{32}$$

for $n=1, 2, 3, \cdots$

with $h_0(f) = \frac{A\tau}{2^{n-1}} \operatorname{sinc}(2\tau f)$ (33)

Applying the formula in (32) with (33) to derive the Fourier transforms of the functions in (31) and calculating by means of the *Property 2* and *3* in section Ⅱ, we can see that the analytical processes for compact forms of the transforms are very much simpler than the conventional methods discussed in section Ⅲ. Moreover, the equation (32) with (33) can be easily solved by computer-aided numerical methods because of its recursive characteristics [7].

Ⅴ. Separation of coefficients

Glancing at (20)-(22) and after a few inductive steps, we can say that the construction of the transform derived from the relation (32) with (33) is a product of a constant determined by the order n and a sum of shifted sinc functions without loss of generality for n=1, 2, 3, ⋯. Therefore, we can write the relationship given in (32) as follows :

$$P_n(f) = C_n \cdot R_n(f) \quad (34)$$

where $C_n = \frac{A\tau}{2^{n-1}}$ (35)

$$R_n(f) = R_{n-1}\left[f + \frac{1}{4\tau}\right] + R_{n-1}\left[f - \frac{1}{4\tau}\right] \quad (36)$$

for n=1, 2, 3, ⋯ with $R_0(f) = \operatorname{sinc}(2\tau f)$ (37)

The relation (36) with (37) have no coefficients which hinder us from solving the difference equation resulting in a more compact representation. Developing the relationship in (36) for some n and comparing with the expansion of $(a+b)^n$, we can find very interesting facts. The binomial theorem, which can be proved by mathematical induction, gives the general expression for the expansion of $(a+b)^n$ [8].

$$(a+b)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} a^{n-i} b^i = \begin{bmatrix} n \\ 0 \end{bmatrix} a^n b^0 + \begin{bmatrix} n \\ 1 \end{bmatrix} a^{n-1} b^1 + \cdots + \begin{bmatrix} n \\ n \end{bmatrix} a^0 b^n \quad (38)$$

where the binomial coefficients are given by

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{n!}{i!\,(n-i)!}, \quad \text{for } i = 0, 1, 2, \cdots, n. \quad (39)$$

In addition, if we define the exponetial difference between a and b for the i-th term of the expansion (38) as

$$d_i \equiv (\text{exponent of a}) - (\text{exponent of b}) = n - 2i, \text{ for } i = 0, 1, 2, \cdots, n. \quad (40)$$

we obtain a simple and useful representation of $R_n(f)$. We arrange these interesting things as a proposition as follows :

(*Proposition 2*)

$$Q(n) : R_n(f) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} R_0\left[f + \frac{d_i}{4\tau}\right] = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \operatorname{sinc}\left[2\tau f + \frac{d_i}{2}\right] \quad (41)$$

for n=1, 2, 3, ⋯ with d_i in (40).

This proposition will be proved in Appendix B.

We now arrive at the final result from all the facts discussed above. The Fourier transform of the n-th order cosine-pulse given in (1) can be represented as

$$P_n(f) = \frac{A\tau}{2^{n-1}} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \operatorname{sinc}\left[2\tau f + \frac{d_i}{2}\right] \quad (42)$$

for n=1, 2, 3, ⋯

where d_i is given by (40).

We can deal with the result (42) as three parts, namely, the constant part, the binomial coefficient part, and shifted sinc-function part. The constant part is obtained directly from (35) for any n. But the binomial coefficients are given differently for the powers of (a+b). Fortunately, it is well known that the coefficients of the successive powers of (a+b) can be arranged in a triangular array of numbers, called Pascal's triangle [8]. The Pascal's triangle has the following interesting properties.

1)The first number and the last number in each row is 1.

2)Every other number in the array can be obtained by adding the two numbers appearing directly above it (see TABLE I).

Therefore, we can have also easily the binomial coefficient part of (42) from the Pascal's triangle discussed above. Finally, we know that the shifted sinc-function part can be completely determined by the values of d_i because the remainings of the part are all fixed. The values of d_i are given by the exponential differences (40) in the expansion of $(a+b)^n$. From these analyses, we submit TABLE I for some order. The table can be easily extended to any order by simple arithmetics.

Table. 1 Parameters for the Fourier Transforms of the n-th order cosine-pulses

Order n	Constant	Pascal's Triangle	Exponential difference d_i
0	$2A\tau$	1	0
1	$A\tau$	1 1	1 -1
2	$A\tau/2$	1 2 1	2 0 -2
3	$A\tau/2^2$	1 3 3 1	3 1 -1 -3
4	$A\tau/2^3$	1 4 6 4 1	4 2 0 -2 -4
5	$A\tau/2^4$	1 5 10 10 5 1	5 3 1 -1 -3 -5
6	$A\tau/2^5$	1 6 15 20 15 6 1	6 4 2 0 -2 -4 -6
7	$A\tau/2^6$	1 7 21 35 35 21 7 1	7 5 3 1 -1 -3 -5 -7
8	$A\tau/2^7$	1 8 28 56 70 56 28 8 1	8 6 4 2 0 -2 -4 -6 -8

Ultimately, we need to calculate tediously to obtain the Fourier transforms of the functions like (1) any more. Only we need are the order of problem function and TABLE I. For example, given 5-th order of (1), we can find all the parameters needed on the 5-th row in TABLE I resulting in

$$P_5(f)=\frac{A\tau}{2^4}\Big\{\mathrm{sinc}\Big[2\tau f+\frac{5}{2}\Big]+5\,\mathrm{sinc}\Big[2\tau f+\frac{3}{2}\Big]+10\,\mathrm{sinc}\Big[2\tau f+\frac{1}{2}\Big]+10\,\mathrm{sinc}\Big[2\tau f-\frac{1}{2}\Big]+5\,\mathrm{sinc}\Big[2\tau f-\frac{3}{2}\Big]+\mathrm{sinc}\Big[2\tau f-\frac{5}{2}\Big]\Big\} \quad (43)$$

Applying the *Property 3* to the result (43) and after a few steps, we can obtain the transform in an one-term style.

Ⅵ. CONCLUSION

This paper has proposed a new and easy recursive method to derive the Fourier transform of the n-th order cosine-pulse using modified class-I PRS system model. We have discovered that the procedure is much more compact and simple than the conventional methods using consecutive differentiations or the convolution theorem in both analytical and numerical points of view. The formula derived for each order consists of a sum of two functions which are easily obtained from the step for the order just before and we can easily fabricate the transform resulting in an one-term style because the formula have only the sinc function shifted by symmetrical factors. We have also given some properties of the sinc functions for the conveniences of such calculations.

In addition, we have split up the final result into three parts by the separation of coefficients and we have shown that the parameters for each part can be obtained easily from a table which has been constituted using the well-known Pascal's triangle with (35) and the expansion of $(a+b)^n$. Now, we need not to perform complex calculations but need only to fetch the parameters for each order from the corresponding row by the table look-up processes.

APPENDIX A

Proff of the (*Proposition 1*)

From the structure of the system model for the n-th order function given in Fig.1, we have a Fourier transform pair for the transversal filter

$$k_n(t) = \delta(t+T_1/2) + \delta(t-T_1/2)$$
$$\Leftrightarrow K_n(f) = 2\cos \pi f T_1, \text{ for all } n=1, 2, 3, \cdots \quad (A.1)$$

and from the convolution theorem for the system theory [7], the system functions for the model constitute another pair as follows :

$$h_n(t) = k_n(t) * g_n(t) = k_n(t) * h_{n-1}(t) = h_{n-1}(t+T_1/2) + h_{n-1}(t-T_1/2)$$
$$\Leftrightarrow H_n(f) = K_n(f)\, G_n(f) = K_n(f)\, H_{n-1}(f) = 2\cos \pi f T_1\, H_{n-1}(f), \text{ for all } n=1, 2, 3, \cdots \quad (A.2)$$

where $H_0(f)$ and $h_0(t)$ are given in (24) and (25) respectively and the symbol '$*$' denotes the convolution integral of the two functions on both sides.

Next, we shall prove the proposition by the principle of mathematical induction in the following manner [6].

Basis step : When $n=1$, using the assumptions (24), (25) and the relation (A.2), we have

$$H_1(f) = 2\cos \pi f T_1\, H_0(f) = 2T_2 \cos \pi f T_1\, \Pi(T_1 f)$$
$$\Leftrightarrow \quad h_1(t) = h_0(t+T_1/2) + h_0(t-T_1/2)$$

which indicates that the statement $P(1)$ is true.

Induction step : If we assume that the statement $P(m)$ is true for some integer $m \geq 1$, we have a pair

$$H_m(f) = 2^m\, T_2\, (\cos \pi f T_1)^m\, \Pi(T_1 f)$$
$$\Leftrightarrow h_m(t) = h_{m-1}(t+T_1/2) + h_{m-1}(t-T_1/2)$$

And after one more step further with (A.1) and (A.2), we can obtain the relation for $m+1$ as follows :

$$H_{m+1}(f) = 2\cos \pi f T_1\, H_m(f) = 2^{m+1}\, T_2\, (\cos \pi f T_1)^{m+1}\, \Pi(T_1 f)$$
$$\Leftrightarrow \quad h_{m+1}(t) = h_m(t+T_1/2) + h_m(t-T_1/2)$$

which proves the validity of $P(m+1)$.

By induction, the statement $P(n)$ is true for all $n=1, 2, 3, \cdots$ (Q.E.D.).

APPENDIX B

Proof of the (*Proposition 2*)

This proposition can also be proved by the princeple of mathematical induction in the following manner [6].

Basis step : When $n=1$, we can obtain the following relationships using (36) and (37)

$$R_1(f) = R_0(f+1/4\tau) + R_0(f-1/4\tau)$$
$$\text{with } R_0(f) = \text{sinc}\,(2\tau f)$$

and they are written as

$$R_1(f) = \sum_{i=0}^{1} \begin{bmatrix} 1 \\ i \end{bmatrix} R_0\left[f + \frac{d_i}{4\tau} \right] = \sum_{i=0}^{1} \begin{bmatrix} 1 \\ i \end{bmatrix} \text{sinc}\left[2\tau f + \frac{d_i}{2} \right]$$

with $d_i = 1-2i\,(i=0, 1)$ from (40). Thus the statement $Q(1)$ is true.

Induction step : Suppose that the statement $Q(m)$ is true for some $m \geq 1$, then we have

$$R_m(f) = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix} R_0\left[f + \frac{d_i}{4\tau} \right] = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix} \text{sinc}\left[2\tau f + \frac{d_i}{2} \right]$$

with $d_i = m-2i\,(i=0, 1, 2, \cdots, m)$ from(40).

Applying again (36) and (37), we have an expression for $(m+1)$ as follows:

$$R_{m+1}(f)=R_m\left[f+\frac{1}{4\tau}\right]+R_m\left[f-\frac{1}{4\tau}\right]$$

$$=\sum_{i=0}^{m}\begin{bmatrix}m\\i\end{bmatrix}R_0\left[f+\frac{d_i+1}{4\tau}\right]$$

$$+\sum_{i=0}^{m}\begin{bmatrix}m\\i\end{bmatrix}R_0\left[f+\frac{d_i-1}{4\tau}\right]$$

where $d_i=m-2i$, for $i=0, 1, 2, \cdots, m$.

Expanding the summations in the above expression and binding the terms which have the same delay factors, we get

$$R_{m+1}(f)=\left\{\begin{bmatrix}m\\1\end{bmatrix}+\begin{bmatrix}m\\0\end{bmatrix}\right\}R_0\left[f+\frac{m-1}{4\tau}\right]$$

$$+\left\{\begin{bmatrix}m\\2\end{bmatrix}+\begin{bmatrix}m\\1\end{bmatrix}\right\}R_0\left[f+\frac{m-3}{4\tau}\right]$$

$$+\circ\circ\circ\circ+\left\{\begin{bmatrix}m\\s\end{bmatrix}+\begin{bmatrix}m\\s-1\end{bmatrix}\right\}R_0$$

$$\left[f+\frac{m-2s+1}{4\tau}\right]+\circ\circ\circ\circ$$

$$+\left\{\begin{bmatrix}m\\m-1\end{bmatrix}+\begin{bmatrix}m\\m-2\end{bmatrix}\right\}R_0\left[f-\frac{m-3}{4\tau}\right]$$

$$+\left\{\begin{bmatrix}m\\m\end{bmatrix}+\begin{bmatrix}m\\m-1\end{bmatrix}\right\}R_0\left[f-\frac{m-1}{4\tau}\right]$$

$$+\begin{bmatrix}m\\0\end{bmatrix}R_0\left[f+\frac{m+1}{4\tau}\right]$$

$$+\begin{bmatrix}m\\m\end{bmatrix}R_0\left[f-\frac{m+1}{4\tau}\right]$$

Using the theorem [8]

$$\begin{bmatrix}m\\s\end{bmatrix}+\begin{bmatrix}m\\s-1\end{bmatrix}=\begin{bmatrix}m+1\\s\end{bmatrix},\ \text{for } s=1, 2, 3, \cdots, m$$

and an identity

$$\begin{bmatrix}m\\0\end{bmatrix}=\begin{bmatrix}m+1\\0\end{bmatrix}=\begin{bmatrix}m\\m\end{bmatrix}=\begin{bmatrix}m+1\\m+1\end{bmatrix}$$

we can obtain a simple result

$$R_{m+1}(f)=\sum_{i=0}^{m+1}\begin{bmatrix}m+1\\i\end{bmatrix}R_0\left[f+\frac{d_i}{4\tau}\right]$$

were $d_i=(m+1)-2i$, for $i=0, 1, 2, \cdots, (m+1)$.

With the help of (37), we arrive at the final result as follows:

$$R_{m+1}(f)=\sum_{i=0}^{m+1}\begin{bmatrix}m+1\\i\end{bmatrix}R_0\left[f+\frac{d_i}{4\tau}\right]$$

$$=\sum_{i=0}^{m+1}\begin{bmatrix}m+1\\i\end{bmatrix}\mathrm{sinc}\left[2\tau f+\frac{d_i}{2}\right]$$

These prove the statement $Q(m+1)$ is also true.

Therefore, the statement $Q(n)$ is true for all $n=1, 2, 3, \cdots$. (Q.E.D.).

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